rather than analytically, can do by Simpson's rule if rewrite as
\[ \frac{1}{h_0} \int_{0}^{1} \frac{dx}{\sqrt{2m(x_0) + x^3 \lambda(x_0)}} \]

take \[ f(x) = x^{1/2} \left[ 2m(x_0) + x^3 \lambda(x_0) \right]^{-1/2} \]

and use Simpson's rule which says with \[ A = \frac{1}{3} h \left[ (y_0 + y_{2m}) + 4(y_1 + y_3 + \cdots + y_{2m-1}) + 2(y_2 + y_4 + \cdots + y_{2m-2}) \right] \]

\[ h = \text{separation of points in } x \text{ i.e. } \frac{1}{2m-1} \]

i.e. if do 111 pts., \[ \frac{1}{10} = 0.1 = 1 \]

i.e. 0, 0.1, 0.2, 0.3, ..., 1.0 = 10 + 1 points

\[ \left( \frac{1}{3}, 5, 3, 9 \right) \]

\[ \left( 2, 4, 6, 8 \right) \]
In last case \( R_{\lambda}(0) = 0 \), \( R_{\nu}(0) < 1 \), \( \epsilon > \frac{1}{2} \), \( k = -1 \)

\[
\Theta = \sqrt{5}, \quad 1 - \cos \Theta = \left( \frac{\Theta}{\Theta_0} \right) \frac{R(t)}{R_0}
\]

\[
H_0 t = \frac{\Theta_0}{2} (1 - \Theta_0)^{-3/2} \sinh \sqrt{5} - \sqrt{5}
\]

\[
\cosh \sqrt{5} - 1 = \frac{4}{\left( \frac{\Theta}{\Theta_0} \right) \frac{R(t)}{R_0}}
\]

As \( t \to \infty \)

\[
\frac{R(t)}{R_0} \to \frac{\Theta_0}{4} (1 - \Theta_0)^{-1} e^{\sqrt{5}}
\]

\[
\to (1 - \Theta_0)^{1/2} H_0 t
\]

\[
\cosh \left( \frac{\Theta}{\Theta_0} \right) = \left( \frac{2}{\Theta_0} - 1 \right) \left[ R(t) = R_0 \right]
\]

and \( t_0 = H_0^{-1} \left[ (1 - \Theta_0)^{-1} - \frac{\Theta_0}{2} (1 - \Theta_0)^{-3/2} \sinh^{-1} \left( \frac{\Theta}{\Theta_0} - 1 \right) \right] \)

Take \( \Theta_0 = 0.025 \) (from Weinberg)

or \( \sqrt{5} \times 5 \) and \( t_0 \approx 0.96 H_0^{-1} \)

Point to note as before: except for the \( \lambda = 0 \), \( \nu = 0 \)

\( k = 0 \) case, \( \gamma \) value today is just an "incident" i.e. \( \gamma_0 \) varies with time.
\text{travel} = \text{time} - \text{time particle emitted}

In particles or light, if we were far away, time emitted
would be large, but finite
so travel \& time

now if travel is less than \( R(t') \), then
light has time to arrive and
we'll see. Here \( R(t') = \) some radius in the
future.

now if we've got as function of development and
position \( R(t') \) a little tricky,
if we remember we started with
\[
\frac{R^2}{R_0^2} = H_0^2 \left[ 1 - 2 \frac{\mathcal{E}}{\mathcal{E}_0} + 2 \frac{\mathcal{E}}{\mathcal{E}_0} \frac{d\mathcal{E}}{dt} \right]
\]
we can derive
\[
r = \frac{1}{H_0 R_0 (1+z)}
\]
we see that
we had an integral with
\[
\int_0^r \frac{dt'}{(1-\mathcal{E}'(1+z)^2)^{1/2}} = \frac{1}{R_0 H_0} \int_0^1 \frac{dt}{(1+z)^{1/2}}
\]
this gives us a distance between "here" and
"there", now if we ask how far due to
this is as \( R(t) \) increases, we multiply
\( r \) by \( R(t') \) where \( t' = \) our new age
of universe \( \Rightarrow r' = R(t') = \frac{R(t')}{R(0) H_0} \frac{2 \mathcal{E}_0 \mathcal{H}_0}{(1+z)^2} \)
\[ \text{now we see } \gamma_0 = \frac{1}{\gamma_0} \]
\[ \text{and } \frac{t \, R(t)}{c} = \frac{R(t)}{R_0 \, H_0} = \frac{1}{H_0 (1+z)} \]
\[ \Rightarrow \text{as } \theta = \pi - \min \theta \text{ exceeds } 1 - \cos \theta \]
occurs at about \( \theta = 2.39 \text{ radians} \)
\[ \gamma_0 = 1 \text{ as } \frac{\text{tage}}{2 \pi m} = \frac{2.39 - \sin (2.39)}{2 \pi} \approx \frac{1}{4} \]
\[ \text{now for } \gamma_0 < \frac{1}{2} \]
we have \( t = \frac{1}{(8)^{3/2}} \left( \left( e^{\psi} - e^{-\psi} \right) \right) \)
\[ \left( \frac{e^{\psi} + e^{-\psi}}{2} - 1 \right) \left( \frac{0.1}{0.8} \right) \]
and \( \psi > 1 \) we see that \( t = \frac{0.1}{(0.8)^{3/2}} \left( \frac{e^{\psi} - e^{-\psi}}{2} \right) \)
\[ \frac{t \, R(t)}{c} = \frac{e^{\psi}}{2} \left( \frac{0.1}{0.8} \right) \text{ and } \frac{t \, R(t)}{c} < \text{tage} \]
and we see that we’ll eventually see all that is to see in the \( q_0 \leq \frac{1}{2} \) case

\[
\left\{ \begin{array}{l}
q_0 = \frac{1}{2} \\
\frac{2 \left( \frac{R(t)}{R_0} \right)^{\frac{3}{2}}}{3 H_0 (R_0)} = \text{tape}
\end{array} \right.
\]

versus \( \frac{1}{c} \frac{R(t)}{H_0 R_0} \)

\[
= \left( \frac{3 H_0 t^2}{2} \right)^{\frac{1}{3}}
\]

let’s see \( \frac{\text{tape}}{(\frac{3}{2} H_0 t^2)^{\frac{1}{3}}} = \left( \frac{2}{3} \right)^{\frac{1}{3}} t^{\frac{1}{3}} H_0^{-\frac{1}{3}} \)

since \( \text{tape} > H_0 \) by def. \( H_0 \to 0 \)

and \( \frac{\text{tape}}{(\frac{3}{2} H_0 t^2)^{\frac{1}{3}}} < 1 \)

we see again \( \text{tape} < \frac{R(t)}{H_0} \), \( \checkmark \) ok

now what about the crunch, i.e., when is the time of the last event we’ll be able to see from our most distant spot going to occur (only applies \( q_0 > \frac{1}{2} \))
The answer is $t_m + 2 = t_{	ext{now}}$

since this is the time from when the $\epsilon$ region crosses over to the big crunch

or $82 \times 10^9 - 20 \times 10^9 \approx 60 \times 10^9$.

Again, question doesn't come up in $z_0 < \frac{1}{2}$ Universe.

Note some reality checks:

1) For light, we can't see both beyond $z = 1000$

2) Can't see back to $z = 0$ because $d_L = \frac{R(t)}{L(1+z)} \rightarrow \infty$ and hence

\[ R/L = \frac{L}{d_L} \rightarrow 0. \]

3) Note as time goes forward, the $R(t)$ (decelerating) remains filled (this is time since big bang) so

$\frac{1}{1+z} = \frac{R(t)}{R(t_0)}$, will get bigger forever

in the $z_0 < \frac{1}{2}$ Universe.
and although the "size" of the universe will increase, we won't see any more stuff - just a lower density universe.

For $g_0 > \frac{1}{3}$, $R(t_0)$ increases to $t \to t_m$.

Then, worry, $H(t_0)$ changes sign! (and goes through 0 of key).

and $\frac{R(t_0)}{R(t_{decoupling})}$

apparent CMB will increase until we hit $R(t_0) = R(t_{decoupling})$ where the physics will change.

(4) we can see out like this

![Diagram]

now what about somebody that lives here?
well they see us at \( 1 + z = 1000 \)

and they'd never see beyond that point, because we were optically thin at \( z = 1000 \). But if they could, they would see us at \( \frac{(10000)^{1/2}}{1+1000} = \frac{z}{10^6} \) 

while with \( z = 5 \), let's look at age of Universe at decouply, when \( t = \frac{L_0}{c} \) 

energy density

take this to be \( t_{decoup} = 5 \times 10^5 \) years \( = 1.5 \times 10^7 \) sec

now what is angular scale? let \( \theta_0 = \frac{z}{2} \)

for simplicity we have \( R(\theta_0, z) \) proper distance is \( C \) terminal \( \frac{1}{2} R(\theta_0, z) \) is what counts, to terminal \( \frac{C}{2} \)

\[ d_A = \frac{c}{H_0 (1+z)^2} \]

\[ \theta_0 = \frac{z}{2} d_A = \frac{c}{H_0 (1+z)^3} \frac{2}{H_0 (1+z)} \]

\[ c_{terminal} = c \left( \frac{\frac{2}{3}}{H_0 (1+z)} \right)^{2/3} \]

\[ \Rightarrow \frac{c_{terminal}}{d_A} = \frac{4}{3} (1+z)^{1/2} \]

\[ = \frac{4}{3} \left( \frac{1}{1800} \right)^{1/2} \text{ radians} \]
$z \approx 2^\circ$ = right order of magnitude

We took $t_{\text{final}} = \text{age of universe at } z = 1500$

Now on to cosmological constant

There's a lot of 'fun' cases but we'll not do those.

Instead 1st

Look at what Einstein did

$p \rightarrow p - \frac{\Lambda c^4}{8\pi G}$ and $e \rightarrow e + \frac{\Lambda c^2}{8\pi G} \equiv \bar{e}$

Now in matter dominated universe $p_{\text{matter}} = 0$

so $p_{\Lambda} > 0 \rightarrow p < 0$. Does this make universe expand or contract?

Answer is, it makes it expand!

How does negative pressure make expansion?
The answer is: 

\[ dU = -dW \quad (dQ = 0) \]

but \( dW = -p dV \)

and we see that we gain energy (i.e., we don't need to put any in to regulate) if \( p \) is negative and \( dV \) is positive.

The way to think about this is: universe does work on itself to expand, so if \( p \) is positive, universe must do work to expand, and positive \( p \) then allows expansion.

Now let's suppose \( r_c = \frac{c^2 \Lambda}{8 \pi G} \approx 0.5 \times 10^{-30} \) m.

So we have \( \frac{1}{2} + \frac{1}{2} r_m + r_c \)

Then what is value of \( \frac{c^2 \Lambda}{8 \pi G} \) or \( \Lambda \)?

Take \( c = \frac{3}{8 \pi G} \), \( H_0 = 50 \) km s\(^{-1}\), \( Mpc = 1.6 \times 10^{18} \)

\[ c = 5 \times 10^{-30} \text{ g} / \text{cm}^3 \] or \( \frac{c^2 \Lambda}{8 \pi G} \approx 2.5 \times 10^{-30} \text{ g} / \text{cm}^3 \)

or \( \Lambda \) = some really, really small number