rather than analytically can do by simpson's rule if rewrite as

$$\frac{1}{H_0}\int_0^1 \frac{x^{+1/2}\quad dx}{[\Omega_m(0) + x^3\Omega(0)_\Lambda]^{1/2}}$$

take $f(x) = x^{1/2}\,[\Omega_m(0) + x^3\Omega_\Lambda(0)]^{-1/2}$

and use simpson's rule
which says integral =

$$A = \tfrac{1}{3}h\,[(y_0 + y_{2m}) + 4\{y_1 + y_3 \cdots y_{2m-1}\} + 2\{y_2 + y_4 + \cdots y_{2m-2}\}]$$

h = separation of points in x i.e. $= \frac{1}{2m-1}$

i.e. if do 11 pts., $\frac{1}{10} = 0.1 = 1$

i.e. 0, 0.1, 0.2, 0.3, ... 1.0 = 10+1 points

$$\begin{pmatrix} 1, 3, 5, 7, 9 \\ 2, 4, 6, 8 \end{pmatrix}$$

for last case $\Omega_\Lambda(0) = 0 \quad \Omega_m(0) < 1 \quad (q_0 < \frac{1}{2}, k = -1)$

$$\theta = i\Psi \quad ; \quad 1 - \cos\theta = \left(\frac{2(\Omega_0 - 1)}{\Omega_0}\right)\frac{R(t)}{R_0}$$

$$H_0 t = \frac{\Omega_0}{2}(1-\Omega_0)^{-3/2}\sinh\Psi - \Psi$$

$$\cosh\Psi - 1 = 2\left(\frac{1-\Omega_0}{\Omega_0}\right)\frac{R(t)}{R_0}$$

for $t \to \infty$

$$\frac{R(t)}{R_0} \to \frac{\Omega_0}{4}(1-\Omega_0)^{-1}e^{\Psi}$$

$$\to (1-\Omega_0)^{1/2}H_0 t$$

$$\cosh\Psi_0 = \left(\frac{2}{\Omega_0} - 1\right) \qquad [\, R(t) = R_0 \,]$$

and $t_0 = H_0^{-1}\left[(1-\Omega_0)^{-1} - \frac{\Omega_0}{2}(1-\Omega_0)^{-3/2}\cosh^{-1}\left(\frac{2}{\Omega_0} - 1\right)\right]$

take $\Omega_0 = $ ~~0.0344~~ 0.028 (from Weinberg)
so $\Psi \approx 5$ and $t_0 \approx 0.96\, H_0^{-1}$

point to note as before: except for ~~k~~ $\Omega_\Lambda = 0$ $\Omega_r = 0$ $k=0$ case, q_0 value today is just an "accident" i.e. q_0 varies with time

$t_{arrival} = t_{age} - t_{\text{time particle emitted}}$

for particles or light that are far away, time emitted ≈ 0 or z large but finite

so $t_{arrival} \approx t_{age}$

now if $t_{arrival}$ is less than $\frac{R(t_0')\, r_1}{c}$ then light has time to arrive and we'll see. Here $R(t_0')$ = some radius in the future.

now t we've got as function of development angle — for $r_1 R(t_0')$ a little tricky.

if we remember we started with

$$\frac{\dot{R}^2}{R_0^2} = H_0^2 \left[1 - 2q_0 + 2q_0 \frac{R_0}{R}\right], \text{ we}$$

to derive $r_1 = \dfrac{(\qquad)}{H_0 R_0 (1+z)} \leftarrow$ stuff,

we see that

we had an integral with

$$\int_0^{r_1} \frac{dr}{(1-kr^2)^{1/2}} = \frac{1}{R_0 H_0} \int_{\frac{1}{(1+z)}}^{1} [\qquad]$$

this gives us a distance between "here" and "there", now if we ask how far here to there is as $R(t)$ increases, we multiply r_1 by $R(t_0')$ where t_0' = our new age of universe $\Rightarrow r_1 R(t_0)' = \dfrac{R(t_0')}{R_0 H_0} \dfrac{z q_0 + \cdots}{q_0^2 (1+z)}$

note if wanted to do "all" in future is necessary because q_0 and H_0 both vary with time so $H_0' \to 0$ but $q_0' \to$ (33)

now we see $t_{age} = \frac{1}{H_0}\, \theta - \sin\theta$ for $q_0 = 1$

and $\frac{r_1 R(t)}{c} = \frac{R(t)}{R_0 H_0} \quad \frac{z}{H_0(1+z)}$

$$\approx \frac{R(t)}{R_0 H_0} \qquad \text{for } z \gg 1$$

$$= \frac{1-\cos\theta}{H_0} \qquad \text{for } q_0 = 1$$

$\Rightarrow$ ask where $\frac{\theta - \sin\theta}{1-\cos\theta}$ exceeds

occurs at about $\theta = 2.39$ radians

$q_0 = 1$ or $\frac{t_{age}}{2t_m} = \frac{2.39 - \sin(2.39)}{2\pi} \approx \frac{1}{4}$ ✓

now for $q_0 < 1/2$

we have $t = \frac{.1}{(.8)^{3/2}}\left(\frac{e^{\psi} - e^{-\psi}}{2} - \Psi\right) \qquad q_0 = 0.1$

so $\left(\frac{e^{\psi} + e^{-\psi}}{2} - 1\right)\left(\frac{0.1}{.8}\right) \qquad r_1 R(t)$

and for $\psi \gg 1$ we see that $t \approx \frac{0.1}{(0.8)^{3/2}}\left(\frac{e^{\psi}}{2} - 2\psi\right)$

so $r_1 R(t) = \frac{e^{\psi}}{2}\left(\frac{0.1}{0.8}\right)$ and

$r_1 R(t) > t_{age}$

and we see that we'll eventually see all there is to see in the $q_0 \le 1/2$ case

$$\left\{ \quad [q_0 = 1/2] \quad \frac{2}{3H_0}\left(\frac{R(t)}{R_0}\right)^{3/2} = t_{age} \right.$$

$$\text{versus} \quad \frac{r_1 R(t)}{C} = \frac{R(t)}{H_0 R_0}$$

$$= \frac{\left(\frac{3}{2} H_0 t_{age}\right)^{2/3}}{H_0}$$

let's see
$$\frac{t_{age}}{\left(\frac{3}{2} H_0 t_{age}\right)^{2/3}} = \left(\frac{2}{3}\right)^{2/3} \times t^{1/3} H_0^{-2/3}$$

since $t_{age} > H_0$ by def. $\quad H_0 t_0 > 1$

and
$$\frac{t_{age}}{\left(\frac{3}{2} H_0 t_{age}\right)^{2/3}} < 1 \quad \text{and}$$

we see again $t_{age} < R(t) r_1$ ✓ okay

now what about big crunch i.e. when is the time of the last event we'll be able to see from our most distant spot going to occur (only applies $q_0 > 1/2$)

the answer is $t_{m} * 2 - t_{cross}$

since this is the time from when the 2 regions cross over to the big crunch or $82 \times 10^9 - 20 \times 10^9 \approx 60 \times 10^9$

again, question doesn't come up in $q_0 < 1/2$ universe.

note some reality checks:

(1) for light we can't see back beyond $z \approx 1000$

(2) can't see back to $z = \infty$ because $d_L = R_0 r_1 (1+z) \rightarrow \infty$ and hence
$$Flux = \frac{L}{d_L^2} \rightarrow 0!$$

(3) note as time goes forward the $R(t_{decoupling})$ remains fixed (this is time since big bang) so $\frac{1}{1+z} = \frac{R(t_0')}{R(t)}$ will get bigger for ever in the $q_0 < 1/2$ case

and although the "size" of the universe will increase, we won't see any more stuff – just a lower density universe.

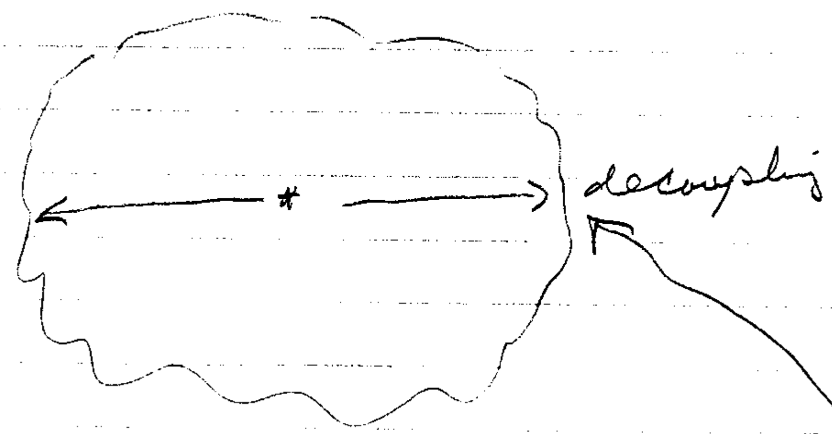
for $q_0 > 1/2$ $R(t_0')$ increases to $t \to t_m$,

then, wow, $H(t_0')$ changes sign! (and goes through 0 of $\dot{R}$ eq.)

and $\frac{R(t_0')}{R(t_{decoupling})}$ will decrease and

apparent CMB will increase until we hit $R(t_0') = R(t_{decoupling})$ where the physics will change!

(4) we can see out like this



now what about somebody that lives here?

well they see us at $1+z \approx 1000$!

and they'll never see beyond that point to the other side, because we were optically thick at $z = 1000$!, but if they could, they would see us at $\left(\frac{1}{1+1000}\right) * \frac{1}{1+1000} \approx z = 10^6$ not $> c$ ✓

(5) while we're at it, let's look at age of universe at decoupling when $e_\gamma \sim \rho_m c^2$
energy density

take this to be $t_{decoup} \approx 5 \times 10^5$ years $\approx 1.5 \times 10^{12}$ sec

now what is angular scale?
let $q_0 = 1/2$ for simplicity

we have $r_1 R(t_{decoup}) \approx$ proper distance
is $<$ $t_{arrival}$ so $r_1 R(t_{decoup})$ is what counts, to $t_{arrival} * c$

$$d_A = \frac{d_L}{1+z^2} = \frac{r_1 R_0 (1+z)}{(1+z)^2} \quad \text{or} \quad d_A = \frac{r_1 R_0}{1+z}$$

$$q_0 = 1/2 \qquad d_A \approx \frac{c \frac{z}{2} \times 4}{H_0 (1+z)^2} \approx \frac{2c}{H_0(1+z)}$$

$$c\, t_{arrival} \approx \frac{c}{}\left(\frac{1}{H_0(1+z)}\right)^{3/2} \frac{2}{3}$$

$$\Rightarrow \Rightarrow \frac{c\, t_{arrival}}{d_A} \approx \frac{4}{3}(1+z)^{-1/2}$$

$$= \frac{4}{3}\left(\frac{1}{1500}\right)^{1/2} \text{ radians}$$

$\approx 2^\circ \approx$ right order of magnitude
(Weinberg gets 3° for $q_0 = 1/2$)

we took $t_{arrival}$ = age of universe at
$z = 1500$

now on to cosmological constant
there a lot's of "fun" cases but we'll

not do those:

instead, 1st

look at what Einstein did

$$p \to p - \frac{\Lambda c^4}{8\pi G} \quad \text{and} \quad \rho \to \rho + \frac{\Lambda c^2}{8\pi G} \equiv \tilde{\rho}$$

now in matter dominated universe $p_{matter} \approx 0$

so for $\Lambda > 0$ $p < 0$ does this make universe
expand or contract?

answer is, it makes it expand!
how does negative pressure make expansion?

answer is:

$$dU = -dW \qquad (dQ=0)$$

but $dW = -p\,dV$

and we see that we gain energy (i.e. we don't need to put any in to system) if p is negative and dV is positive

The way to think about this is Universe does work on itself to expand, so if p is positive, Universe must do work to expand and positive p then alows expansion!

now let's suppose $\Omega_\Lambda = \dfrac{c^2\Lambda}{8\pi G\,\rho_c} \simeq 0.5$

so we have $1/2 + 1/2 \quad \Omega_m + \Omega_\Lambda$

then what is value of $\dfrac{c^2\Lambda}{8\pi G}$ or Λ?

take $\rho_c = \dfrac{3H_0^2}{8\pi G}$, $H_0 = 50\ \mathrm{km\ s^{-1}\ Mpc} = 1.6\times10^{-18}\ \mathrm{sec}$

$\simeq 5\times10^{-30}\ \mathrm{g/cm^3}$ or $\dfrac{c^2\Lambda}{8\pi G} \simeq 2.5\times10^{-30}\ \mathrm{g/cm^3}$

or Λ = som really, really small number