Orders of Magnitude (20 points): simply circle the correct answer.

1. The brightest supergiants, which have absolute magnitudes $M_V \simeq -6$, can be seen with the naked eye out to a distance of about
   \begin{align*}
   12 \text{ AU} & \quad 12 \text{ pc} & \quad 0.5 \text{ kpc} & \quad 2.5 \text{ kpc}
   \end{align*}

2. A large telescope with an angular resolution of 0.1′ is used to observe a binary system in which the two stars are orbiting each other at 10 AU. The maximum distance out to which the two stars in this binary system can be resolved by the telescope is about
   \begin{align*}
   1 \text{ pc} & \quad 10 \text{ pc} & \quad 100 \text{ pc} & \quad 1 \text{ kpc}
   \end{align*}

3. A typical $1 M_\odot$ red giant expands to a maximum radius of about
   \begin{align*}
   1 R_\odot & \quad 10 R_\odot & \quad 1 \text{ AU} & \quad 1 \text{ pc}
   \end{align*}

4. The age of the universe in seconds is about
   \begin{align*}
   1.2 \times 10^{23} & \quad 4 \times 10^{17} & \quad 2 \times 10^{10} & \quad 3 \times 10^7
   \end{align*}

5. Because of differential rotation in the Galactic disk, stars near the Sun in the direction of galactic longitude $l \simeq 30^\circ$ have their spectra
   \begin{align*}
   \text{red shifted} & \quad \text{blue shifted} & \quad \text{absorbed} & \quad \text{horribly distorted}
   \end{align*}

6. The characteristic luminosity $L_\ast$ in the Schechter luminosity function is of order
   \begin{align*}
   L_\odot & \quad 100 L_\odot & \quad 10^6 L_\odot & \quad 10^{10} L_\odot
   \end{align*}

7. A large globular cluster contains about how many stars?
   \begin{align*}
   10^{10} & \quad 10^{6} & \quad 10^3 & \quad 10^2
   \end{align*}

8. Gravity deflects starlight that just grazes the surface of Jupiter by about
   \begin{align*}
   20 \text{ mas} & \quad 2^\circ & \quad 2 \text{ mas} & \quad 2^\circ
   \end{align*}

9. The fraction of $M_\odot c^2$ that the Sun will radiate during its entire life as a star is of order
   \begin{align*}
   10^{-9} & \quad 10^{-6} & \quad 10^{-3} & \quad 0.007
   \end{align*}

10. In a star cluster with velocity dispersion $\sigma \simeq 30 \text{ km s}^{-1}$, the distance of closest approach for strong encounters between solar-like stars is of order
    \begin{align*}
    1 \text{ AU} & \quad 1 \text{ pc} & \quad 1 R_\odot & \quad 10 \text{ km}
    \end{align*}
Problem 1. Gravitational Potentials. (30 points)

For each part of this problem it is crucial that you explain clearly your reasoning, using words and pictures as necessary.

(a) Using Gauss’s law, prove Newton’s theorem about the gravitational potential inside a hollow spherical shell of matter.

(b) Consider a mass distribution concentrated in a very long, thin, and straight filament with mass per unit length $\lambda$. (Examples could be cosmic strings or filaments in the large scale structure of the universe.) Derive an expression for the gravitational potential at a distance $R$ from the filament.

(c) In cylindrical coordinates $(R, z)$ the potential of a Kuzmin disk is given by

$$\Phi_K = -\frac{GM}{[R^2 + (a_K + |z|)^2]^{1/2}}.$$ 

Derive the expression for the corresponding surface mass density $\Sigma_K(R)$.

(d) How would the surface gravity $g = GM_{\oplus}/R_{\oplus}^2$ at a point $P$ on the surface of the Earth be changed if a hollow spherical cavity of radius $R_h$ were present at depth $d$ just below this point? Take the center of the cavity to be at a distance $d$ below the surface and on the radius from the center of the Earth to point $P$, with $R_h < d$. Also assume that the density inside the Earth is a constant $\rho_{\oplus} = M_{\oplus}/(4\pi R_{\oplus}^3/3)$. 

Problem 2. Gravitational lensing. (30 points)
Recall that the relation between the apparent angular position $\theta$ and the true position $\beta$ of a gravitationally lensed star can be written

$$\theta - \beta = \frac{\theta_E^2}{\theta},$$

where $\theta_E$ is the Einstein radius, given by

$$\theta_E^2 = \frac{4GM}{c^2} \frac{d_{LS}}{d_L d_S}$$

for a lens of mass $M$ at distance $d_L$; the other two distances are the distance to the source, $d_S$, and the distance from the lens to the source, $d_{LS} = d_S - d_L$.

(a) Derive the expression for the total magnification $\mathcal{M}$ in the limit where $\beta \ll \theta_E$. Explain clearly your reasoning in words and pictures.

(b) For a stationary source derive an expression for the shape of the microlensing lightcurve (magnification as a function of time) of a strong event near its peak, for a MACHO moving through the Galactic halo with proper motion $\mu$. Take $t = 0$ to be at the peak, where $\beta = \beta_0 \ll \theta_E$ for a strong microlensing event (with large magnification).

(c) Estimate roughly the total duration of a microlensing event for MACHOs in our Galactic halo. Scale your result to $(M/M_\odot)$. If repeated photometric observations of each source star are done once per day, what limit does this set on the mass of MACHOs that could be detected? (Useful numbers: $GM_\odot/c^2 \simeq 1.5$ km; 1 pc $= 3 \times 10^{18}$ cm)
Problem 3. Dynamical Mass Estimates. (20 points)

(a) From some of the data on satellites of the Milky Way in Table 4.1 of S&G (appended), obtain a dynamical mass estimate for our Galaxy, and calculate the corresponding mass-to-light ratio ($M/L_V$ in solar units). Explain how you would do a better job if you had more time.

(b) Think carefully and list all the assumptions you had to make in part (a).
Additional space: