Name:

This is a closed-book test. No notes, books, or calculators allowed.

Orders of Magnitude (20 points): simply circle the correct answer.

1. A binary star system with a circular orbit of radius 100 AU is observed at a distance of 10 pc from earth. The maximum angular separation between the two stars is

   $1 \text{ rad} \quad 10'' \quad 1'' \quad 0.1''$

2. Recall that the absolute magnitude of the sun is about 5. Within roughly what distance from earth would a star like the sun become the brightest star in the night sky?

   $0.1 \text{ pc} \quad 1 \text{ pc} \quad 10 \text{ pc} \quad 100 \text{ pc}$

3. The hydrogen mass fraction in the sun is about

   $0.7 \quad 0.3 \quad 0.02 \quad 0.007$

4. The effective surface temperature of an M5 dwarf is about

   $10^7 \text{ K} \quad 10^4 \text{ K} \quad 3,000 \text{ K} \quad 2.7 \text{ K}$

5. The total luminosity of the Milky Way in Watts is of order

   $10^{37} \quad 10^{33} \quad 10^{26} \quad 10^{23}$

6. The radius of the stellar disk of the Milky Way is roughly

   $20 \text{ AU} \quad 20 \text{ Mpc} \quad 15 \text{ pc} \quad 15 \text{ kpc}$

7. The value of the Hubble constant $H_0$ is probably about 70 in what units?

   $\text{km s}^{-1} \quad \text{km s}^{-1} \text{ Mpc}^{-1} \quad \text{km s}^{-1} \text{ kpc}^{-1} \quad \text{kpc yr}^{-1}$

8. Starlight that just grazes the sun’s surface is deflected by about

   $2^\circ \quad 2' \quad 2'' \quad 2\text{ mas}$

9. The typical relaxation time in a galaxy like the Milky Way is about

   $10^8 \text{ yr} \quad 10^9 \text{ yr} \quad 10^{10} \text{ yr} \quad 10^{13} \text{ yr}$

10. The redshift $z$ of a galaxy at a distance of $10^{10}$ light-years is about

    $1 \quad \infty \quad 1500 \quad 0.1$
Problem 1. Relaxation Time. (20 points)

Considering *only strong encounters*, show explicitly that for any self-gravitating system containing $N$ point masses in virial equilibrium,

$$\frac{t_{\text{relax}}}{t_{\text{cross}}} \sim N,$$

where $t_{\text{relax}}$ is the relaxation time (mean time between encounters) and $t_{\text{cross}}$ is the crossing time (dynamical time) of the system.

Justify clearly and explain in words each step in your derivation. State all your assumptions explicitly. Ignore all numerical coefficients of order unity.
Problem 2. Gravitational lensing by galaxies. (30 points)
Consider the gravitational lensing of a distant source (e.g., a quasar or other bright galaxy) at distance $d_S$ by an extended lens (i.e., not a point mass as we considered previously) with a spherically symmetric distribution of mass (e.g., a galactic halo or a cluster of galaxies) at distance $d_L$. As before we will denote by $d_{LS} = d_S - d_L$ the distance between the source and the lens.

It can be shown that the deflection angle at distance $b$ from the center of this extended lens is given by

$$\alpha(b) = \frac{4G}{c^2} \frac{M(< b)}{b},$$

where $M(< b)$ is the projected mass within $b$ (i.e., the total mass contained within a cylinder of radius $b$, where the axis of the cylinder is the line of sight through the center of the lens).

(a) Show that the relation between the apparent angular position $\theta$ of an image and the true position $\beta$ of the source can be written

$$\theta \left[ 1 - \frac{\Sigma(< b)}{\Sigma_{\text{crit}}} \right] = \beta, \quad (1)$$

where the average surface density $\Sigma(< b) \equiv M(< b)/(\pi b^2)$ and

$$\Sigma_{\text{crit}} \equiv \frac{c^2}{4\pi G} \frac{d_S}{d_L d_{LS}}$$

is called the critical surface density.

(b) In general the surface density $\Sigma(< b)$ of a galaxy tends to a constant value $\Sigma_c$ near the center, and declines outside the center. For example it is easy to show that for a Plummer model $\Sigma(< b) = M/[\pi (a_p^2 + b^2)]$. Sketch the behavior of the LHS of eq. (1) as a function of $\theta = b/d_L$ for both $\Sigma_c < \Sigma_{\text{crit}}$ and $\Sigma_c > \Sigma_{\text{crit}}$ and conclude that lensing can occur only in the latter case.

(c) Find the angular radius of the Einstein ring, obtained when $\beta = 0$ and $\Sigma_c > \Sigma_{\text{crit}}$.

(Observations of distant background galaxies lensed into long arcs when seen through a foreground dense cluster of galaxies provides one of the main methods for directly measuring the total amount of mass in a cluster of galaxies. See §6.5.3 of S&G for more details.)
**Problem 3. Impulsive mass loss.** (30 points)

Consider a star cluster initially in virial equilibrium. Suppose that a fraction $f$ of the cluster’s mass is suddenly removed at each radius (this could represent, for example, a large number of supernova explosions taking place in the cluster).

(a) Show that if $f > 0.5$ the stars are no longer bound together and the cluster will disperse.

(b) Assuming $f < 0.5$, use the virial theorem to show that, after the remaining stars have come to a new equilibrium, the characteristic radius of the system will be larger by a factor $(1 - f)/(1 - 2f)$. What additional assumption did you have to make about the virialization process in order to get this simple result?

(c) If a star cluster is born with an initial stellar mass function given by $\xi(m)$ between $m_{\text{min}}$ and $m_{\text{max}}$, and $\tau(m)$ gives the total lifetime of a star born with mass $m$, explain how you would calculate (and give a general expression for) the mass loss fraction $f$ after some time $t$. Assume that all stars leave behind a remnant of mass $m_r \ll m$ at the end of their life.

If time permits, and for extra credit, you may want to try to estimate numerically the mass loss fraction $f$ in a typical globular cluster after, say, the first $\sim 10$ Myr of stellar evolution. Take, for example, $m_{\text{min}} = 0.5 M_\odot$ and $m_{\text{max}} = 100 M_\odot$ as reasonable values, and assume a steep initial mass function with $\xi(m) \propto m^{-3}$. 