more
home work [set #2) April 5

1. show that my form and the forms used by Perlmutter & Schmidt are equivalent to mine in the Case where Ω_m is the only term

2. for Ω_m only versus $\Omega_m + \Omega_\Lambda$

compare d_L values for different $\Omega_T = \Omega_m + \Omega_\Lambda$

for $z = 0.5$, i.e. in principle how accurately do you need to determine d_L to distinguish?

[note to do correctly we really need a "K" correction for band width effects, but leave that to later]

3. show that if $T = (1+z)T_0$, $\nu = (1+z)\nu_0$ that a black body spectrum maintains its BB form for all z
use $B_\nu \propto \nu^3 c^{-2} / [e^{h\nu/kT} - 1]$
also show the BB energy density $= \int_0^\infty B_\nu \, d\nu$
~~goes~~ as $(1+z)^4$

(over)

4. do the binomial expansion of equation 42
5. do " " " of inside the integral of equation 41

6. compare the radiation density and matter density today and derive where they are about equal

use $H_0 = 50 \text{ km s}^{-1} \text{Mpc}$

$\Omega_0 = \Omega_m(0) = 0.1$

to get ρ_0 matter

and for ρ_{rad} use $T = 2.7$ K

Spring 1999

Cosmology Class C-30 D-30

Outline, "Myth" + Concepts

Cosmology is all about the structure and evolution of the Universe

In our model of the Universe, we live on a 4 dimensional surface that is expanding. The key cosmology question is how are we going to expand - forever or will we collapse.

Other questions we ask are
what is the mass + energy density of the Universe and what is the make up of this mass + energy density
Second, in our standard model, the Universe starts out very smooth but ended up clumpy. How did this happen?

Other questions: What was the initial composition of H, D, ^3He, ^4He, and ^7Li (Big Bang nucleosynthesis)
Particle Astrophysics ⇔ inflation and explains why slight over density of matter or anti-matter

Spring 1999

First a little "history", then what are some things we want to measure (and how)

Old, Old Cosmology, nothing but ordinary matter and all we needed were the equations for (non relativistic) matter dominated universe and then we had this nice relation:

$$q_0 = \tfrac{1}{2}\Omega_0 \quad ; \quad 0 < q_0 < 1/2 \;,\; k = -1$$
$$q_0 = 1/2 \quad k = 0$$
$$q_0 > 1/2 \quad k = 1$$

q_0 = deacceleration parameter

$$\Omega_0 = \frac{\rho_0}{\rho_c} \qquad \text{where } \rho_c = \frac{3 H_0^2}{8\pi G}$$

and the equations you needed to determine the luminosity distance and the angular distance and volume were are straight forward

The Universe might have seemed a "little" smooth in the past compared to today, and there was some dark matter, but overall, BBNS + total Ω_0 (dark baryonic matter included) was not too bad

then we had inflation ⇒ demanded $\Omega_0 = 1$
and demanded we learn how to derive $\Omega(z)$
[aside, $1+z = \frac{R_0}{R(t)}$ where R = scale factor of

the universe and $1+z$ also = λ emitted from an
object at time t whose light is just reaching us
now / λ observed i.e. = λ_e / λ_0, more later]

also, when combined with BBNS demanded that
there be ~ 90% of universe with non-baryonic
matter which is either non-relativistic
(CDM, i.e. WIMPS = Weakly Interacting
Massive Particles] or "neutralino's" such as
neutrinos, axinos or gravatinos = Hot
Dark Matter (see enclosed table from Nalikar for detail)
then we had to go back and revise our
equations for d_L etc to include a $\Omega_{relativistic}$
[more later].
One nice thing (the only?) nice thing about
inflation is that by invoking CDM and HDM,
it helped explain why, when we measured
the smoothness of the early universe, we were able
to form the structures we see today

For, in the mean time, the measurements of the CMB had gotten better so it became impossible to find a solution that gives a smooth background and a clumpy universe today without evoking "magic"

magic consists of either non-baryonic matter [WIMPS etc.] or "iso-curvature" – some how causes clumps prior to dissociation, but radiation has not followed the matter clumps and is smooth, or textures [defects in space based on field theory]

so What were the observational issues:

where / how is the dark matter distributed

how can we directly determine the non-baryonic nature of the dark matter?

how does the CMB vary on finer scales so as to distinguish, in principle, textures versus CDM vs CDM + HDM etc.

what is the "true measure" of the clumpiness of the universe to compare with more detailed predictions from the best fit to the CMB can we at least distinguish between Ω_r and Ω_m?

what limits can we place on Ω_{Bary} by measuring H and He, D, ^3He and Li?

in the fat chance but Nobel prize category:
directly detect the WIMPS
" " ~1.6 eV neutrinos from BB
[" " anti-matter nuclei in cosmic rays - indirectly related to cosmology in that we're not sure if any primordial anti matter was left over]

in the detail checking category there is measuring the age directly and measuring the implied age via the expansion rate and the Ω

evolution of galaxies & clusters of galaxies to be consistent with theory have to form early (but not too early)

can we measure how structure forms
can we check and show CMB is there at higher T as predicted

Then to make matters worse, a few years ago, in an attempt to measure Ω via d_L, astronomers claim to have found they need to invoke yet another parameter = Λ, the cosmological constant, so now there is Ω (total) Ω_m, Ω_r and Ω_v v = vacuum energy density
⇒ the measure of everything gets complicated and needs numerical integrations and now the relationship between q_0 and Ω_0 is totally "lost."
⇒ need to measure d_L accurately with respect to z so can delimit curvature and best hope is gravitational lenses and here can use QSO statistics etc.
Is there any consensus right now?

Observers would say they see dark matter and an $\Omega_0 < 0.2$. CMB smooth, galaxies clumpy so need some "magic": $H_0 \sim 50-70$ km/sec-Mpc, just might be able to get Ω_0 observed self consistent between DM observations and He, D abundances. Age of universe as measured versus implied from H_0, Ω_0 is okay.

Theoreticians consensus is $\Omega_0 = 1$
Inflation rules but needs modification of general
perturbations and physics [but never mind] and
then would like H_0 of 30 to fit age of universe.
Or allow $\Lambda \neq 0$ (cf. Figure 3.6 from book)
* **But** note, physics of $\Lambda \neq 0$ is very difficult.
i.e. "natural" explanations don't work.
By natural, I mean "first principle" estimate of
value of Λ (if not 0) from particle physics
yields $\Lambda = e^{137}$ (or so, i.e. very large)
so must resort to "exotica" to make things
come out right

$\Rightarrow$ Except for "fringe" can still set $\Lambda = 0$
and take $\Omega_0 = \Omega_m = 0.05 - 1$
and $H_0 \sim 50$ and will be okay most of
time.

now goals of course:
(1) to understand what is interesting to measure
(2) some idea of how to measure
(3) learn "applied" geometry i.e. d_L, d_A,
$dV(z)$, how to measure H_0, and if we
have time, how to measure "P_K" or "Δ_K" for
galaxies and clusters of galaxies, K correction
(4) learn standard "myth" —

of going from $t = 10^{-43}$ sec to present.
and why inflation people like inflation.

To summarize what we need to "explain" and measure better:

We see that universe is expanding, $H_0 \sim 50$ km/sec-Mpc
Radius of Universe $\sim 10^{28}$ cm
Age of " $\sim 15 \times 10^9$ yrs
He/H by number $\sim 10\%$, D/H = ? Li/H = ?
Cosmic Microwave Background exists,
is very smooth + shows fluctuation power spectrum (with θ)
Galaxies evolve, with most distant ones seen at $z \gtrsim 5.4$
Clusters of galaxies evolve with most distant at $z \sim 0.9$
QSO's evolve with closest at $z \sim 0.16$ (0.158)
most distant at $z \sim 5$
→ large and small scale structure exist
~ 100 Mpc ~ 100 kpc
$\Lambda \neq 0$? (probably not)

And without evoking some non-laboratory physics cannot explain smooth + clumpy, but if assume $\Omega_0 = 1$, then must necessarily require "magic"

Other handy units to know

Mpc = Mega parsec = 3×10^{24} cm
$L_\odot$ = solar luminosity = 4×10^{33} erg s^{-1}
$M_\odot$ = solar mass = 2×10^{33} g

Overview of Theory Section:
Geometry + GR ⇒ lots of models of expansion of universe
⇒ Inflation ⇒ new physics ⇒ standard Big Bang ⇒ connection to observables
(Ω_0, H_0, d_L, d_A, dV, P_k or Δ_k, CMB etc)

Concept of GR: Takes away Gravity as a force and replaces it with "geometry"
e.g. no gravity (or mass) ⇒ flat (local) space where mass and gravity yeild <u>curved</u> space.

⇒ Universe is curved in space and time
[time and distance are related]
all "distances" are caculated with the motion of a light beam and the travel time.
Since ligth is energy, light feels the pull of gravity ⇒ light travels a "curved" path
(the stronger the pull, the more curved, e.g. lenses or higher density universe)

The Roberson-Walker metric is based solely on the Cosmological Principle = The Universe is homogeneous and isotropic (perfectly)
metric means "measure" in such a way that light has measure 0
in simple form (we ignore θ & ϕ terms)

for light $d\tau^2 = c^2 dt^2 - \frac{R^2(t) dr}{1 - kr^2} = 0$ [1]

or $c^2 t^2 = \frac{R^2(t)\, dr}{1 - kr^2}$

this is equivalent to $v \times T = D$, but in a curved space [Einstein' GR + Roberson Walker metric are based on a 4 dim <u>surface</u> and the 4 dim are space + time

asides: $R(t)$ = scale factor of universe at the time light is emitted and t is measured from $R = 0$, $t = 0$ point.
"r" has no unit, R has unit of length
k " " " $= -1, 0, 1$

as noted earlier, $Z = \frac{\lambda_0 - \lambda_1}{\lambda_1} = \frac{R(t_0)}{R(t)} - 1$ [2]

note our goal is to derive how $R(t)$ evolves with time and to relate $R(t)$, z, $\Omega(z)$ together to be able to derive d_L, d_A, K correction etc.

This requires a knowledge of the equation of state and the all important physical assumption that the expansion of the universe is <u>adiabatic</u>

comment on book comment on

$$1 + z \neq \text{really} \sqrt{\frac{1 + v/c}{1 - v/c}} \quad \text{because says}$$

we are not really moving away. However, net effect is there and observable universe stops where $v = c$. Whatever,
maybe most important is for small z we <u>do</u> measure relative motions, e.g. 2 clusters of galaxies with $10^{15} M_\odot$ each

so be ware that $z \approx v/c$ but for $z \gtrsim 0.01$ still need to use $1 + z = \sqrt{\frac{1+v/c}{1-v/c}}$

so if you measure v you must use above to get z and vice versa we will ignore Λ to start and then try to include Λ and other "exotica"

now we will write down Einstein Field equation which looks just like newtonian escape equation with an extra term

$$\dot{R}^2 + kc^2 = \frac{G 8\pi}{3} \rho R^2 \qquad [3]$$

1st concept just from here is critical density

$$\Rightarrow \text{today} \quad \frac{\dot{R}_0^2}{R_0} + \frac{kc^2}{R_0^2} = \frac{G 8\pi}{3} \rho_0 \qquad [4]$$

and we see that $k = 0$ when

$$\rho_c \frac{G 8\pi}{3} = \frac{\dot{R}_0^2}{R_0^2} \equiv H_0^2 \text{ or } \rho_c = \frac{3}{8\pi G} \frac{\dot{R}_0^2}{R_0^2} \qquad [5]$$

$$\boxed{\rho_c = \frac{3 H_0^2}{8\pi G} \quad [6]}$$

for, H_0 = Hubble constant $\equiv \frac{\dot{R}_0}{R_0}$ [7]
and this is measurable as

v = Distance $*$ H_0 for v small enough so "t" is negligibly different from t_0
but D not so close that local gravity dominates
$z \gtrsim 0.02$

$\Rightarrow$ easy to see how $k=-1$ $\quad \rho < \rho_c$ $\quad$ [8.a]

$k = 0$ $\quad \rho = \rho_c$ $\quad$ [8.b]

$k = 1$ $\quad \rho > \rho_c$ $\quad$ [8.c]

aside for later $\Omega_0 = \rho_0/\rho_c$

$\Omega_n(0)$ = different Ω's at $z = 0$ $\quad$ [9]

and n = different equations of state so

[10] $\Omega_n(0) = \dfrac{\rho_n(0)}{\rho_c}$ $\quad$ and $\Omega_n(0)(1+z)^n \equiv \rho_n(0)(1+z)^n$

$\boxed{\text{note } \Omega(1+z) = \text{function of } z \text{ is not } = \Sigma\Omega_n^{(0)}(1+z)^n}$

$\rho_n = \rho_n(0)(1+z)^n$ is how ρ_n varies with z see below

we can rewrite "[4]" as

$$\text{first } \frac{kc^2}{R_0^2} = H_0^2\left(\rho_0/\rho_c - 1\right) = H_0^2(\Omega_0 - 1) \quad [11]$$

from here we can see that H depends on z if ρ has different equations of state so start with

$$\frac{kc^2}{R^2} = H^2\left(\frac{\rho}{\rho_c'} - 1\right) \quad [12]$$

which is first eq. "11" with the "0" removed

$$\rho_c' = \frac{3H^2}{8\pi G} \; ; \; \rho_c = \frac{3H_0^2}{8\pi G} \quad [14]$$

$$\Rightarrow \frac{\rho_c'}{\rho_c} = \frac{H^2}{H_0^2} \text{ or } \rho_c' = \rho_c\frac{H^2}{H_0^2}$$

note, we can see slight typo in book for eq. 3.32

since $\rho_c' = \frac{3H^2}{8\pi G}$ and $\rho_c = \frac{3H_0^2}{8\pi G}$ $\rho_c' = \rho_c \frac{H^2}{H_0^2}$

can replace ρ_c' in equation "12" by $\rho_c \frac{H^2}{H_0^2}$

$$\Rightarrow \frac{kc^2}{R^2} = H^2\left(\frac{\rho}{\frac{H^2}{H_0^2}\rho_c} - 1\right) \qquad [13]$$

but re write ρ as $\Sigma \rho_n(0)(1+z)^n$

and using $\Omega_n(0) = \frac{\rho_n(0)}{\rho_c}(1+z)^n$

we can replace $\frac{\rho}{\rho_c}$ with $\Sigma \Omega_n(0)(1+z)^n$

$$\text{or} \quad \frac{kc^2}{H^2R^2} = \frac{H_0^2}{H^2}\left(\Sigma \Omega_n(0)(1+z)^n - \frac{H^2}{H_0^2}\right) \qquad [14]$$

compare with 3.32

$$\Rightarrow H^2 + \frac{kc^2}{R^2} = H_0^2\left(\Sigma \Omega_n(0)(1+z)^n\right) = \frac{8\pi G\rho}{3} \qquad [15]$$

compare with 3.31

Table 1

from book by Eric Linder

$$\rho_n \propto \left(\frac{R_0}{R}\right)^{3(1+\sigma) \equiv n}$$

n	σ	Form of energy density
3	0	non-relativistic matter
4	1/3	relativistic matter
6	1	free massless scalars, e.g. "shear" energy
2	−1/3	Cosmic strings
1	−2/3	Domain Walls
0	−1	Cosmological constant

now let's go back to equation 12
re written as $\frac{\dot{R}^2}{R^2} + \frac{kc^2}{R^2} = \frac{8\pi G \rho}{3}$

and take simple case first of $\rho = \rho_0 (1+z)^3 = \frac{R_0^3}{R^3}$
[matter dominated $p = 0$ case, see below]

$$\Rightarrow \quad \frac{\dot{R}^2}{R^2} + \frac{kc^2}{R^2} = \frac{8\pi G\rho_0}{3} \frac{R_0^3}{R^3}$$

for $k = 0$ $\quad \frac{R}{R_0^3}\left(\frac{dR}{dt}\right)^2 = \frac{8\pi G \rho_0}{3} = \Omega_0 H_0^2$
but $\Omega_0 = 1$
by definition!

or $\quad \frac{R^{1/2}\,dR}{H_0 R_0^{3/2}} = dt \qquad [16]$

and can integrate and find age of universe
$= \quad t_0 = \frac{2}{3}\frac{1}{H_0}$ [17], but we see if

$k \neq 0$ we need another equation to get rid of the kc^2 term

$\Rightarrow$ adiabatic expansion

two asides: the above integral is done from 0 to t
(beginning of universe when $R = 0$ to some t up to t_0 [or beyond]
the other side? Well use $x = \frac{1}{1+z}$ then
$\frac{dR}{R_0} = d\left(\frac{1}{1+z}\right) = dx$ and x goes from 1 (at $z=0$ today)
to $1+z$ where

relates to $t = 0$, so in this case, $z \to \infty$ and $x \to 0$

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then A+A "
" send query

so $\int_0^{t_0} dt = \int_{R=0}^{R_0} = \int_0^1 \frac{x^{1/2}\,dx}{H_0}$ in the above example

and we see that again $t_0 = \frac{2}{3}\frac{1}{H_0}$

we also see that the age of universe at some arbitrary time is just $t = \frac{2}{3}\frac{1}{H_0}\left(\frac{1}{1+z}\right)^{3/2}$

and if we want a "look back" i.e., how long it took light to get here or how much younger the object at $1+z$ is compared to today, we need

$t_0 - t$ which in this case is

$$\frac{2}{3}\frac{1}{H_0} - \frac{2}{3}\frac{1}{H_0}\left(\frac{1}{1+z}\right)^{3/2}$$

or $\frac{2}{3}\frac{1}{H_0}\left(1 - \left(\frac{1}{1+z}\right)^{3/2}\right) \Rightarrow z = 5$

means $\left(1 - \left(\frac{1}{6}\right)^{3/2}\right) = 0.93$ or we are looking back to see (in time) 93% of universe!

if instead, looking at

$$\int_0^{r_1} \frac{dr}{(1-kr^2)^{1/2}} = \int_{t_1}^{t_0} = \int_{R_1}^{R_0} = \int_{\frac{1}{1+z}}^{1} \frac{c\,dt}{R(t)}\frac{dR}{dt}\frac{1}{\dot{R}}\frac{R_0}{R_0}$$

and the limits on the left are from the object to us.
on the right, the object is at $1+z$ so that is where the scale factor starts off when light left point and $1 = \frac{1}{1+0=z}$ when light reaches us

$$dU = dQ - dW$$

but for adiabatic expansion, $dQ = 0$

so $dW = p\,dV = p\,d(R^3)$ [18]

$dU = dE = dmc^2 = d(\rho c^2 R^3)$ [19]

so using $dU + dW = 0$ and dividing by dt,

$$\frac{d\rho c^2 R^3}{dt} + p\frac{dR^3}{dt} \quad [20]$$

or

$$3p\dot{R}R^2 + \dot{\rho}c^2R^3 + 3\rho c^2\dot{R}R^2 = 0 \quad [21]$$

solving from $\dot{\rho}$ and substituting into "11" or "3"

we have

$$[22] \quad \boxed{2\dot{R}\ddot{R} = 2 * \frac{8\pi G}{3}\dot{R}R\rho + \frac{8\pi G R^2}{3}\left(-3\frac{p\dot{R}}{Rc^2} - 3\rho\frac{\dot{R}}{R}\right)}$$

now trick is to equate p in terms of ρ.

"old fashioned" way of thinking of this was

$p = (\gamma - 1)\rho c^2$, γ = ratio of specific heats
but $\gamma = 4/3$ relativistic
matter $\gamma = 1$ for "isothermal"
today when $p \approx 0$
and $\gamma = 0$ for $\Lambda \neq 0$

in parlance of modern day
simply write $p = c^2 \sigma \rho$ as in table 1
and we see $\sigma = \gamma - 1$ (so we can derive
other pressures, if
need be)

and $p = \Sigma p_i = \sum_i c^2 \sigma_i \rho_i$

so $-\ddot{R} = -\frac{8\pi}{3} G R \rho + \frac{8\pi}{3} G R^2 \left(+ \frac{3}{2} \frac{p}{c^2 R} + \frac{3\rho}{2R} \right)$ [23]

or $-\frac{\ddot{R} R}{\dot{R}^2} = q = -\frac{8\pi}{3} G \rho \frac{R^2}{\dot{R}^2} + \frac{8\pi}{3} G \frac{R^2}{\dot{R}^2} \left(\frac{3}{2} \frac{p}{c^2} + \frac{3}{2}\rho \right)$ [24]

simple case 1st, which is $p = 0$, at $t = t_0$ today,
then $q_0 = -\frac{2}{2}\left(\frac{8\pi G \rho_0}{3 H_0^2}\right) + \frac{3}{2}\left(\frac{8\pi G}{3 H_0^2} \rho_0\right) = \frac{1}{2} \frac{\rho_0}{\rho_c}$ ← [25]

so

$$q_0 = \frac{-8\pi G}{3H_0^2}\rho_T + \frac{3}{2}\left(\frac{8\pi G}{3H_0^2}\left[\rho_T + \frac{1}{3}\rho_r - \rho_\Lambda\right]\right] \qquad [26]$$

$$= -\Omega_m - \Omega_r - \Omega_\Lambda + \frac{3}{2}\Omega_m + \frac{3}{2}\Omega_r + \cancel{\frac{3}{2}\Omega_\Lambda} + \frac{1}{2}\Omega_r - \cancel{\frac{3}{2}\Omega_\Lambda}$$

$$\boxed{q_0 = \frac{1}{2}\Omega_m^{(0)} + \Omega_r^{(0)} - \Omega_\Lambda^{(0)}}$$ [27] cf equation 3.34 in book

but what we want, remember is
$t_0 = f(H_0, \Omega_0)$ and

$d_L = f(H_0, \Omega_0, 1+z)$ and we won't work much with q_0 much even though book implies something else also book is wrong ~~again~~ for equation on q – see below
[more for curiosity, go back to

$$q = -\frac{8\pi G\rho}{3}\frac{R^2}{\dot{R}^2} + \frac{8\pi G}{3}\frac{R^2}{\dot{R}^2}\left(\frac{3}{2}\frac{p}{c^2} + \frac{3}{2}\rho\right)$$

and plug in for $\rho = \dfrac{3H^2}{8\pi G} + \dfrac{kc^2}{R^2}\dfrac{3}{8\pi G}$

first combine the ρ's

$$q = \frac{8\pi G}{3H^2}\left[\frac{1}{2}\rho + \frac{3}{2}\frac{p}{c^2}\right]$$

then have $q = \frac{8\pi G}{3 H^2}\left[\frac{1}{2}\frac{3H^2}{8\pi G} + \frac{kc^2}{R^2}\frac{3}{8\pi G} + \frac{3}{2}\frac{p}{c^2}\right]$ [28]

then set $k = 0$ $\qquad p = \rho_m^{\to 0}\ \frac{1}{3}\rho_r c^2 - \rho_\Lambda c^2$

$$q = \frac{1}{2} + \frac{\rho_r}{2\rho_c'} - \frac{3\rho_\Lambda}{2\rho_c'}$$

or $q = \frac{1}{2} + \frac{\Omega_r}{2} - \frac{3}{2}\Omega_\Lambda$ for all time for $k = 0$

Compare with eq. 3.34a

note if $\Omega_r = 0$ $\Omega_\Lambda = 0$ $q = 1/2$ for all time in my equation, which it should

now turning to $\displaystyle\int \frac{dr}{(1-kr^2)^{1/2}} = \int \frac{dt}{R(t)}$

we need to find $R(t)$ as function of t or dt as function on R so can integrate the right side. The left side integrates to $\sin(r)$, $\sinh(r)$ depending on k ($k \neq 0$) so to go through ~~too~~ get the final result is a big pain,

the answer, written commonly as $R_0 r_1$ is written in the book as $R_0 S_k(r)$, eq. 3.78

$$\int_0^{r_1} \frac{dr}{(1-kr^2)^{1/2}} = \int_{\frac{1}{1+z}}^{1} c \, \frac{x^{-1}dx}{\dot{R}} \quad ; \text{ now need to} \qquad [31]$$

have $\dot{R}$ in terms of $1+z$ or x and we're almost all set

do this with $\dot{R}^2 + kc^2 = \frac{8\pi G}{3} \rho R^2$

and $\dot{R}_0^2 + kc^2 = \frac{8\pi G \rho_0 R_0^2}{3}$

and remove the kc^2

$$\Rightarrow \frac{\dot{R}^2}{\dot{R}_0^2} = H_0^2 \left[\frac{R^2}{R_0^2} \frac{\rho}{\rho_c} - \frac{\rho_0}{\rho_c} + 1 \right]$$

$$\text{or } \dot{R} = R_0 H_0 \left[x^2 \sum_n \Omega_0 \, x^{-n} - \Omega_0 + 1 \right]^{1/2} \qquad [32]$$

$n = 3, 4, 0$ for ... in general case, and in simple case first, $n = 3$

$$\text{or } \dot{R} = R_0 H_0 \left[\Omega_0 x^{-1} - \Omega_0 + 1 \right]^{1/2} \qquad [33]$$

for $n = 3$ or 0, next simplest

$$\dot{R} = R_0 H_0 \left[\Omega_0 x^{-1} + x^2 \Omega_0 - \Omega_0 + 1 \right]^{1/2}$$

now do simplest case first which is $n = 3$ only

$$\Rightarrow \begin{array}{ll} \sin^{-1} r & (k=1) \\ r & (k=0 \\ \sinh^{-1} r & (k=-1) \end{array} = \frac{c}{H_o R_o} \int_{\frac{1}{1+z}}^{z} \frac{x^{-1} dx}{[\Omega_o x^{-1} - \Omega_o + 1]^{1/2}} \quad [34]$$

$$= \frac{c}{H_o R_o} \int \frac{dx}{[\Omega_o x + x^2(1-\Omega_o)]^{1/2}} \quad [35]$$

[36] use $\cos^{-1}\left(\frac{a-x}{a}\right) = 2\sin^{-1}\sqrt{\frac{x}{a}}$

or $\sinh x = \cosh^{-1}\sqrt{x-1}$ [37]

- $\sin(A-B) = \sin A \cos B - \sin B \cos A$
or $\sinh^{-1}(x-y) = \sinh A \cosh B - \sin B \cosh A$

and $$\int \frac{1}{\sqrt{bx+ax^2}} = \frac{1}{\sqrt{a}} \sinh^{-1}\left(\frac{2ax+b}{-b^2}\right), \quad a>0 \quad [38]$$

$$= \frac{1}{\sqrt{-a}} \sin^{-1}\left(\frac{-2ax-b}{b}\right), \quad a<0 \quad [39]$$

$\Rightarrow$ we see the $a<0$ case $\sim \Omega_o > 1, k=1$

now also need $\frac{k}{(\Omega_o - 1)^{1/2}} = \frac{R_o H_o}{c}$ $\quad k=1$, $k=-1$

which is true regardless of matter type!

need this to get rid of the $\frac{c}{H_o R_o}$

then right hand side in case of sin for example ($k = +1$) is (to get r_1 on left, take sin of both sides)

[40] $\sin\left[\frac{c}{R_0 H_0} * \sqrt{\Omega_0 - 1} \, \sin^{-1}[\ ] - \sin^{-1}[\ ]\right]$

has $x = 1$ has $x = \frac{1}{1+z}$

then we see $\frac{c}{R_0 H_0} \sqrt{\Omega_0 - 1} = 1$

and we have our desired $\sin(A - B)$ form and we just have to switch between $\cos^{-1}$ and $\sin^{-1}$ as necessary! ✓

$\Rightarrow$ for $\Lambda \neq 0$ we have (see Schmidt et al or Perlmutter et al)

[41] $$D_L = \frac{1}{H_0}(1+z)|K_0|^{-1/2} S$$

$$* \left\{ |K_0|^{1/2} \int_0^z dz' \left[\sum_i \Omega_i (1+z')^{3+3\sigma_i} - K_0 (1+z')^2 \right]^{-1/2} \right\}$$

$$K_0 \equiv \frac{k c^2}{R_0^2 H_0} = (\Omega_0 - 1)^{1/2}$$

$S(x) \equiv \sin(x)$, x or $\sinh(x)$ $k = 1, 0, -1$

and if $z < 1$ we can use

$$D_L H_0 = \frac{1}{q_0^2}\left[q_0 z + (q_0 - 1)(\sqrt{1 + 2 q_0 z} - 1\right]$$

where

$$q_0 = \frac{1}{2}\Omega_m^{(0)} + \Omega_r^{(0)} - \Omega_\Lambda^{(0)} \quad \checkmark \text{ as defined}$$

2 references are

Schmidt et al Ap. J. <u>507</u>, pages 46-63, 1998, Nov 1

Perlmutter et al Ap. J. <u>483</u> pages 565-581, 1997, July 10

so note, we need D_L very accurately or we need to do the true form of D_L with $z \gtrsim 1$!

we see that from homework

$$\Omega = \frac{\Omega_0}{\Omega_0 + (1-\Omega_0)\left(\frac{R_0}{R}\right)^{2-n}}$$ so for $n = 3$ or 4 (matter or rel

$\Omega \to 1$ at $z = \frac{R_0}{R} \to \infty$

and if $\Omega_0 = 1$ $\Omega = 1$ for all time!

if $\Lambda \neq 0$ then tricky, if we start with $\Omega_0 = 1$ and go back, we see $\Omega = 1$ for all time, but if $\Omega_0 \neq 1$ today, as we go back, $\Omega \to 0$
as the book notes this is not suprising because

$$\frac{\dot{R}^2}{R^2} + \frac{kc^2}{R^2} = \frac{8\pi G}{3} \rho_0 \left(\frac{R_0}{R}\right)^n$$ and for $n \doteq 3$ or 4

(radiation dominates at about $z = 1000$)

we have $\frac{kc^2}{R^2}$ term becomes negligible.

distances we need to know - note we'll add in K correction later but also look at papers by Schmidt & Perlmutter

$$d_L = d_P(1+z)$$

$$d_A = d_L(1+z)^{-2}$$

$$d_M = d_L(1+z)^{-1}$$

cf Weinberg

and for Ω_m, $p=0$ (FRW models)

$$r_p = R_0 r_1 = \frac{2z\Omega_0 + 2\left(\frac{\Omega_0}{2}-1\right)\left(-1+\sqrt{\Omega_0 z+1}\right)}{H_0\,\Omega_0^2} \qquad [42]$$

$$q_0 = \tfrac{1}{2}\Omega_0$$

if use binomial expansion of $\Omega_0 z + 1$ term can get case of $\Omega_0 z \ll 1$ so can get $\Omega_0 \sim 0$, $z < \infty$; this is because you only have Ω_0^2 or higher order Ω_0 terms on top ✓