1. The virial theorem states that for a star in hydrostatic equilibrium \(< P > = -\frac{1}{3} \frac{E_{gr}}{V}\), where \(< P >\) is the volume average pressure, \(V\) is the volume, and \(E_{gr}\) is the gravitational potential energy.

   a. For a star supported by thermal gas pressure \(P = nk_BT\), what is the relationship between the internal energy \(E_K\) and \(E_{gr}\).

   b. Assume the Sun is supported by thermal gas pressure and is composed entirely of ionized hydrogen. Estimate the typical (virial) temperature \(< T >\) in the Sun.

   c. For a star dominated by radiation pressure, what is the relationship between \(E_k\) and \(E_{gr}\)? What is the total \(E_{tot}\) in this case?

   d. Find an expression for the mass of a star at which radiation pressure begins to dominate. Don’t worry about numerical factors, express your answer in terms of the physical constants \(c, h, G\) and \(m_H\).

2. **Kelvin-Helmholtz time** Derive an expression for the Kelvin Helmholtz time for which the Sun could be shining at its present luminosity due to the release of gravitational energy. Calculate this time in years.

3. Imagine the Sun were to lose all pressure support. Estimate how long it would take to collapse gravitationally?

4. For single-line spectroscopic binary star, the maximal radial velocity \(v_{1obs}\) for the observed primary star is given by the expression:

\[
(m_1 + m_2) \sin^3 i = \frac{P|v_{1obs}|^3}{2\pi G} \left(1 + \frac{m_1}{m_2}\right)^3,
\]

where \(P\) is the binary period, \(m_1\) is the primary mass, \(m_2\) is the secondary mass and \(i\) is the inclination angle. Use this expression to estimate the “wobble velocity” \(v_{1obs}\) induced on a 1\(M_\odot\) star which is orbited by a Jupiter mass \((10^{-3}M_\odot)\) star at distance of 1 AU. What is the size of the Doppler shift \(\Delta\lambda/\lambda\) expected for this wobble velocity?

5. The luminosity of the Sun can be written as:

\[
L = \frac{U}{t_d},
\]

where \(U\) is the total energy in radiation in the star, and \(t_d\) is the typical time it takes a photon to diffuse in a random-walk from the center of the surface.

   a. Estimate the mean-free path of the photons between scattering in the Sun.
b. Estimate $t_d$ for the Sun, in years.

c. Is $t_d$ larger or smaller than the Kelvin-Helmholtz time $t_{KH}$? Explain.

6. Assume scattering of photons is via Thomson scattering with free electrons. Estimate the relation between stellar mass and luminosity in this case.

7. Write down the nuclear reaction in the p-p chain.

8. Estimate what fraction of the proton rest mass energy is released per fusion event.

9. Estimate how long the Sun can shine at its present luminosity via p-p reactions.

10. The nuclear reaction rate between two species A and B is:

$$P_{AB} = 6.5 \times 10^{-18} \frac{n_A n_B}{A_r Z_A Z_B} S(E_0) \left( \frac{E_G}{4k_BT} \right)^{2/3} \exp \left\{ -3 \left( \frac{E_G}{4k_BT} \right)^{1/3} \right\}$$

where $E_G = (\pi \alpha Z_A Z_B)^2 2\mu c^2$, $\alpha$ is the fine structure constant, $A_r$ is the reduced mass in units of the proton mass, and $\mu = A_r m_p$. Assume the p-p reaction occurs near a core temperature $10^7$ K. What is the sensitivity of the stellar luminosity to the core temperature (express as $L \propto T^a$, and find $a$)?

11. Recall that the Fermi momentum for an electron gas with density $n_e$ is defined by the equation:

$$n_e = \frac{8\pi P_f^3}{3 h^3}.$$  \hspace{1cm} (4)

a. Derive an expression for the degeneracy pressure for non-relativistic electrons given by $n_e$.

b. Derive the mass-radius relation for white dwarfs supported by non-relativistic degeneracy pressure. Don’t worry about numerical factors, express your answer as a relation between $R, M$ and the constants $\hbar, G, m_e$ and $m_p$.

12. Derive an expression for the degeneracy pressure for relativistic electrons.

13. Estimate the Chandrasekhar Mass for the maximum mass of white-dwarfs, assuming that electrons become relativistic as $R \to 0$. Again, ignore numerical factors, just keep physical constants $\hbar, c, G, m_p$.

14. Estimate the radius of a neutron star supported by degenerate non-relativistic neutrons. How much gravitational energy is released in the formation of a neutron star? If this occurs on a free-fall time what is the luminosity released?

15. The Friedmann equations are:

$$\left( \frac{\dot{R}}{R} \right)^2 = \frac{8\pi G}{3} \rho - \frac{k c^2}{R^2}$$

\hspace{1cm} (5)
\[ \frac{\dot{R}}{R} = \frac{8\pi G}{3} \left( \frac{\rho + 3P}{c^2} \right) \]  
\[ 0 = \dot{\rho} + 3 \frac{\dot{R}}{R} \left( \rho + \frac{P}{c^2} \right) \]  

(i) A galaxy is observed in which the hydrogen Ly\(\alpha\) line appears at 6000 Angstrom. By what factor has the universe expanded since the light was emitted by the galaxy?

(ii) For “radiation dominated” universe where \(P = \frac{1}{2} \rho_r c^2\), how does the scale factor vary with time?

(iii) Assume a flat matter-dominated universe.

a. How does the scale factor vary with time?

b. If the present day Hubble constant equals 72 km s\(^{-1}\) Mpc\(^{-1}\), what is the age of the Universe?

c. A galaxy is observed with redshift \(z = 2\). How old was the Universe when the light was emitted?