$$\int_0^{r_1} \frac{dr}{(1-kr^2)^{1/2}} = \int_{\frac{1}{1+z}}^{1} c\,\frac{x^{-1}dx}{\dot{R}}$$; now need to [31]

have $\dot{R}$ in terms of $1+z$ or x and we're almost all set

do this with $\dot{R}^2 + kc^2 = \frac{8\pi G}{3}\rho R^2$

and $\dot{R}_0^2 + kc^2 = \frac{8\pi G \rho_0 R_0^2}{3}$

and remove the kc^2

$$\Rightarrow \frac{\dot{R}^2}{\dot{R}_0^2} = H_0^2 \left[\frac{R^2}{R_0^2}\frac{\rho}{\rho_c} - \frac{\rho_0}{\rho_c} + 1 \right]$$

or $\dot{R} = R_0 H_0 \left[x^2 \sum_n \Omega_0\, x^{-n} - \Omega_0 + 1 \right]^{1/2}$ [32]

$n = 3, 4, 0$ for us in general case, and in simple case first, $n = 3$

or $\dot{R} = R_0 H_0 \left[\Omega_0 x^{-1} - \Omega_0 + 1 \right]^{1/2}$ [33]

for $n = 3$ & 0, next simplest

$\dot{R} = R_0 H_0 \left[\Omega_0 x^{-1} + x^2 \Omega_0 - \Omega_0 + 1 \right]^{1/2}$

now do simplest case first which is $n = 3$ only

$$\Rightarrow \quad \begin{matrix} \sin^{-1} r & (k=1) \\ r & (k=0 \\ \sinh^{-1} r & (k=-1) \end{matrix} \quad = \frac{c}{H_0 R_0} \int_{\frac{1}{1+z}}^{1} \frac{x^{-1}\, dx}{[\Omega_0 x^{-1} - \Omega_0 + 1]^{1/2}} \qquad [34]$$

$$= \frac{c}{H_0 R_0} \int \frac{dx}{[\Omega_0 x + x^2 (1-\Omega_0)]^{1/2}} \qquad [35]$$

[36] use $\cos^{-1}\left(\frac{a-x}{a}\right) = 2 \sin^{-1} \sqrt{\frac{x}{a}}$

or $\sinh x = \cosh^{-1} \sqrt{x-1}$ [37]

- $\sin(A-B) = \sin A \cos B - \sin B \cos A$

or $\sinh^{-1}(x-y) = \sinh A \cancel{\cosh} \cosh B - \sin B \cosh A$

and $$\int \frac{1}{\sqrt{bx + ax^2}} = \frac{1}{\sqrt{a}} \sinh^{-1}\left(\frac{2ax+b}{-b^2}\right), \quad a>0 \quad [38]$$

$$= \frac{1}{\sqrt{-a}} \sin^{-1}\left(\frac{-2ax-b}{b}\right), \quad a<0 \quad [39]$$

$\Rightarrow$ we see the $a<0$ case $\sim \Omega_0 > 1, k=1$

now also need $\frac{(k)^{1/2}}{(\Omega_0 - 1)^{1/2}} = \frac{R_0 H_0}{c}$ $\quad k=1$, $k=-1$

which is true regarless of matter type!

need this to get rid of the $\frac{c}{H_0 R_0}$

then right hand side in case of sin for example ($k=+1$) is (to get r_1 on left, take sin of both sides)

[40] $$\sin\left[\frac{c}{R_0 H_0} * \sqrt{\Omega_0 - 1}\ \sin^{-1}[\ \ ] - \sin^{-1}[\ \ ]\right]$$

has $x=1$ has $x=\frac{1}{1+z}$

then we see $\frac{c}{R_0 H_0}\sqrt{\Omega_0-1} = 1$

and we have our desired $\sin(A-B)$ form and we just have to switch between $\cos^{-1}$ and $\sin^{-1}$ as necessary!

note $R_0 r_1$ = distance to r_1 without relativity correction = proper distance to z and $R_0 r_1(1+z) = D_L$

$\Rightarrow$ for $\Lambda \neq 0$ we have (see Schmidt et al or Perlmutter et al)

[41] $$D_L = \frac{1}{H_0}(1+z)|K_0|^{-1/2} S$$

$$* \left\{ |K_0|^{1/2} \int_0^z dz' \left[\sum_i \Omega_i (1+z')^{3+3\sigma_i} - K_0(1+z')^2\right]^{-1/2} \right\}$$

$$K_0 \equiv \frac{kc^2}{R_0^2 H_0} = (\Omega_0 - 1)^{1/2}$$

$S(x) \equiv \sin(x)$, x or $\sinh(x)$ $k = 1, 0, -1$

and if $z < 1$ we can use

$$D_L H_0 = \frac{1}{q_0^2}\left[q_0 z + (q_0 - 1)(\sqrt{1+2q_0 z} - 1)\right]$$

where

$$q_0 = \frac{1}{2}\Omega_m^{(0)} + \Omega_r^{(0)} - \Omega_\Lambda^{(0)} \quad \checkmark \text{ as I derived}$$

2 references are

Schmidt et al Ap. J. <u>507</u>, pages 46–63, 1998, Nov 1

Perlmutter et al Ap. J. <u>483</u> pages 565–581, 1997, July 10

so note, we need D_L very accurately or we need to do the true form of D_L with $z \gtrsim 1$!

we see that from homework ↙ correct, same as book

$$\Omega = \frac{\Omega_0}{\Omega_0 + (1-\Omega_0)\left(\frac{R_0}{R}\right)^{2-n}}$$

so for $n = 3$ or 4 (matter or rel matter).

$\Omega \to 1$ at $z = \frac{R_0}{R} \to \infty$

and if $\Omega_0 = 1$ $\Omega = 1$ for all time!

if $\Lambda \neq 0$ then tricky, see (25A) i.e. need to put in another form - effectively this is a "divide by zero" proplem

as the book notes this is not surprising for ordinary matter ($\Lambda = 0$) because

$$\frac{\dot{R}^2}{R^2} + \frac{kc^2}{R^2} = \frac{8\pi G}{3}\rho_0\left(\frac{R_0}{R}\right)^n$$

and for $n \doteq 3$ or 4 (radiation dominates at about $z = 1000$)

we have $\frac{kc^2}{R^2}$ term becomes negligible but ! $$r-1 = \frac{r_0 - 1}{1 - r_0 + \varepsilon\left(\frac{R_0}{R}\right)^{n-2}(r_0)}$$ for $n<2$

$n = 0, 3, 4$

$\left(a = \frac{1}{1+z} \text{ in book}\right)$

is the same thing, and is this form we can see that as $\frac{R_0}{R} \to \infty$ that $r-1 \to 0$ and r still goes to 1

but in this form of only $n = 3$ or 4

this looks like $r-1 \to -1$ (ouch!)

so be careful! problem is can't divide by 0 and one way or other this is what is happening i.e. use the above if $\Lambda \neq 0$ and use my form of $\Lambda = 0$

$$r - 1 = \frac{r_0 - 1}{(\quad)} \Rightarrow$$

$$r = \frac{\cancel{r_0 - 1} + \cancel{1 - r_0} + \varepsilon\left(\frac{R_0}{R}\right)^{2-n} r_0}{1 - r_0 + \varepsilon\left(\frac{R_0}{R}\right)^{2-n} r_0}$$

$$r = \frac{r_0}{(1 - r_0)\,\varepsilon\left(\frac{R_0}{R}\right)^{n-2} + r_0}$$

which is my form ✓ which is good for $n \geq 3$

distances we need to know — note we'll add in K correction later but also look at papers by Schmidt & Perlmutter

$d_L = d_p (1+z)$

$d_A = d_L (1+z)^{-2}$

$d_M = d_L (1+z)^{-1}$

cf Weinberg

and for Ω_m, $p=0$ (FRW models)

$$r_p = R_0 r_1 = \frac{2z\Omega_0 + 2\left(\frac{\Omega_0}{2} - 1\right)\left(-1 + \sqrt{\Omega_0 z + 1}\right)}{H_0 \Omega_0^2} \qquad [4\,2]$$

$$q_0 = \tfrac{1}{2}\Omega_0$$

if use binomial expansion of $\Omega_0 z + 1$ term can get case of $\Omega_0 z \ll 1$ ~~so~~ can get $\Omega_0 \sim 0$ $z < \infty$; this is because you only have Ω_0^2 or higher order Ω_0 terms on top ✓

reminder then:

$$\dot{R} = R_0 H_0 \left[\frac{R^2}{R_0^2} \frac{\rho}{\rho_c} - \frac{\rho_0}{\rho_c} + 1 \right]^{1/2}$$

will have $1+z$ dep.

Ω_0 and no $1+z$ dep!

so writing $x = \frac{1}{1+z} = \frac{R}{R_0}$

$$\Rightarrow \quad \Omega_n = \Omega_n(0)\, x^{-n}$$

$$\text{or } \Omega = \frac{\rho}{\rho_c} = \sum_n \Omega_n(0)\, x^{-n} \qquad n = 0, 3, 4$$

for us

$$\Rightarrow \dot{R} = R_0 H_0 \left[x^2 \sum_n \Omega_n x^{-n} - \Omega_0 + 1 \right]^{1/2}$$

$$\text{or} \quad \int_0^1 \frac{dx}{[\quad]} = \int_0^t dt$$

where $\frac{dR}{R_0} = d\left(\frac{1}{1+z}\right) = d\left(\frac{R}{R_0}\right)$

and $x = \frac{1}{1+z}$

now remember that we started with $\int_{t_1}^{t_0} dt$

for previous calculation of r, R_0

suppose we want age of universe?
then make $t_1 \to 0$
and or $\frac{1}{1+z} \to 0$ ($z \to \infty$) and then

we need to integrate again over

$$\frac{1}{H_0}\int_0^1 \frac{dx}{\left[x^2 \sum_n \Omega_0 x^{-n} - \Omega_0 + 1\right]^{1/2}}$$ [see eq 32 on page 21]

except now we won't take a sin when we are done ⇒ messy even for Ω_m only
⇒ derive development θ (cf notes from Weinberg) define $1 - \cos\theta = \left(\frac{2(\Omega_0 - 1)}{\Omega_0}\right)\frac{R(t)}{R_0}$

⇒ $k = +1$ $\quad \Omega_0 > 1$

$$H_0 t = \frac{\Omega_0}{2}(\Omega_0 - 1)^{-3/2}(\theta - \sin\theta) \qquad 0 \le \theta \le 2\pi$$

θ_m when $\theta = \pi$

$$t_m = \frac{\pi \Omega_0}{2 H_0 (\Omega_0 - 1)^{3/2}} \qquad R(t_m) = \frac{\Omega_0 R_0}{\Omega_0 - 1}$$

note using $\Omega = 2q$ we can see that $q = \infty$ at $R = R(t_m)$

$\cos\theta_0 = \frac{2}{\Omega_0} - 1$ today (because $R(t) = R_0$)

$$\Rightarrow t_0 = H_0^{-1} \frac{\Omega_0}{2} (\Omega_0 - 1)^{-3/2} \left[\cos^{-1}\left(\frac{2}{\Omega_0} - 1\right) - \frac{2}{\Omega_0}(\Omega_0 - 1)^{1/2} \right]$$

if $\Omega_0 \simeq 2$ ($q_0 \simeq 1$)

$$t_0 \simeq \left(\frac{\pi}{2} - 1\right) H_0^{-1}$$

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if $\Omega_0 = \Omega_m = 1$, $k = 0$

then $\frac{R(t)}{R_0} = \left(\frac{3H_0 t}{2}\right)^{2/3}$ or $\boxed{t_0 = \frac{2}{3}\frac{1}{H_0}}$ for $R(t) = R_0$

note another homework problem, let $k = 0$

$\Omega_0 = \Omega_m^{(0)} + \Omega_\Lambda^{(0)} = 1$, Ω_r etc $= 0$

$\Updownarrow$

then find t_0 as function of Ω_m

(cf figure 3.6 in book)

$\Omega_0 = 1$ so $\Omega_0 - 1$ term is gone

i.e. start with $\frac{1}{H_0}\int_0^1 \frac{dx}{[\Omega_m^{(0)}/x + x^2\Omega_\Lambda^{(0)}]^{1/2}}$

$$t = \frac{1}{H_0}\int_0^1 \frac{dx}{[\Omega_m^{(0)} x^{-1} + x^2\Omega_\Lambda^{(0)}]^{1/2}}$$

can see if $\Omega_\Lambda = 0$

$\int x^{1/2} = \frac{2}{3}$ ✓