From *Numerical Recipes*:

- Chapter 9: sections 0, 1, 4, and 6.
- Chapter 9 functions: rtbis.c, rtnewt.c, rtsafe.c
4.1 Each of two bodies in a two-body system in a bound gravitational equilibrium state follow Keplerian orbits, i.e., ellipses, with the other body at the focus of the ellipse. Such orbits are characterized in general by the orbital semimajor axis $A$ and the orbital eccentricity $e$. These two quantities effectively define the orbital energy (depends only on the masses and the orbital semi-major axis) and angular momentum (depends on the masses, orbital semi-major axis, and eccentricity). Such elliptical orbits are described by the famous Kepler equation:

$$
\omega t = \psi - e \sin \psi,
$$

where $\omega$ is the orbital angular velocity (equal to $2\pi/P$, where $P$ is the orbital period), $\omega t$ (called the mean anomaly) and $\psi$ (called the eccentric anomaly) run from 0 to $2\pi$ in the course of a complete orbital revolution. The position of each body along the orbit can be described in polar coordinates $r$ and $\theta$ given by the following equations:

$$
\cos \theta = \frac{\cos \psi - e}{1 - e \cos \psi},
$$

$$
r = A(1 - e \cos \psi).
$$

$A$ and $P$ are connected by Kepler's third law:

$$
\frac{G(M_1 + M_2)}{4\pi^2} = \frac{A^3}{P^2},
$$

where $G$ is the gravitational constant, and $M_1$, $M_2$ are the masses of the objects. Assuming that the masses of the two bodies are chosen in such a way that $A = 1$ and $P = 1$:

1. With the above units plot $\omega t - \psi + e \sin \psi$ as a function of $\psi$ (from 0 to $2\pi$) for $t = 0.0, 0.5, 1.0$ and for $e = 0$ and 0.6.

2. Write a code that uses the Newton-Raphson method to solve the Kepler equation for $t$ in the range 0 to 1 and for a given value of $e$. Use the solution for $\psi$ to calculate $r$, $\theta$, and the cartesian coordinates $x = r \cos \theta$, $y = r \sin \theta$ as a function of $t$.

3. Plot the elliptical orbits (using the cartesian coordinates) for $e = 0, 0.3, 0.6, 0.9$

4. Plot $r$ and $\theta$ as a function of $t$, for $e = 0, 0.3, 0.6, 0.9$

5. To plot the shape of the orbits, is it necessary to solve the Kepler equation?
4.2 In quantum mechanics, a particle in a square potential well of finite depth has a discrete spectrum of bound energy states \((E < 0; E_0, E_1, E_2, E_3, \ldots)\). After solving the Schrödinger equation for such a particle it can be shown that the *even* \((E_0, E_2, E_4, \ldots)\) energy levels are determined by the following equation:

\[
\sqrt{-E} = \sqrt{E - V} \tan \left( \frac{a}{\hbar/2\pi} \sqrt{2m(E - V)} \right),
\]

(1)

where \(m\) is the mass of the particle, \(\hbar\) is the Planck constant, \(a\) is the half-width of the well, and \(V < 0\) is the depth of the well. Choosing \(V = -1\) and \(\alpha = a/\left(\sqrt{-2mV}\right)^{-1}\), the equation becomes:

\[
\sqrt{-E} = \sqrt{E + 1} \tan (\alpha \sqrt{E + 1}).
\]

(2)

Note that in the chosen units \(E\) lies in the range \(-1\) to 0. The discretization of energy levels comes from the periodicity (branches of the tangent function). Constraining the tangent argument \((\alpha \sqrt{E + 1})\) to lie between \(-\pi/2\) and \(\pi/2\) gives us \(E_0\), between \(\pi/2\) and \(3\pi/2\) gives us \(E_2\), between \(3\pi/2\) and \(5\pi/2\) gives us \(E_4\), etc. Whether these ranges for the tangent argument are allowed depends on the value of \(\alpha\), i.e., the width of the well for a given depth \((V = -1)\).

1. Plot the left and right hand sides of equation (2) as a function of the tangent argument \((\alpha \sqrt{E + 1})\), for \(\alpha = 1.0, 5.0, 10.0\). For each \(\alpha\) value plot the two curves on the same plot.

2. What is the the minimum value of \(\alpha\) in integer multiples of \(\pi/2\), so that at least 4 energy levels of even order \((E_0, E_2, E_4, E_6)\) exist.

3. Adopt this minimum value of \(\alpha\) and use bisection to solve the above equation and calculate the first 4 even energy levels.

NOTE: To find the various energy levels, you need to search for solutions of \(E\) within certain angle (the tangent argument) ranges. Remember that angles are measured in radians in C. However, also remember that the tangent cannot be evaluated at exactly these values (it approaches infinity). Instead you should bracket the solution for \(E\) using values slightly different from the exact boundaries mentioned above.