Computational Solutions To The Quasi-static Tidal Interactions In White Dwarf Binaries

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Introduction

Astronomers count down to the expected launch of NASA and ESA’s multi-million dollar gravitational wave project, the Laser Interferometer Space Antenna or LISA. Once launched, scientists hope that a whole new window of observational astronomy will open up and reveal the hidden truth behind some of the most spectacular cosmic phenomena. Compared to the problematic interface of ground based observatories such as the currently running Laser Interferometer Gravity Observatory (LIGO), LISA will be far away from seismic interference, and should have a clean look at phenomena within a large range of gravitational wavelength.

One group of gravitationally radiating phenomena that occur relatively frequently in the cosmos are binary systems where both stars are in the white dwarf (WD) stage of stellar evolution. In fact, so many of these systems are hypothesized to exist that the signals of these systems will likely dominate LISA’s output. It is therefore important to understand precisely the nature of these systems so that we are better able to filter them out of LISA’s signal.

Thus far, models have been markedly simple, treating the white dwarfs as point masses, and ignoring complications that arise from tidal interactions, among other processes. In this thesis, I will address the numerical difficulties of solving a system of differential equations describing a WD-WD binary’s tidal interaction.

In particular, I will emphasize the computational methodology of solving the tides, comparing the consistency of different approaches to issue a precise statement on the reliability of the results. In Chapter 1, I describe the process of generating the composite polytope models we needed to use as initial data sets. In Chapter 2, I introduce and motivate a modern approach of understanding the physics of tides in a binary star system, leading up to a final system of differential equations which we can solve for the total perturbation of gravitational potential. Chapter 3 outlines the computational
methodology used in finding solutions to the differential equations of white dwarf binaries. In Chapter 4 I discuss results, emphasizing maintenance of consistency between solutions obtained by different techniques. I particularly compare semi-analytical solutions to numerical solutions, and consider different choices of independent variables, and numerical integrators.
Chapter 1
Stellar Structure Models

In order to study tides, input models describing the density, \( \rho \), mass, \( m \), and pressure, \( P \) as a function of radius, \( r \) along the stellar interior are required. Since we have no way of obtaining these models observationally or experimentally, we must generate them theoretically.

Because stars are by their very nature objects that are governed by the statistical interactions of many particles, approximations must be made when we model them. The process of modeling stellar interiors numerically is a refined science that has been developed sufficiently to produce sophisticated, descriptive data about the stellar interior. In our application, however, we allow ourselves to assume hydrostatic equilibrium, and neglect other complications such as magnetic fields. With this assumption, and after applying the mass continuity condition, the following differential relations hold true: (Hansen 1994)

\[
\frac{dP}{dr} = -\frac{Gm}{r^2}\rho, \tag{1.1}
\]
\[
\frac{dm}{dr} = 4\pi r^2\rho. \tag{1.2}
\]

1.1 Single polytropes

One simple, early stellar model that follows the above requirements is known as a polytrope. A polytropic model is created by defining the density, and the pressure to follow the characteristic power laws:

\[
\rho = \lambda \theta^n, \tag{1.3}
\]
\[
P = \kappa \rho^{\frac{n+1}{n}}. \tag{1.4}
\]
where \( n \) is the so-called polytropic index. If we set \( \theta \) at the center equal to 1, we see that \( \lambda \) represents \( \rho_c \), the density at the center of the star, and \( \kappa \) represents \( P_c \), the pressure at the center of the star. We call the function \( \theta \) the Lane-Emden variable. If we now assume that density and pressure go to zero at the surface, it is possible to derive a single differential equation describing the state of the polytrope that is commonly known as the Lane-Emden equation:

\[
\frac{1}{\xi^2} \frac{d}{d\xi} \left( \xi^2 \frac{d\theta}{d\xi} \right) + \theta^n = 0,
\]

where we use \( \xi \) as a normalized radius variable:

\[
\xi = r \left[ \frac{4\pi G \rho^2}{(n+1) P_c} \right]^{1/2}.
\]

We can integrate the Lane-Emden equation over \( \xi \) from the center of the star where \( r = \xi = 0 \) and \( \theta_0 = 1 \) to the surface, which is defined by \( \theta_s = 0 \) and \( \xi = \xi_s \). It is important to note that \( \xi \) at zero induces a singular point in Eq. (1.5), therefore it is necessary for us to adopt a Taylor series approximation for \( \theta \) near the origin:

\[
\theta = \theta_0 - \xi^2 \frac{\theta_0^2}{6} + \xi^4 \frac{n \theta_0^{2n-1}}{120} - n(8n - 5) \xi^6 \frac{\theta_0^{3n-2}}{15120} + \ldots
\]

We define \( \alpha \)

\[
\alpha = \frac{R}{\xi_s}.
\]

The following quantities follow, where the subscript \( c \) denotes that the value at the center of the star is to be used, \( R \) is the total radius, and \( M \) is the total mass.

\[
\lambda = \rho_c = \frac{M}{4\pi R^3} \frac{\xi_s}{\left( \frac{d\theta}{d\xi} \right)_s},
\]

\[
\kappa = P_c = \frac{4\pi G}{n+1} \alpha^2 \lambda \frac{n+1}{n}. \tag{1.10}
\]

By definition, we already know the behavior of pressure and density, and now we can also derive the radius, mass, and gravity in terms of \( \xi \) and \( \theta \):

\[
r = R \frac{\xi}{\xi_s}, \tag{1.11}
\]

\[
m(r) = M \frac{1}{\xi_s^2} \frac{\left( \frac{d\theta}{d\xi} \right)_s}{\xi^2} \frac{d\theta}{d\xi}. \tag{1.12}
\]
1.1. SINGLE POLYTROPES

\[ g = \frac{GM}{R^2} \left( \frac{\text{d} \theta}{\text{d} \xi} \right)_s \text{d} \theta \text{d} \xi. \]  
(1.13)

These quantities can be made dimensionless by dividing them by reference values:

\[ r^* = \frac{r}{R^*}, \]  
(1.14)

\[ \rho^* = \frac{\rho}{M/4\pi R^3}, \]  
(1.15)

\[ P^* = \frac{P}{GM^2}, \]  
(1.16)

\[ m^* = \frac{m}{M}, \]  
(1.17)

\[ g^* = \frac{g}{GM/R^2}. \]  
(1.18)

Dimensionless quantities are particularly useful as they are inherently more inclined to bring all quantities to comparable orders of magnitude, and remove bulky normalization constants from equations. For the remainder of this thesis, we consider only dimensionless quantities, unless otherwise noted.

In the context of the later chapters of this thesis project, in which we will study tidal interactions in white dwarf stars with outer convective envelopes, we require the derivation of additional quantities. These are the square of the isentropic sound speed \( c^2 \), the Brunt-Vaisala Frequency \( N^2 \), and turbulent viscosity coefficient \( \nu_0 \), that are defined as:

\[ c^2 = \Gamma_1 \frac{P}{\rho}, \]  
(1.19)

\[ N^2 = -g \left( \frac{g}{c^2} + \frac{1}{\rho} \frac{d\rho}{dr} \right), \]  
(1.20)

\[ \nu_0 = \left( \frac{P}{\rho g} \right)^2 |N^2|^{1/2}, \]  
(1.21)

where \( \Gamma_1 \) is the first generalized isentropic coefficient that represents an aspect of the thermodynamic properties of the system. We approximate the star to be composed of a monatomic ideal gas, thus allowing us to simply take:

\[ \Gamma_1 = \frac{5}{3}. \]  
(1.22)
CHAPTER 1. STELLAR STRUCTURE MODELS

1.2 Composite Polytropes

The polytropic model’s strength lies in the simplicity of choosing one polytropic index, \( n \) to parameterize the state of the system, and obtaining a full table of computed stellar quantities over the entire stellar radius. However, many stellar objects, including the majority of white dwarfs, are found to have several regions, each of which is dominated by different physical processes. In the context of polytropic stellar models, this means that the regions are differentiated by different polytropic indices. A white dwarf consists of a large radiative zone in the deeper interior of the star, and a thin convective zone near the surface. In order to properly model this feature of white dwarfs with polytropic models, a switch of polytropic indices must occur at the interface between the interior and the envelope. Such models are known as “composite polytropes”.

In order to tabulate a composite polytrope, we choose the two polytropic indices, \( n_1 \) and \( n_2 \), and a radial value for the interface \( r_b \). Where the subscript \( 'b' \) denotes boundary. We start with the original Lane-Emden equation and integrate it from \( \xi_1 = 0 \) to \( \xi_1 = \xi_1 b \), which is the value of \( \xi_1 \) that corresponds to our chosen boundary, \( r_b \). Note that the subscript 1 is used for all core quantities.

\[
\frac{1}{\xi_1^2} \frac{d}{d\xi_1} (\xi_1^2 \frac{d\theta_1}{d\xi_1}) + \theta_1^{n_1} = 0. \tag{1.23}
\]

When we get to the interface, the polytropic index is switched from \( n_1 \) to \( n_2 \), thus changing the inherent differential equation. We therefore now denote all envelope quantities with a 2 subscript. The newly constructed differential equation, that appears to have the same form as the Lane-Emden equation but has different boundary conditions, is integrated until \( \theta_2 = 0 \) at some final value of \( \xi_2 \).

\[
\frac{1}{\xi_2^2} \frac{d}{d\xi_2} (\xi_2^2 \frac{d\theta_2}{d\xi_2}) + \theta_2^{n_2} = 0. \tag{1.24}
\]

Note that while Eq. (1.24) has similar form to Eq. (1.23), \( \theta_2 \) and \( \xi_2 \) no longer correspond to the conditions defining the polytrope that were prescribed at the center for integration, implying that Eq. (1.24) is not strictly a Lane-Emden equation. To keep this model physically consistent, core quantities and envelope quantities must now be defined separately, as functions of their corresponding values of \( \xi \) and \( \theta \). To continue solving the system, we impose mass, radius, pressure, and density to be continuous. This gives us four equations in terms of the Lane-Emden variables:
1.2. COMPOSITE POLYTROPES

Figure 1.1: This is an image of a 1-D representation of a composite polytrope, showing the boundary between the two regions, and the equations of state that apply in each region.

mass density, $\rho : \lambda_1 \theta_{1b}^{n_1} = \lambda_2 \theta_{2b}^{n_2}$, \hfill (1.25)

Pressure, $P : \frac{\alpha_1^2 \lambda_1^2}{n_1 + 1} \theta_{1b}^{n_1 + 1} = \frac{\alpha_2^2 \lambda_2^2}{n_2 + 1} \theta_{2b}^{n_2 + 1}$, \hfill (1.26)

Radius, $r : \alpha_1 \xi_{1b} = \alpha_2 \xi_{2b}$, \hfill (1.27)

Mass, $m : \alpha_3^3 \lambda_1 \xi_{1b}^2 \left( \frac{d\theta_1}{d\xi_1} \right)_b = \alpha_2^3 \lambda_2 \xi_{2b}^2 \left( \frac{d\theta_2}{d\xi_2} \right)_b$. \hfill (1.28)

These four conditions can be reduced to two major constraints for the Lane-Emden variables.

$$\xi_{1b} \frac{\theta_{1b}^{n_1}}{\left( \frac{d\theta_1}{d\xi_1} \right)_b} = \xi_{2b} \frac{\theta_{2b}^{n_2}}{\left( \frac{d\theta_2}{d\xi_2} \right)_b},$$ \hfill (1.29)

$$\left( n_1 + 1 \right) \xi_{1b} \frac{\left( \frac{d\theta_1}{d\xi_1} \right)_b}{\theta_{1b}} = \left( n_2 + 1 \right) \xi_{2b} \frac{\left( \frac{d\theta_2}{d\xi_2} \right)_b}{\theta_{2b}}. \hfill (1.30)$$

We note that there is one degree of freedom we can use to normalize the equations at the boundary while maintaining validity in our method, so long as we define everything in accordance to the normalization. Many traditional approaches, by Chandrasekhar (1958) for instance, choose to set $\xi_{1b} = \xi_{2b}$. Algebraic manipulation around this presumption leads to the following conditions on the boundary value for $\theta_2$

$$\theta_{2b} = \left[ \frac{n_1 + 1}{n_2 + 1} \theta_{1b}^{n_1 - 1} \right]^{\frac{1}{n_2 - n_1}},$$ \hfill (1.31)

$$\left( \frac{d\theta_2}{d\xi_2} \right)_b = \left( \frac{d\theta_1}{d\xi_1} \right)_b \frac{n_1 + 1}{n_2 + 1} \theta_{2b}. \hfill (1.32)$$

We first considered using this approach, but found that Eq. (1.32) formulae to become singular for $n_2 = 1$, which corresponds to a potentially relevant
index for surface convective zones. Therefore, we instead choose the condition \( \theta_{2b} = \theta_{1b} \). This time algebraic manipulation leads to:

\[
\xi_{2b}^2 = \frac{n_1 + 1}{n_2 + 1} \theta_{1b}^{n_1 - n_2} \xi_{1b}^2, \quad (1.33)
\]

\[
\left( \frac{d\theta_2}{d\xi_2} \right)_b^2 = \frac{n_1 + 1}{n_2 + 1} \theta_{1b}^{n_2 - n_1} \left( \frac{d\theta_2}{d\xi_2} \right)_2^2. \quad (1.34)
\]

Now, we can begin to integrate Eq. (1.24). We define the end point of integration, where \( \theta_{2s} = 0 \) and \( \xi = \xi_{2s} \) to be the surface, therefore we have

\[
\alpha_2 = \frac{R}{\xi_{2s}}. \quad (1.35)
\]

We define \( \lambda_2 \) to be the usual polytropic scaling factor for density, however noting that it does not correspond to any central density:

\[
\lambda_2 = \frac{M}{4\pi R^3} \left( \frac{d\theta_2}{d\xi_2} \right)_0 \xi_{2s}. \quad (1.36)
\]

From continuity of density and \( \theta \) we compute \( \lambda \) for the core:

\[
\lambda_1 = \frac{M}{4\pi R^3} \left( \frac{d\theta_2}{d\xi_2} \right)_x \theta_{1b}^{n_2 - n_1}. \quad (1.37)
\]

From continuity of radius we derive \( \alpha_1 \)

\[
\alpha_1 = \frac{\xi_{2b}}{\xi_{1b}} \frac{R}{\xi_{2s}}. \quad (1.38)
\]

It is important to notice that with this method, the radius in the core is defined to be a function of \( \xi_{2b} \) and \( \xi_{2s} \), which are both undetermined upon runtime. It is therefore necessary to iterate through values of \( \xi_{1b} \), where the first integration stops, and the boundary conditions are imposed until the final solution meets the condition:

\[
\frac{r_b}{R} = \frac{\xi_{2b}}{\xi_{2s}}. \quad (1.39)
\]

It is now possible to tabulate all stellar quantities for the envelope and core. In the envelope:

\[
r_2 = \frac{\xi_2}{\xi_{2s}}, \quad (1.40)
\]
\[\rho_2 = -\xi_{2s} \frac{\theta_{1b}^{n_2}}{\left(\frac{d\theta_{1b}}{d\xi_{1b}}\right)_{s}},\]  
\[P_2 = \frac{1}{n_2 + 1} \left[\frac{1}{\xi_{2s} \left(\frac{d\theta_{1b}}{d\xi_{1b}}\right)_{s}}\right]^{2} \theta_{2}^{n_2+1},\]  
\[m_2(r) = \frac{1}{\xi_{2s} \left(\frac{d\theta_{1b}}{d\xi_{1b}}\right)_{s}} \xi_{2}^{2} \frac{d\theta_{2}}{d\xi_{2}},\]  
\[\frac{d\rho_2}{dr_2} = -\xi_{2s} \xi_{1b} \frac{n_2 \theta_{n_2-1}}{\xi_{2s}} \frac{d\theta_{2}}{d\xi_{2}},\]  
\[g_2 = \frac{m_2(r)}{r_2^2},\]  
\[(c_2^2) = \Gamma_1 \frac{P_2}{\rho_2},\]  
\[(N_2^2) = -g_2 \left[\frac{g_2}{(c_2^2)} + \frac{1}{\rho_2} \frac{d\rho_2}{dr_2}\right],\]  
\[(\nu_0)_2 = \left(\frac{P_2}{\rho_2 g_2}\right)^2 |(N_2^2)|^{1/2}.\]  

In the core:

\[r_1 = \frac{\xi_1 \xi_{2b}}{\xi_{2s} \xi_{1b}},\]  
\[\rho_1 = -\xi_{2s} \frac{\theta_{1b}^{n_1}}{\left(\frac{d\theta_{1b}}{d\xi_{1b}}\right)_{s}} \theta_{1b}^{n_{2b}-n_1},\]  
\[P_1 = \frac{1}{n_2 + 1} \left[\frac{\theta_{1b}^{n_{2b}-n_1}}{\xi_{1b} \left(\frac{d\theta_{1b}}{d\xi_{1b}}\right)_{s}}\right]^{2} \theta_{2}^{n_2+1} \left(\frac{\xi_{2b}}{\xi_{1b}}\right)^{2},\]  
\[m_1(r) = \frac{\theta_{1b}^{n_{2b}-n_1}}{\xi_{2s} \left(\frac{d\theta_{1b}}{d\xi_{1b}}\right)_{s}} \xi_{2}^{2} \frac{d\theta_{2}}{d\xi_{2}} \left(\frac{\xi_{2b}}{\xi_{1b}}\right)^{3},\]  
\[\frac{d\rho_1}{dr_1} = -\xi_{2b} \xi_{1b} \frac{\rho_1 \theta_{1b}}{\xi_{2b}} \frac{n_1 \theta_{1b}^{n_1-1} \frac{d\theta_{1b}}{d\xi_{1b}}}{\theta_{1b}^{n_{2b}-n_1}},\]  
\[g_1 = \frac{m_1(r)}{r_1^2}.\]
\[ (c_1^2) = \Gamma_1 \frac{P_1}{\rho_1}, \]  
(1.55)  

\[ (N_1^2) = -g_1 \left( \frac{g_1}{c_1^2} + \frac{1}{\rho_1} \frac{d\rho_1}{dr_1} \right), \]  
(1.56)  

\[ (\nu_0)_1 = \left( \frac{P_1}{\rho_1 g_1} \right)^2 |(N_1^2)|^{1/2}. \]  
(1.57)  

Again, we assume monatomic ideal gas, thus we can take the adiabatic coefficient, \( \Gamma_1 \) to be:  
\[ \Gamma_1 = \frac{5}{3}. \]  
(1.58)  

### 1.2.1 Figures

As we will see in subsequent sections, a large part of the motivation behind developing the composite polytrope model was to create a predictable expression for \( \nu_0 \), and the ability to tune the width of the convection zone. The following plots illustrate the functional form of the turbulent viscosity coefficient, as well as the Brunt-Vaisala Frequency, \( N^2 \), the density \( \rho \), and the gravity \( g \) as a function of radius. The models used here are all composite polytropes with a polytropic index of \( n_1 = 3 \) for the core, polytropic indices in the range of \( 1.0 \leq n_2 \leq 1.5 \) for the envelope, and a selection of values for the boundary of the two regions, \( r_b \).
Figure 1.2: This is a plot of the Brunt-Vaisala frequency, $N^2$ for $n_2 = 1.0$
$r_b = .1, .3, .6, .9$ note that the plots have different scales for the y axis.
Figure 1.3: This is a plot of the turbulent viscosity coefficient, $\nu_0$ for $n_2 = 1.0$ $r_b = .1, .3, .6, .9$ note that the plots have different scales for the y axis
Figure 1.4: This is a plot of the logarithm of mass density, \( \log \rho \), gravity, \( g \), Brunt-Vaisala frequency, \( N^2 \), turbulent viscosity coefficient, \( \nu_0 \) as a function of radius \( r \) for \( r_b = 0.1, n_2 = 1.0 \).
Figure 1.5: This is a plot of the logarithm of mass density, log $\rho$, gravity, $g$, Brunt-Vaisala frequency, $N^2$, turbulent viscosity Coefficient, $\nu_0$ as a function of radius $r$ for $r_b = 0.1$, $n_2 = 1.4$. 
Figure 1.6: This is a plot of the logarithm of mass density, $\log \rho$, gravity, $g$, Brunt-Vaisala frequency, $N^2$, turbulent viscosity coefficient, $\nu_0$ as a function of radius $r$ for $r_b = 0.9$, $n_2 = 1.0$. 
Figure 1.7: This is a plot of the logarithm of mass density, $\log \rho$, gravity, $g$, Brunt-Vaisala frequency, $N^2$, turbulent viscosity coefficient, $\nu_0$ as a function of radius $r$ for $r_b = 0.9$, $n_2 = 1.4$. 
Figure 1.8: This is a plot of the logarithm of mass density, $\log \rho$, gravity, $g$, Brunt-Vaisala frequency, $N^2$, turbulent viscosity coefficient, $\nu_0$ as a function of radius $r$ for $r_b = 0.999$, $n_2 = 1.0$. This figure demonstrates the difficulty associated with reading the plots for $N^2$ and $\nu_0$ as a function of radius, when the boundary between the convective and radiative zones occurs close to the surface. $\nu_0$ does not even show up.
Chapter 2

Equations governing quasi-static tides in close binaries

2.1 Basic Theory of Tides

Tidal forces, most familiar as alternating depth changes on the beach, have a prescribed, non-trivial effect on the evolution of binary stars. This effect can be seen by studying the motion of the mass elements on the star’s surface, and inductively by studying the motion of the two binary components in a perturbed gravitational orbit.

The mechanism of the tidal force in binary star systems is analogous to that of the earth moon system. The direct effect of the force itself is to cause a small deformation on the surface of one of the stars, removing it from spherical symmetry. Due to friction, this bulge does not stay symmetric with regard to the axis joining the centers of the stars, and undergoes a phase shift. The shifted mass component then exerts a torque on the star, prompting angular momentum transfer between the spin and the orbit. The tidal effect is divided into two components - A hydrostatic adjustment due to the perturbing force of the companion that is referred to as the static tide, and a dynamical response based on resonance with with oscillation modes known as the dynamic tide. (Zahn 1989).

To review the theory necessary for our basic study of tides, we start from a star in local thermal and hydrostatic equilibrium. We adopt a spherical coordinate reference frame, with the coordinate grid corotating with the per-
turbed star. With these assumptions, we can write down the equations for hydrostatic equilibrium, the energy generation equation, and the equation of Poisson. Note that in this section, dimensions are kept in the equations for the sake of consistency with cited papers:

\[
\frac{dP}{dr} = -\rho g, \tag{2.1}
\]
\[
\frac{dM_r}{dr} = 4\pi r^2 \rho, \tag{2.2}
\]
\[
\frac{dL_r}{dr} = 4\pi r^2 \rho \epsilon_N, \tag{2.3}
\]
\[
\nabla^2 \Phi = 4\pi G \rho, \tag{2.4}
\]

where \( \Phi \) is the potential of self-gravitation, \( P \) is pressure, \( M_r \) is the mass, \( L_r \) is the luminosity, \( \epsilon_N \) is the energy generation constant.

We examine one star, and note that the companion has a pull on any element that will perturb it in all three spherical dimensions, and will locally perturb the pressure, the density, and therefore, the potential for self-gravitation. We denote the last of these quantities as \( \Phi_T' \). Carrying out the perturbations into the equations for motion of a mass element, while also adding in thermal processes, give rise to high-order partial differential equations. Assuming that the tide generating potential is small compared to the star’s gravitational potential at the surface, the equations for the tidal motions of the mass elements in the spherical coordinates are given by (Willems 2000):

\[
\frac{\partial^2}{\partial t^2} (\delta r)_T = -\frac{\partial \Psi_T}{\partial r} + \frac{\rho_T'}{\rho} \frac{dP}{dr} - \frac{1}{\rho} \frac{\partial P_T'}{\partial r}, \tag{2.5}
\]
\[
\frac{\partial^2}{\partial t^2} [r (\delta \theta)_T] = -\frac{1}{r} \frac{\partial}{\partial \theta} \left( \Psi_T + \frac{P_T'}{\rho} \right), \tag{2.6}
\]
\[
\frac{\partial^2}{\partial t^2} [r \sin \theta (\delta \phi)_T] = -\frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} \left( \Psi_T + \frac{P_T'}{\rho} \right), \tag{2.7}
\]
\[
\frac{\rho_T'}{\rho} + \frac{1}{\rho} \frac{d\rho}{dr} (\delta r)_T = -\alpha_T, \tag{2.8}
\]
\[
\frac{P_T'}{P} + \frac{1}{P} \frac{dP}{dr} (\delta r)_T - \Gamma_1 \left[ \frac{\rho_T'}{\rho} + \frac{1}{\rho} \frac{d\rho}{dr} (\delta r)_T \right] = 0, \tag{2.9}
\]
\[
\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \Psi_T}{\partial r} \right) - \frac{1}{r^2} L^2 \Psi_T = 4\pi G \rho_T'. \tag{2.10}
\]

\( \alpha_T \) is the divergence of the tidal displacement field, \( L^2 \) is the Legendrian, perturbed coordinates are referred to with \( \delta \), and perturbed quantities are
2.2. EQUATIONS FOR GRAVITATIONAL POTENTIAL FOR QUASI-STATIC TIDES

We also define the total perturbation of gravitational potential, which is the quantity I will be examining most closely in subsequent sections of this thesis:

$$\Psi_T = \Phi_T' + \epsilon_T W. \tag{2.11}$$

$\epsilon_T W$ is the tide generating potential. Since the tide generating potential generally represents an oscillatory perturbation, its relative effect can be described by a Fourier expansion. (Polfliet et al. 1990)

$$\epsilon_T W(r, t) = \epsilon_T \sum_{l=2}^{4} \sum_{m=-l}^{l} \sum_{k=-\infty}^{\infty} \frac{GM_1}{R_1} c_{l,m,k} \left(\frac{r}{R_1}\right)^l Y_l^m(\theta, \phi)(r) \exp\left[i(\sigma_{T;m,k} t - kn \tau)\right], \tag{2.12}$$

where $k$ is the Fourier index, $l$ is the spherical harmonic index, and $m$ is the spherical harmonic azimuthal number. $\epsilon_T$ represents the ratio of the stars tidal force to its surface gravity at the star’s equator, $\sigma_{T;m,k}$ is the forcing angular frequency, $n$ the mean motion, $\tau$ is a time of periastron passage. $M_1$ is the mass of the perturbed star, $R_1$ is the equilibrium radius of the perturbed star, $G$ is the Gravitational constant.

$$\sigma_{T;m,k} = m\Omega + kn, \tag{2.13}$$

where $\Omega$ is the angular velocity of the perturbed star in equilibrium. $Y_l^m(\theta, \phi)$ are the spherical harmonics. The value of the fourier coefficients, $c_{l,m,k}$ are given by the following integral:

$$c_{l,m,k} = \frac{(l - |m|)!}{(l + |m|)!} P_l^m(0) \left(\frac{R_1}{a}\right)^{l-2} \frac{1}{\pi} \int_0^{\pi} \left[\frac{u(M)}{a}\right]^{-(l+1)} \cos[kM + mv(M)]dM, \tag{2.14}$$

where $P_l^m$ are the Legendre polynomials, and $u$ and $v$ are the radial distance and true anomaly of the companion.

2.2 Equations for gravitational potential for quasi-static tides

As discussed earlier, the tidal effect is resolved into a static and a dynamic component. For stars with convective zones, the static tidal mechanism is
Dominant (Zahn 1966). White dwarfs that are modeled by the polytropes discussed in Chapter 1 have such a convective zone, however it was realized that the thinness of this convective zone in white dwarfs rendered the quasi-static tides relatively unimportant compared to the dominant dynamical mechanism (Willems et al. (in preparation)). Nonetheless, we wish to explore the physics associated with the quasi-static tidal mechanism, and explore the numerical intricacies involved in solving the problem.

In order to simplify the problem, we only choose to consider the low frequency limit of oscillations. $\Psi_T$ can then be approximated by a Taylor expansion around $\sigma_T = 0$.

$$\Psi_T = \Psi_T^{(0)} + \sigma_T \Psi_T^{(1)} + \sigma_T^2 \Psi_T^2 + \ldots$$

(2.15)

Note that $\sigma_T = 0$ represents fully static tides.

In the regime of quasi-static tides, in the limit of $\sigma_T << 1$, the equations for the total perturbation of the gravitational potential in the low frequency limit, that is the zeroth and first order terms of the taylor expansion, are given by (Willems et al. (In preparation)):

$$\frac{d^2 \Psi_T^{(0)}}{dr^2} + \frac{2}{r} \frac{d \Psi_T^{(0)}}{dr} - \left[\frac{1}{g} \frac{d \rho}{dr} + \frac{\ell(\ell + 1)}{r^2}\right] \Psi_T^{(0)} = 0,$$

(2.16)

$$\frac{d^2 \Psi_T^{(1)}}{dr^2} + \frac{2}{r} \frac{d \Psi_T^{(1)}}{dr} - \left[\frac{1}{g} \frac{d \rho}{dr} + \frac{\ell(\ell + 1)}{r^2}\right] \Psi_T^{(1)} = i \frac{1}{gr^2} \frac{d}{dr} \left[\rho r^2 \nu \frac{d}{dr} \left(-\frac{\Psi_T^{(0)}}{g}\right)\right],$$

(2.17)

with boundary conditions, taken at $r = 1$, where we impose continuity of the gravitational potential and its gradient (Willems et al. 2000)

$$\left(\frac{d \Psi_T^{(0)}}{dr}\right)_{r=1} + (\ell + 1)(\Psi_T^{(0)})_{r=1} + \left(-\frac{\rho}{g} \Psi_T^{(0)}\right)_{r=1} = -\epsilon_T (2\ell + 1) c_{l,m,k},$$

(2.18)

$$\left(\frac{d \Psi_T^{(1)}}{dr}\right)_{r=1} + (\ell + 1)(\Psi_T^{(1)})_{r=1} + \left(-\frac{\rho}{g} \Psi_T^{(1)}\right)_{r=1} =$$

$$-i\left\{\frac{1}{N^2 r^2} \frac{d}{dr} \left[\rho r^2 \nu \frac{d}{dr} \left(-\frac{\Psi_T^{(0)}}{g}\right)\right]\right\}_{r=1},$$

(2.19)

where the 1 and 0 superscripts refer to first and zeroth order approximations, note that we have returned to dimensionless quantities in this section. For the
2.3. ALTERNATIVE FORM OF THE EQUATIONS FOR THE TIDAL PERTURBATION OF THE GRAVITATIONAL POTENTIAL

sake of computational ease, the derivative on the right hand side of Eq(2.17) is evaluated and simplified, we denote it as $RHS(\Psi^{(0)})$:

$$RHS(\Psi^{(0)}) = \frac{1}{gnr^2} \frac{d}{dr} \left[ \rho n^2 \nu \frac{d}{dr} \left( -\frac{\Psi^{(0)}_T}{g} \right) \right] =$$

$$-\frac{\rho \nu}{g^2} \left( \frac{1}{\rho} \frac{d \rho}{dr} + \frac{4}{r} + \frac{1}{\nu} \frac{d \nu}{dr} - \frac{2 \rho}{g} \right) \frac{d \Psi^{(0)}_T}{dr} +$$

$$\left\{ \frac{\rho \nu}{g^2} \left( \frac{\rho}{g} - \frac{2}{r} \right) \left( \frac{1}{\rho} \frac{d \rho}{dr} + \frac{1}{\nu} \frac{d \nu}{dr} \right) +$$

$$\frac{\rho \nu}{g^2} \left[ \frac{2 \rho}{g} \left( \frac{4}{r} - \frac{\rho}{g} \right) - \frac{6}{r^2} - \frac{l(l+1)}{r^2} \right] \right\} \Psi^0. \quad (2.20)$$

The right hand member of boundary condition eq(2.19) is identical to eq(2.17) times a factor of $\frac{g}{N^2}$, we denote this as the quantity $f$:

$$f(\Psi^{(0)}_T) = \frac{g}{N^2} RHS(\Psi^{(0)}_T). \quad (2.21)$$

We note that the homogeneous part of the first order equation is identical in form to the zeroth order. Solutions of such second order differential equations can be written in the form:

$$\Psi^{0}_T = Ay_1 + By_2, \quad (2.22)$$

where $y_1$ and $y_2$ are two linearly independent solutions to the zeroth order equations and

$$\Psi^{1}_T = Cz_1 + Dz_2 + z_p, \quad (2.23)$$

where $z_1$ and $z_2$ are two linearly independent solutions to the homogeneous part of the first order equations, while $z_p$ is a particular solution to the inhomogeneous part of the first order equation equation. Using a Taylor expansion at the origin, it is derived that one solution to the zeroth order and the homogeneous part of the first order equation, say $y_1$ and hence $z_1$ behaves as $r^\ell$, and the other, $y_2$ and $z_2$ as $r^{-(\ell+1)}$.

### 2.3 Alternative form of the equations for the tidal perturbation of the gravitational potential

The structure of a white dwarf star is best described as being composed mostly of a radiative core with only a small convective envelope region on
its exterior. Only the outer 0.1% of the star represents the outer convective zone. This implies that numerical routines must be driven to incredibly dense grids near the surface, otherwise, the relatively small size of the convective region will prove to be a source of inconsistency when processed through black box numerical integrators. Our inability to always directly control the way that adaptive routines behave leaves us in the dark about whether our code is truly resolving the thin surface convection region, which is a must to obtain reliable numerical results. In addition, interpreting plots of affected quantities near the surface on a normal radial scale becomes cumbersome (see Figure 2.5 comparing it with Figure 1.8). Taking this into consideration, it becomes helpful to alter the independent variable in the equations to a logarithmic mass variable that varies very rapidly near the surface, therefore naturally lending itself to take the right number of steps. As is customary in the modelling of white dwarfs, we define this dimensionless variable to be $\xi$ such that:

$$\xi = - \ln \left(1 - m\right).$$  \hspace{1cm} (2.24)

We note that this $\xi$ in this section is not at all related to the $\xi$ discussed in the previous section regarding the normalized radius variable for polytropes. In terms of this variable, the differential equations governing quasi-static tides are altered to reflect this change of independent variables. This is done with straightforward use of the chain rule.

Using:

$$\frac{dm}{dr} = \rho r^2,$$  \hspace{1cm} (2.25)

we derive:

$$\frac{d\xi}{dr} = \left(1 - m\right)^{-1} \rho r^2,$$  \hspace{1cm} (2.26)

$$\frac{d^2\xi}{dr^2} = \left(1 - m\right)^{-1} \frac{d\rho}{dr} r^2 + \left(1 - m\right)^{-1} \rho 2r + \left(1 - m\right)^{-2} \rho^2 r^4,$$  \hspace{1cm} (2.27)

We apply the chain rule to reparameterize the second derivative of $\Psi$ so that it is in terms of $\xi$

$$\frac{d\Psi}{dr} = \frac{d\xi}{dr} \frac{d\Psi}{d\xi},$$  \hspace{1cm} (2.28)

$$\frac{d^2\Psi}{dr^2} = \left(\frac{d\xi}{dr}\right)^2 \frac{d^2\Psi}{d\xi^2} + \frac{d^2\xi}{dr^2} \frac{d\Psi}{d\xi},$$  \hspace{1cm} (2.29)

We can then rewrite the equations for the total gravitational perturbation of
2.3. ALTERNATIVE FORM OF THE EQUATIONS FOR THE TIDAL PERTURBATION OF THE

gravitational potential, as well as the boundary conditions:

\[
\frac{d^2 \Psi^{(0)}_T}{d\xi^2} \left( \frac{d\xi}{dr} \right)^2 + \frac{d^2 \xi}{dr^2} \frac{d \Psi^{(0)}_T}{d\xi} + 2 \frac{d \Psi^{(0)}_T}{d\xi} \frac{d\xi}{dr} - \left[ \frac{1}{g} \frac{d \rho}{d\xi} \frac{d\xi}{dr} + \frac{\ell (\ell + 1)}{r^2} \right] \Psi^{(0)}_T = 0,
\]

(2.30)

\[
\frac{d^2 \Psi^{(1)}_T}{d\xi^2} \left( \frac{d\xi}{dr} \right)^2 + \frac{d^2 \xi}{dr^2} \frac{d \Psi^{(1)}_T}{d\xi} + 2 \frac{d \Psi^{(1)}_T}{d\xi} \frac{d\xi}{dr} - \left[ \frac{1}{g} \frac{d \rho}{d\xi} \frac{\ell (\ell + 1)}{r^2} \right] \Psi^{(1)}_T = \text{RHS}_\xi(\Psi^{(0)}_T),
\]

(2.31)

\[
\left( \frac{d \Psi^{(0)}_T}{d\xi} \frac{d\xi}{dr} \right)_{\xi=\infty} + (\ell + 1)(\Psi^{(0)}_T)_{\xi=\infty} + \left( -\frac{\rho}{g} \Psi^{(0)}_T \right)_{\xi=\infty} = -\epsilon_T (2\ell + 1)c_{l,m,k},
\]

(2.32)

\[
\left( \frac{d \Psi^{(1)}_T}{d\xi} \frac{d\xi}{dr} \right)_{\xi=\infty} + (\ell + 1)(\Psi^{(1)}_T)_{r=1} + \left( -\frac{\rho}{g} \Psi^{(1)}_T \right)_{\xi=\infty} = f_\xi(\Psi^{(0)}_T),
\]

(2.33)

with

\[
\text{RHS}_\xi(\Psi^{(0)}_T) = i \frac{1}{g r^2} \frac{d\xi}{dr} \frac{d}{d\xi} \left[ \rho r^2 \frac{d\xi}{dr} \frac{d}{d\xi} \left( -\frac{\Psi^{(0)}_T}{g} \right) \right],
\]

(2.34)

and

\[
f(\Psi^{(0)}_T) = \frac{g}{N^2} \text{RHS}_\xi(\Psi^{(0)}_T).
\]

(2.35)

To motivate the importance of this transformation, we examine some figures from the polytropic models like those at the end of chapter 2, except now using \( \xi \) as the independent variable rather than \( r \).
CHAPTER 2. EQUATIONS GOVERNING QUASI-STATIC TIDES IN CLOSE BINARIES

Figure 2.1: These are plots of Brunt-Vaisala frequency, $N^2$, turbulent viscosity coefficient, $\nu_0$ as a function of logarithmic mass $\xi$ for $r_b = 0.1$, $n_2 = 1.0$ (Compare with Figure 1.4)

Figure 2.2: These are plots of Brunt-Vaisala frequency, $N^2$, turbulent viscosity coefficient, $\nu_0$ as a function of logarithmic mass $\xi$ for $r_b = 0.1$, $n_2 = 1.4$ (Compare with Figure 1.5)
2.3. ALTERNATIVE FORM OF THE EQUATIONS FOR THE TIDAL PERTURBATION OF THE GRAVITATIONAL POTENTIAL

Figure 2.3: These are plots of Brunt-Vaisala frequency, $N^2$, turbulent viscosity coefficient, $\nu_0$ as a function of logarithmic mass $\xi$ for $r_b = 0.9$, $n_2 = 1.0$ (Compare with Figure 1.6)

Figure 2.4: These are plots of Brunt-Vaisala frequency, $N^2$, turbulent viscosity coefficient, $\nu_0$ as a function of logarithmic mass $\xi$ for $r_b = 0.9$, $n_2 = 1.4$ (Compare with Figure 1.7)
Figure 2.5: These are plots of Brunt-Vaisala frequency, $N^2$, turbulent viscosity coefficient, $\nu_0$ as a function of logarithmic mass $\xi$ for $r_b = 0.999$, $n_2 = 1.0$. This figure demonstrates the relative ease associated with reading plots for $N^2$ and $\nu_0$ as a function of $\xi$, when the boundary between the convective and radiative zones occurs close to the surface. (Compare with Figure 1.8)
Chapter 3

Numerical solution methods

3.1 A description of numerical routines

A computational code called Triton was developed by Bart Willems, and is continually being refined by myself and Bart Willems. The object of the code is to primarily resolve the solutions to Eq. (2.16) - Eq.(2.19) with a white dwarf-like input model. A secondary goal is to test different numerical routines in the context of this problem to better assess their effectiveness in the context of application to physical problems of this type.

To solve these equations numerically, we invoke routines developed by the Numerical Algorithms Group (NAG). The NAG library routines applicable to ordinary differential equations can be categorized as either boundary value problem (BVP) or initial value problem (IVP) routines, and subcategorized as either shooting or relaxation methods.

Initial value problems and boundary value problems are differentiated based on whether constraints for the value of the dependant variables of the equations are given at one or both endpoints of integration. The formulation of the tidal potential problem in Chapter 2. naturally presents itself as a BVP, therefore we first chose to examine NAGs BVP solving algorithms.

BVP shooting algorithms function by estimating a solution from an initial value problem, then integrating to the final point for a second solution. The routine equates the boundary values to the integrated solution to form a system of non-linear equations that it can solve by Newton’s method. These routines are generally said to be convergent unless the system is unstable or good initial estimates to the free parameters cannot be found.
BVP collocation methods work by constructing a mesh of piecewise polynomials approximating solution parameters, adjusting the coefficients of the polynomials to satisfy the differential equations and boundary conditions on each mesh subinterval, and then refining the mesh as a whole, with the aim of evenly distributing error over all elements.

BVP relaxation methods work by constructing a piecewise mesh of finite difference equations. A trial solution that does not necessarily solve any of the boundary conditions or the said finite difference equations is first adjusted, 'relaxing' down towards the true solution. Relaxation methods are most effective if used with a good approximation for the initial guess, on problems where boundary conditions are sensitive. It is not as effective in dealing with oscillatory and rapidly changing solutions.

It was observed upon analyzing results from initial runs that varying the initial values of boundary condition parameters, as well as routine parameters such as accuracy, could significantly affect the outcome of the solution in an unpredictable manner. This problem most likely arose due to the nature of our white dwarf input models: the right hand member of the first order equation becomes non-zero for only a small interval near the surface of the star, and has a value of relatively small order of magnitude compared to other problem parameters, thus provoking numerical instability in the boundary region. It is likely that this element was able to throw off any of the three types of routines, as it caused natural instability and rapid change within the routine. NAG’s BVP solving algorithms thus had to be removed from our consideration.

In the next section, we will develop a method that is able to instead use IVP solving algorithms to work through this problem. The Triton code provides three NAG IVP routines to determine $y_1$, $y_2$, $z_1$, $z_2$, and $z_p$, as they are defined in Eq. (2.22) and (2.23):

**D02BJF**

D02BJF is a runge-kutta integrator that looks for a zero of the dependent variable of our ODE within a specified interval. Out of the three IVP routines, D02BJF is the simplest to use and define, and is the most familiar to me, therefore making it my test routine for most of the code I have written thus far. Hypothetically, so long as the system is not considered stiff, and the accuracy isn’t driven too high, this routine should be the most efficient
3.2. INTERIOR MATCHING TECHNIQUE

as well. In light of potential numerical problems affecting the results that will be discussed in Chapter 4, however, it will be worth re-evaluating the effectiveness of this routine in future work.

**D02CJF**

D02CJF uses a variable-order, variable-step Adams method. This method has an efficiency advantage over D02BJF when high accuracy over a long range of values is demanded, but requires constructing a Jacobian for the boundary conditions in order to be implemented.

**D02EJF**

D02EJF uses a variable-order, variable-step method that contains backward differentiation formulae. This routine is known to be effective for stiff systems with rapidly decaying transient components. For non-stiff systems, however, it is generally a less time-efficient algorithm than the other two described.

All of the above routines are described in the NAG contents.

3.2 Interior matching technique

While a semi-analytical approach that will be developed in Section 3.3 offers a comprehensive solution to the problem we have proposed, it is worth considering a second, fully numerical solution strategy. The computational cost of calculating the integrals in the semi-analytical problem has the potential of becoming immense for high precision runs. The numerical solution method also offers a way for us to check the solution for consistency, therefore enabling us to make a definitive statement about the accuracy of our results. Finally, and perhaps most importantly, this technique is a fully numerical routine in which no additional assumptions about the magnitude of the forcing angular frequency $|\sigma_T|$ is made. It therefore allowing us to potential study any systems where $0 < |\sigma_T| < 1$

In light of the NAG libraries’ routines for BVP shortcomings for solving these problems, a different approach had to be developed. Instead of using a BVP routine to solve the BVP, we instead look at each boundary separately, using our knowledge of the boundary conditions and the overall form of the solution to cast the problem as two IVPs, that are then matched at an arbitrary interface.
We integrate a set of solutions to Eq. (2.16)-Eq(2.19) from the stars center, and another set from the stars surface to a point \( r_0 \) where \( 0 < r_0 < 1 \). For each case, we obtain two linearly independent solutions to Eq. (2.16) the homogeneous part of Eq. (2.17). We also obtain a particular solution to the inhomogeneous component of Eq. (2.17) and Eq. (2.19), \( z_p \) for both the surface and the center solution. Individually, the surface and center solutions will have forms identical to Eq. (2.22) and Eq.(2.23). We will denote surface solution quantities with the subscript \((s)\) and center solution quantities with the subscript \((c)\).

At the interior matching point \( r_0 \), we combine the solutions and impose continuity of \( \Psi \) and its first derivative to extract a system of equations with the unknown parameters being the coefficients \( A, B, C, \) and \( D \) (see Eq. (2.22) and Eq. (2.23)), which are inherently different for the center and surface solutions. The particular solutions are automatically normalized, as they appear only in the first order equation.

\[
A_c y_{1c} + B_c y_{2c} = A_s y_{1s} + B_s y_{2s}, \tag{3.1}
\]
\[
C_c z_{1c} + D_c z_{2c} + z_{pc} = C_s z_{1s} + D_s z_{2s} + z_{ps}, \tag{3.2}
\]
\[
A_c \frac{d y_{1c}}{d r} + B_c \frac{d y_{2c}}{d r} = A_s \frac{d y_{1s}}{d r} + B_s \frac{d y_{2s}}{d r}, \tag{3.3}
\]
\[
C_c \frac{d z_{1c}}{d r} + D_c \frac{d z_{2c}}{d r} + \frac{d z_{pc}}{d r} = C_s \frac{d z_{1s}}{d r} + D_s \frac{d z_{2s}}{d r} + \frac{d z_{ps}}{d r}, \tag{3.4}
\]

at \( r_0 \).

We recall from discussing Eq.(2.23) that \( y_{1c} \) and \( z_{1c} \), must have the form \( r^\ell \), while \( y_{2c} \) and \( z_{2c} \) must have the form \( r^{-(\ell+1)} \). In order for \( \Psi \) to remain finite at the center we must have:

\[
B_c = 0, \tag{3.5}
\]
\[
D_c = 0. \tag{3.6}
\]

Imposing boundary conditions Eq. (2.18) and Eq(2.19), we obtain:

\[
B_s = -\frac{\epsilon r c_{l,m,k} (2l + 1) + A_s \alpha}{\beta}, \tag{3.7}
\]
\[
D_s = -\frac{\mu + C_s \gamma \delta}{\delta} - \frac{i}{\delta} \left( A_s f_{1s} - B_s f_{2s} \right), \tag{3.8}
\]
where we have introduced:

\[ \alpha = \left( \frac{dy_{1s}}{dr} \right)_{r=1} + \left( l + 1 - \frac{\rho_s}{g_s} \right) y_{1s}(1), \]  
(3.9)

\[ \beta = \left( \frac{dy_{2s}}{dr} \right)_{r=1} + \left( l + 1 - \frac{\rho_s}{g_s} \right) y_{2s}(1), \]  
(3.10)

\[ \gamma = \left( \frac{dz_{1s}}{dr} \right)_{r=1} + \left( l + 1 - \frac{\rho_s}{g_s} \right) z_{1s}(1), \]  
(3.11)

\[ \delta = \left( \frac{dz_{2s}}{dr} \right)_{r=1} + \left( l + 1 - \frac{\rho_s}{g_s} \right) z_{2s}(1), \]  
(3.12)

\[ \mu = \left( \frac{dz_{ps}}{dr} \right)_{r=1} + \left( l + 1 - \frac{\rho_s}{g_s} \right) z_{ps}(1). \]  
(3.13)

We have also brought back the function \( f \) from Section 2.2 defined in eq 2.22 for \( \Psi_{T}^{(0)} \). Here we use it in the context of the individual components of the solution from the surface so that:

\[ f_{1s} = f(y_{1s}) \]  
(3.14)

\[ f_{2s} = f(y_{2s}) \]  
(3.15)

Returning to the continuity conditions, and implementing rigorous algebraic manipulation, we obtain expressions for the remaining parametric coefficient:

\[ A_c = - \frac{\epsilon_T c_{l,m,k}(2l + 1)}{\beta \Delta_1} \left\{ y_{2s}(r_0) \left[ \frac{\alpha}{\beta} \left( \frac{dy_{2s}}{dr} \right)_{r_0} - \left( \frac{dy_{1s}}{dr} \right)_{r_0} \right] - \left( \frac{dy_{2s}}{dr} \right)_{r_0} \left[ \frac{\alpha}{\beta} y_{2s}(r_0) - y_{1s}(r_0) \right] \right\} \]  
(3.16)

\[ A_s = - \frac{\epsilon_T c_{l,m,k}(2l + 1)}{\beta \Delta_1} \left\{ y_{1c}(r_0) \left( \frac{dy_{2s}}{dr} \right)_{r_0} - y_{2s}(r_0) \left( \frac{dy_{1c}}{dr} \right)_{r_0} \right\} \]  
(3.17)

Where:

\[ \Delta_1 = y_{1c}(r_0) \left[ \frac{\alpha}{\beta} \left( \frac{dy_{2s}}{dr} \right)_{r_0} - \left( \frac{dy_{1s}}{dr} \right)_{r_0} \right] - \left( \frac{dy_{1c}}{dr} \right)_{r_0} \left[ \frac{\alpha}{\beta} y_{2s}(r_0) - y_{1s}(r_0) \right] \]  
(3.18)

And:

\[ C_c = \frac{1}{\Delta_2} \left\{ g_1(r_0) \left[ \gamma \left( \frac{dz_{2s}}{dr} \right)_{r_0} - \left( \frac{dz_{1s}}{dr} \right)_{r_0} \right] - g_2(r_0) \left[ \frac{\gamma}{\delta} z_{2s}(r_0) - z_{1s}(r_0) \right] \right\} \]  
(3.19)

\[ C_s = \frac{1}{\Delta_2} \left\{ z_{1c} g_2(r_0) - \left( \frac{dz_{1c}}{dr} \right)_{r_0} g_1(r_0) \right\} \]  
(3.20)
Where:
\[
\Delta_2 = z_{1c}(r_0) \left[ \frac{\gamma}{\delta} \left( \frac{dz_{2s}}{dr} \right)_{r_0} - \left( \frac{dz_{1s}}{dr} \right)_{r_0} \right] - \left( \frac{dz_{1c}}{dr} \right)_{r_0} \left[ \frac{\gamma}{\delta} z_{2s}(r_0) - z_{1s}(r_0) \right] \tag{3.21}
\]

\[
g_1 = z_{ps}(r_0) - z_{pc}(r_0) - \frac{\mu}{\delta} z_{2s}(r_0) - i \frac{1}{\delta} \left[ A_s f_{1s} - \frac{\epsilon_{Tcl,m,k}(2l + 1)}{\beta} f_{2s} \right] z_{2s} \tag{3.22}
\]

\[
g_2 = \left( \frac{dz_{ps}}{dr} \right)_{r_0} - \left( \frac{dz_{pc}}{dr} \right)_{r_0} - \frac{\mu}{\delta} \left( \frac{dz_{2s}}{dr} \right)_{r_0}
- i \frac{1}{\delta} \left[ A_s f_{1s} - \frac{\epsilon_{Tcl,m,k}(2l + 1)}{\beta} f_{2s} \right] \frac{dz_{2s}}{dr} \tag{3.23}
\]

When we compute the coefficients $A_c, A_s$, etc., it becomes possible to write out $\Psi_T$ explicitly over the entire solution space.

It is possible to translate these results to the logarithmic mass variable $\xi$. The transformation is merely application of the chain rule to first order terms, changing every $\frac{d}{dr}$ to $\frac{d}{d\xi}$. This is a redundant process and has not yet been implemented into the Triton code, so I will not show show the explicit formulation.

### 3.3 The semianalytical solution technique

In light of the initial failures of the fully numerical approach, an alternative semi-analytical approach was developed by Bart Willems. The semianalytical approach is only valid for the quasi-static approximation, that is, the regime where $|\sigma_T| << 1$. Compared to the fully numerical interior matching approach discussed in the previous section, the systems that we are able to study using this approach are restricted. Nonetheless, this approach proves to be a very useful tool in serving as a guide for tuning the fully numerical solutions to a level that is more correct.

We start by again noting that the homogeneous part of the solution of Eq. (2.16) and Eq. (2.17) Will have the form described by Eq. (2.22) and Eq. (2.23). We can solve this homogeneous equation the normal manner: we determine the solution components $y_1, y_2, z_1, \text{ and } z_2$, and then impose the surface boundary condition analytically.

\[
\Psi_T^{(0)} = A y_1, \tag{3.24}
\]
3.3. THE SEMIANALYTICAL SOLUTION TECHNIQUE

\[ \Psi^{(1)}_{T,h} = Cz_1. \]  
(3.25)

As in the previous sections, we know that \( B \) and \( D \) are set to 0 in order for the solution to remain finite at the origin.

To solve the inhomogeneous part of the equation, we search for a particular solution with a technique known as variation of the constants:

\[ \Psi^{(1)}_{T,P} = C_r z_1 + D_r z_2, \]  
(3.26)

\[ \frac{d\Psi^{(1)}_{T,P}}{dr} = C_r \frac{dz_1}{dr} + D_r \frac{dz_2}{dr} + \frac{dC}{dr} z_1(r) + \frac{dD}{dr} z_2(r), \]  
(3.27)

where the parameters \( C \) and \( D \) are now able to change as a function of \( r \). Implementing the nuance of the technique, we set the sum of the products of the first derivative of the constants and their corresponding solution equal to zero.

\[ \frac{dC}{dr} z_1(r) + \frac{dD}{dr} z_2(r) = 0, \]  
(3.28)

implying that the second derivative of the particular solution should have this form:

\[ \frac{d^2\Psi^{(1)}_{T,P}}{dr^2} = C_r \frac{d^2z_1}{dr^2} + D_r \frac{d^2z_2}{dr^2} + \frac{dC}{dr} \frac{dz_1}{dr} + \frac{dD}{dr} \frac{dz_2}{dr}. \]  
(3.29)

We then substitute these into Eq. (2.17), finding a convenient expression for \( C_r \) and \( D_r \) in terms of the right hand member.

\[ \frac{dC}{dr} \frac{dz_1}{dr} + \frac{dD}{dr} \frac{dz_2}{dr} = iRHS(Ay_1). \]  
(3.30)

We have brought back the \( RHS \) function from Section 2.2, Eq. (2.21). We now use it in the context of the individual component of the homogeneous part of the solution \( y_1 \) times the corresponding solution parameter, \( A \).

By using Eq. (3.25) in conjunction with Eq. (3.26), we can see that it becomes possible to obtain each of the constants at a given value of \( r \) by integrating:

\[ D_r = i \int_0^r \frac{z_1 RHS(Ay_1)}{z_1 \frac{dz_2}{dr} - z_2 \frac{dz_1}{dr}} dr, \]  
(3.31)
\[ C(r) = -i \int_0^r \frac{z_2 \text{RHS}(Ay_1)}{z_1 \frac{dz_2}{dr} - z_2 \frac{dz_1}{dr}} \, dr. \]  

(3.32)

To compute \( \Psi_T^{(1)} \) we combine the homogeneous and the particular parts of the solution:

\[ \Psi_T^{(1)}(r) = Cz_1(r) + C(r)z_1(r) + D(r)z_2(r). \]  

(3.33)

We still need to solve for the constant \( C \), which is now possible if we implement boundary condition Eq. (2.19). We find:

\[ C = \frac{i \gamma (D_r)_{r=1} + i \delta (C_r)_{r=1} - if(Ay_1)}{\gamma}. \]  

(3.34)

We again use function \( f \) from section 2.2 eq (2.22), this time applying it to the normalized homogeneous part of the solution. \( \gamma \) and \( \delta \) are also used as shorthand, and are defined in a similar way to how we had them in Section 3.2:

\[ \gamma = \left( \frac{dz_1}{dr} \right)_{r=1} + \left( l + 1 - \frac{\rho_s}{g_s} \right) z_1(1), \]  

(3.35)

\[ \delta = \left( \frac{dz_2}{dr} \right)_{r=1} + \left( l + 1 - \frac{\rho_s}{g_s} \right) z_2(1). \]  

(3.36)

We can translate the end product of this derivation to the logarithmic mass variable \( \xi \) discussed in Section 2.3.

We write Eq. (3.26) and Eq. (3.27) as:

\[ \Psi_T^{(1),P} = C_\xi z_1 + D_\xi z_2, \]  

(3.37)

\[ \frac{d\Psi_T^{(1),P}}{d\xi} = \frac{dC_\xi}{d\xi} z_1(\xi) + \frac{dD_\xi}{d\xi} z_2(\xi). \]  

(3.38)

Eq. (3.31) and Eq. (3.32) become:

\[ D(\xi) = i \int_0^\xi \frac{z_1 \text{RHS}(Ay_1)}{\left( \frac{d\xi}{dr} \right)^2 (z_1 \frac{dz_2}{d\xi} - z_2 \frac{dz_1}{d\xi})} \, d\xi, \]  

(3.39)

\[ C(\xi) = -i \int_0^\xi \frac{z_2 \text{RHS}(Ay_1)}{\left( \frac{d\xi}{dr} \right)^2 (z_1 \frac{dz_2}{d\xi} - z_2 \frac{dz_1}{d\xi})} \, d\xi. \]  

(3.40)

We again use, as in Eq. (2.26)

\[ \frac{d\xi}{dr} = (1 - m)^{-1} \rho r^2. \]
We then write Eq. (3.33) as:

$$\Psi_T^{(1)}(\xi) = C z_1(\xi) + C(\xi) z_1(\xi) + D(\xi) z_2(\xi),$$

(3.41)

and Eq. (3.34) becomes:

$$C = \frac{i\gamma(D_{\xi})_{\xi=\xi_s} + i\delta(C_{\xi})_{\xi=\xi_s} - i f(A y_1)}{\gamma}.$$ 

(3.42)
Chapter 4

Results

4.1 Comparison of Numerical and Semi-analytical Methods

We solve equations (2.16) and (2.17) with boundary conditions (2.18) and (2.19) using both the semianalytical method developed in section (3.3) and the interior matching method developed in section (3.2) and compare the outcome. For the input models, we use the composite polytropes discussed in Chapter 1, with $n_1 = 3.0$ to represent the radiative core, and $1.0 < n_2 < 1.5$. One may wonder about the physical consequences of using $n_2 < 1.5$, since $n_2 = 1.5$ represents a convective zone, while $n_2 < 1.5$ does not apply to any physical state we expect to find the white dwarf in. Our choice can only be justified by noting that the convective turnover time scale, which is proportional to $\frac{1}{\sqrt{|N^2|}}$, is infinite for $n_2 = 1.5$. If we want a finite convective turnover time scale, which is a must for tidal dissipation, we must use $n_2 < 1.5$. While this does not prove to be an accurate model of a white dwarf, my main objective at this stage was to test the code with stellar models with thin outer zones that had a sudden onset of the inhomogeneous term of Eq. (2.17), and the composite polytropes met the requirement.

We find that the solution for the zeroth order equation plots consistent with one another. The following figures use an $n_1 = 3.0$, $n_2 = 1.0$, $r_b = .900$ polytope, with the interior matching technique applied at an interface of $r_o = .100$. 
CHAPTER 4. RESULTS

Figure 4.1: Interior matching (left) and semianalytical (right) solutions to the zeroth order component of the total tidal perturbation to the gravitational potential. The zeroth order solution is always fully numerical, and is solved in very similar ways in both versions.

Figure 4.2: relative error between the two solution methods to the zeroth order equations for tidal perturbation to the gravitational potential. We only see slight systematic error. Note that the jump occurs at the boundary that is used in the interior matching technique.
4.1. COMPARISON OF NUMERICAL AND SEMI-ANALYTICAL METHODS

To our dismay, solutions to the first order equations, that is Eq( 2.17 ), are as of right now not matching up, as is shown by Figure (4.3). Effort has been spent to debug the code and come up with potential sources for the error, but we have thus far been unable to resolve these issues. The current consensus is that the semi-analytical solutions, that only rely on computing integrals of completed solution components, are the standard for solution accuracy. This means that something is malfunctioning in the fully numerical approach. Continuing work is being poured into understanding the nature of this problem.

4.1.1 Failure of the interior matching technique to solve the first order equations

The best conjecture for the cause of the difficulties that we can currently come up with is that a deeper rooted problem is imposed with the formulation of our approach in reference to the input physics. At the boundary between the convective and radiative layer of the polytrope, continuity of mass, density, and pressure are imposed as discussed in Section 1.2. What is not imposed, and indeed not accounted for, is the continuity of the first derivative of density. This discontinuity comes in to our formulation of the Brunt-Vaisala frequency, and is therefore present in the first order differential equations.

The fact that the zeroth order solutions match up proves that the interior matching technique is a valid approach for solving a differential equation, and is functional numerically with the NAG routines that we are using. The continuity at the interface (which in the case of this model is found at $r = .100$) of the first order solution as shown in Figure 4.3, and its first derivative (not shown) further provides proof that the continuity conditions relating the boundary parameters to one another, and further to the solution must be valid. What must be going astray is the actual computation of the particular solution via numerical methods due to the discontinuity in our model. This difficulty then translates itself into the formulae computing the constants $C_s$, $C_c$ and so forth, therefore throwing the solution off many orders of magnitude.
CHAPTER 4. RESULTS

Figure 4.3: Fully numerical (left) and Semianalytical (right) solutions to the first order component of the total tidal perturbation to the gravitational potential. It is clear from these plots that the solution methods are not giving consistent results, and suspicion is cast on the interior matching technique.

4.2 Comparison of different independent variables

Due to the inaccuracy of the solutions given by the interior matching method, extensive testing of the translated equations into the logarithmic mass variable has been put on hold.

The transformation has however been implemented into the semi-analytical solution method, and the results obtained from initial runs are satisfactory. These results show that there is potential for the use of the transformed equations in carrying out detailed study of the system as we resolve existing issues with the accuracy of our fully numerical solution methods.

We notice that the transformation appears to work fairly well regardless of choice for the polytropic index of the outer zone, $n_2$, or $r_b$, the location of the interface between the inner and outer zones. Figures 4.4 - 4.12 demonstrate the results attained so far by running the Triton code for three different input models using the semianalytical method with both of the independent variables. It is clear from these figures that we have attained more or less satisfactory numerical convergence between the two solution methods.
4.2. COMPARISON OF DIFFERENT INDEPENDENT VARIABLES

Figure 4.4: $r_b = .900$, $n_1 = 3.00$, $n_2 = 1.00$ polytrope model. The solutions for $\Psi_T^{(0)}$ in both the radial, $r$ (black) and mass, $\xi$ (blue) coordinate. They look identical here.
Figure 4.5: $r_b = 0.900$, $n_1 = 3.00$, $n_2 = 1.00$ polytrope model. The solutions for $\Psi^{(1)}_T$ in both the radial, $r$ (black) and mass, $\xi$ (blue) coordinate. They look identical here.
4.2. COMPARISON OF DIFFERENT INDEPENDENT VARIABLES

Figure 4.6: Relative error between the radial and mass solutions for $\Psi^{(0)}_T$, left, and $\Psi^{(1)}_T$, right, for the above system. We see initial fluctuations that are probably resultant from the implementation of the boundary conditions, followed by a systematic shift that could be due to the inherent difference of the methods.
Figure 4.7: $r_b = 0.990$, $n_1 = 3.00$, $n_2 = 1.20$ polytrope model. The solutions for $\Psi_T^{(0)}$ in both the radial, $r$ (black) and mass, $\xi$ (blue) coordinate. They look virtually identical here.
Figure 4.8: $r_b = .990$, $n_1 = 3.00$, $n_2 = 1.20$ polytrope model. The solutions for $\Psi_1^{(1)}$ in both the radial, $r$ (black) and mass, $\xi$ (blue) coordinate. They look very close, with a slight discrepancy close to the outer boundary.
Figure 4.9: Relative error between the radial and mass solutions for $\Psi_T^{(0)}$, left, and $\Psi_T^{(1)}$, right, for the above system. We see initial fluctuations that are probably numerical, followed by a systematic shift that can perhaps be explained by imprecise treatment of boundary conditions.
4.2. COMPARISON OF DIFFERENT INDEPENDENT VARIABLES

Figure 4.10: \( r_b = 0.600 \), \( n_1 = 3.00 \), \( n_2 = 1.40 \) polytrope model. The solutions for \( \Psi_T^{(0)} \) in both the radial, \( r \) (black) and mass, \( \xi \) (blue) coordinate. They look virtually identical here.
Figure 4.11: $r_b = 0.600$, $n_1 = 3.00$, $n_2 = 1.40$ polytrope model. The solutions for $\Psi_T^{(1)}$ in both the radial, $r$ (black) and mass, $\xi$ (blue) coordinate. They look virtually identical here.
4.2. COMPARISON OF DIFFERENT INDEPENDENT VARIABLES

Figure 4.12: Relative error between the radial and mass solutions for $\Psi^{(0)}_T$, left, and $\Psi^{(1)}_T$, right, for the above system. We see initial fluctuations that are probably resultant from the implementation of the boundary conditions, followed by a systematic shift that could be due to the inherent difference of the methods.
CHAPTER 4. RESULTS

4.3 Future work

While the fully functional semianalytical solution in conjunction with different polytropic models provides us with a useful tool to study some aspects of tidal interactions in white dwarf binaries, the nature of the approach is such that studying the computational aspects of accuracy parameters is not too useful.

Ideally, we wish to go back to the fully numerical interior matching technique, and understand why it is not producing satisfactory results.

It is suspected that numerical difficulties are arising due to the discontinuity in the derivative of density that occurs at the boundary of the two zones in the input model.

Once this code is fully developed, and analysis on computational efficiency for accurate solutions is performed, we can expand our study to systems in the regime outside of the quasi-static tides approximation, where $|\sigma_T|$ is not much smaller than 1.
4.3. FUTURE WORK

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