THE POLARIZATION OF STARLIGHT BY ALIGNED DUST GRAINS

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ABSTRACT

Polarization of starlight may arise from absorption and scattering by elongated dust grains that are at least partially aligned by a magnetic field. We assume that the grains contain mostly compounds of hydrogen with about 12 per cent iron by weight, presumably also in compounds. The theory of Gans is used to compute the scattering and absorption by a spheroidal grain small compared to the wave length, the component of the light with its electric vector parallel to the grain's long axis being preferentially weakened. The polarization and extinction of light are then computed for a cloud of small particles (mean radius 10^{-5}–3×10^{-5} cm) having any specified distribution of orientations.

Our mechanism for orienting the rapidly spinning grains (angular velocities of the order of 10^5–10^6 rad/sec) is the small nonconservative torque due to paramagnetic relaxation in material containing a few per cent of iron. The magnitude of the paramagnetic absorption is estimated from theory and experiment. This torque tends to make the short axis of the grain the axis of rotation, which is still very rapid, and to set this short axis along the magnetic field (orientation rate of the order of 10^{-12}–10^{-13} rad/sec). Thus the maximum extinction coefficient is for light with its electric vector perpendicular to the field. An approximate statistical theory based on a relaxation time (of the order of 10^{13} sec) for the orientation of the grains gives the distribution of orientations to be expected when this tendency toward orientation is balanced by the tendency toward random alignment caused by the bombardment of the grains by the interstellar gas. Small prolate grains of eccentricity such that the ratio of diameters lies in the range 1.2–1.7 will be sufficiently aligned to produce the observed ratio of polarization to color excess, $p/E_1 \leqslant 0.18$, in a magnetic field of 10^{-4}–10^{-5} gauss.

The theory predicts that for uniform magnetic fields and interstellar matter of constant temperature and composition the polarization should be proportional to the color excess. The direction of the observed polarization vectors indicates that over regions of several hundred parsecs in the Milky Way the magnetic field is mainly parallel to the plane of the galaxy, perhaps nearly uniform along a spiral arm or perhaps making random whirls mainly in the plane of the galaxy. Since the interstellar grains are larger than those to which the Gans theory applies and since the predicted wave-length dependence of the extinction is incorrect, it is not surprising that the theory gives an incorrect wave-length dependence of the polarization. Qualitative considerations indicate that the predicted ratio p/E_1 is not too far from the correct order of magnitude and that present observations of the wave-length dependence do not necessarily contradict the rest of our theory, which can be extended when extinction coefficients are known for large grains.

Observational data are summarized in § 1, and the nature of the grains is treated in § 4; our theory is given in outline in § 3, and the mathematical details are worked out in §§ 5–7. In § 8 we show that torques due to eddy currents or to static charges on the grain are unimportant.

1. SUMMARY OF OBSERVATIONAL INFORMATION

The polarization of starlight transmitted through interstellar dust and gas was recently discovered by W. A. Hiltner[1, 2, 3] and John S. Hall.[4, 5, 6] Their data yield three striking characteristics of the polarization.[7]

[1] *Science*, **109**, 165, 1949.

[2] *Ap. J.*, **109**, 471, 1949.

[3] *Phys. Rev.*, **78**, 170, 1950.

[4] *Science*, **109**, 166, 1949.

[5] Hall and Mikesell, *A.J.*, **54**, 187, 1949.

[6] Hall and Mikesell, *Pub. U.S. Naval Obs.*, Vol. **17**, Part I, 1950.

[7] We are greatly indebted to both Hall and Hiltner for permission to use their results before publication.

(i) They have shown that a correlation exists between the available measurements of the polarization p and the photoelectric measurements by Stebbins, Huffer, and Whitford[8] of the interstellar reddening E_1. Large polarization is very rare among stars of small reddening, and a rough upper limit seems to exist for the polarization associated with a given reddening. Stars of large reddening show a range of polarization from nearly zero to this limit. Thus it seems probable that interstellar absorption is a necessary, but not a sufficient, condition for the appearance of polarization. This conclusion is well documented by Hall and Mikesell's detailed review[6] of the correlations of the polarization with other quantities. We shall express this in numerical form, following Hall in using the conventional definition of the polarization, p. As discussed in § 5.3 below, p is 0.46 times the perhaps more convenient definition used by Hiltner. In a plot of p as a function of E_1, the correlation diagram can be bounded by a straight line that gives the normal maximum value of p for a given E_1. This shows that, approximately,

$$p \leqslant 0.18 E_1 , \tag{1}$$

where the natural uncertainties at low values are augmented by the uncertain location of the zero point of the E_1 scale. The mean value of p/E_1 is considerably lower, near 0.09. We do not consider in these estimates the apparently nearly unreddened stars listed by Hall and Mikesell[6] in their Table IV. Since E_1 and the interstellar absorption are very highly correlated, we can introduce the known ratio of the color excess to the absorption at the photographic effective wave length and obtain

$$p \leqslant 0.020 A_{pg} . \tag{2}$$

Under the most favorable conditions, which occur in the galactic plane, this is equivalent to a statement that the extinction coefficient of the dust varies over a range of about 5 per cent with the plane of vibration.

(ii) In general, the vectors representing the polarization of different stars show approximate parallelism for small galactic latitude over a considerable range of galactic longitude. There is, however, only a correlation of directions, not completely uniform alignment. In our discussion we shall describe the polarization in terms of the plane of vibration, the plane in which the electric vector of the radiation has its maximum; we do not use the older "plane of polarization," which lies at right angles to this and which is used by Hall and Mikesell. In the region of the galactic center and in Cygnus there is considerable disorder, but in the Perseus region the planes lie within $\pm 6^\circ$ of parallelism over an area 8° square. According to Hall and Mikesell, the plane of vibration lies approximately in the galactic plane from galactic longitude 80° to 120° and perhaps up to 170°.

(iii) Hiltner[3] has shown that the polarization varies more slowly with wave length than does the interstellar absorption.

We may now briefly discuss the implications of these observations.

(i) The correlation of polarization and color excess indicates that a cloud of interstellar particles can produce a polarization which, under favorable conditions, reaches the upper limit given by equation (2). The fact that the polarization is often less than the maximum can be explained if the mechanism producing the polarization either (1) varies from region to region in strength or effectiveness in producing observable results or (2) has a random orientation from one complex of dust to another, so that the observed net polarization arises from random composition of the polarization vectors. The first possibility is exemplified if the polarization is due to the alignment of dust grains, and the aligning mechanism is more effective in certain regions than in others. As an example of the second possibility, note that, if there are n regions, the average polarization will be $n^{1/2}$ times

[8] *Mt. W. Contr.*, No. 621; *Ap. J.*, **91**, 20, 1940.

that produced by one region, while the reddening is n times. Then p/E_1 will be $n^{-1/2}$ times the ratio in a single cloud.

(ii) The rough parallelism of the planes of vibration suggests that the polarization does not arise in individual stars, circumstellar clouds, or any other small regions of space, since there seems to be no reason why such objects should have parallel axes. In fact, since strong interstellar absorption over the region of approximate parallelism sets in at a distance of 300–500 psc, we conclude that a region with an extension of at least 200 psc across the line of sight produces an essentially constant direction of polarization. In the direction of Sagittarius and Cygnus, where the direction of polarization is more erratic and the ratio p/E_1 somewhat smaller, one might accept the random composition of the effects of small clouds discussed above.

(iii) The color-dependence data imply either that the constituent of interstellar space responsible for polarization is not the one responsible for the main absorption or that, although one constituent does both, the difference in the extinction coefficients of beams of light having different planes of polarization shows less dependence on the wave length than does the sum of the extinction coefficients. Even the former alternative does not imply that dust could not be responsible for the polarization; it could imply that the dust has a wide range of sizes, with different size ranges being dominant in each of the two phenomena.

2. POSSIBLE SOURCES OF POLARIZATION

The polarization must be due to some anisotropy in space. No obvious mechanisms exist that would polarize light by direct interaction with anisotropic gravitational or electromagnetic fields or with matter moving anisotropically. The simplest mechanism is an absorption that depends on the plane of vibration of the light, thus implying some type of alignment in the absorbers. Certain fields with which the absorbers might be aligned cannot be essentially uniform over large enough regions, as in the case of electric fields which are suppressed by the high conductivity of interstellar space. Most other fields produce no alignment of any kind, as in the case of gravitational fields or fields of light-flux. The one exception is a magnetic field, since it is made essentially permanent by the high conductivity and since it exerts an aligning torque on any body with a magnetic moment.

Let us consider sources of absorption which under special conditions might produce polarization. The Thomson scattering by electrons cannot be a source of polarization because of the large mass of ionized material required to produce appreciable absorption. It also does not seem probable that electron spins can be aligned in a magnetic field B unless $Bp \approx kT$, where $p = 9 \times 10^{-21}$ erg/gauss is the magnetic moment of an electron. Fields of $B \approx 10^4 T$ gauss would be required. Also there seems to be no way for aligned electrons to produce polarization by Thomson scattering.

Atomic continua are negligible except shortward of the Lyman limit. The absorption by H^- from its ground state has not as yet been considered in interstellar space. We may estimate the number of H^- ions per cubic centimeter from

$$\frac{n_{H-}}{n_H n_e} = 4.16 \times 10^{-10} \theta_c^{3/2} \theta_e 10^{0.75\theta_c} \left(\frac{T_c}{T_e}\right)^{1/2} \frac{kT_e}{W}, \tag{3}$$

where T_c and T_e are the color and electron temperatures and W is the dilution. In a cool H I region $n_e = 10^{-3} n_H$, and then $n_{H-} = 1.6 \times 10^{-10} n_H^2$; the absorption is $1.5 \times 10^{-5} n_H^2$ per kiloparsec. This is negligible unless $n_H > 10^2$, i.e., unless the average space density along the line of sight is about one hundred times the Oort limit. In a dense dark nebula of large extent, H^- may have to be considered as a source of absorption; unlike dust, its absorption decreases shortward. However, we cannot suggest any mechanism by which H^- can produce differential extinction in a magnetic field.

Dust is believed to be the most important absorbing constituent in space. Polarization is observed to be partially correlated with color excess, which is almost surely due to absorption by dust. Contrary to the cases mentioned above, there exists an obvious mechanism that could produce a polarization of light by aligned dust grains and several mechanisms that might align the grains.

3. THE BASIS OF POLARIZATION BY PARAMAGNETIC RELAXATION: SUMMARY OF RESULTS

In this section we treat the physical basis for the alignment of dust grains by a magnetic field and the resulting polarization of starlight, emphasizing the assumptions and approximations made and summarizing the conclusions that can be drawn. The extensive mathematical derivations are postponed until later sections.

3.1. *The equipartition of angular velocity.*—The essential problem that any theory must meet is the necessity of aligning elongated dust grains in spite of the tendency toward rapid spin caused by collisions with the hydrogen atoms or ions in space. In the absence of any aligning torque, the orientation of the grains would be completely random, and, by the equipartition theorem, their mean kinetic energy of rotation, R_e, would be equal to the mean kinetic energy of the atoms of the interstellar gas. Thus, if T denotes the kinetic temperature of the gas and a the average radius of a grain whose internal density is ρ_g, the mean rotational energy is

$$R_e = \tfrac{3}{2} kT = 2.07 \times 10^{-16} T \text{ ergs}. \tag{4}$$

Since its average moment of inertia is $\bar{I} \approx (\frac{5}{3})\, a^5 \rho_g$, the grain would have a root-mean-square angular velocity, the equipartition angular velocity, of

$$\begin{aligned} \omega_e &= \left(\frac{2R_e}{\bar{I}}\right)^{1/2} = 1.57 \times 10^{-8} \left(\frac{T}{a^5 \rho_g}\right)^{1/2} \text{ rad/sec} \\ &= 5.0 \times 10^5 \text{ rad/sec in } H\,\text{I regions} \\ &= 6.0 \times 10^6 \text{ rad/sec in } H\,\text{II regions}. \end{aligned} \tag{5}$$

The numerical values are for illustration only. For this purpose we shall use the values $a = 10^{-5}$ cm, $T = 10^2$ in H I regions and 10^4 in H II regions. We shall take n_H, the number of atoms per cubic centimeter, as 10 in H I regions and 1 in H II regions. The internal temperature of the grains will be taken to be $T_g = 10°$ K.

Briefly, these figures were selected for the following reasons. The grain is heated by stellar radiation of energy density sufficient to maintain a black body at about 3° K. The low infrared emissivity of most substances results in the particle's heating until it can radiate as much as a black body at 3° K; an estimate of 10°–25° K seems reasonable. The kinetic temperature of the gas, T, will lie in the range 5,000°–10,000° in regions where hydrogen is ionized (H II regions), as it does in emission nebulae. In neutral hydrogen regions (H I) the temperature can be much lower; Spitzer and Savedoff[9] have computed the energy balance in these regions. It seems probable that $T < 200°$ K; $T \approx 100°$ K seems to be a reasonable value to adopt at present. The density in H I regions may be taken as $n_H = 10$ from studies of interstellar lines.[10] The grain size is discussed in § 4.

3.2. *The alignment of spinning bodies.*—It is possible to align a spinning body by applied torques in several distinct ways. First, one can apply a torque that tends to produce

[9] *Ap. J.*, **111**, 593, 1950.

[10] B. Strömgren, *Ap. J.*, **108**, 242, 1948.

the desired alignment directly, as in a compass needle; this is the mechanism of Spitzer and Tukey[11, 12] and of Spitzer and Schatzman.[13] However, unless the torque is relatively large, it will be almost completely counterbalanced by the gyroscopic torque, leaving a precession as the only appreciable effect. More precisely, the work done by the applied torque when the body is rotated through a radian in the most favorable direction must be of the order of magnitude of the mean kinetic energy of rotation if alignment is to be produced in this way. When the applied torque is derivable from a potential, it follows that this potential energy must be of the order of kT. Thus, in order to get large enough torques, Spitzer and Tukey suppose the grains to be ferromagnetic and at fairly low kinetic temperatures; if the particles have a radius of 3×10^{-5} cm and if the kinetic temperature of the gas is 10° K, a field of at least 10^{-4} gauss is necessary. Evaporation of the volatile constituents of some grains is suggested as the source of pure ferromagnetic particles, and the very low kinetic temperature required may be attained in favorable regions of dense *H* I clouds. This hypothesis would naturally receive strong support if the observations should establish that polarization occurs only in neutral hydrogen regions of very low temperature. We adopt the hypothesis that, in general, the grains exist and polarization occurs in both *H* I and *H* II regions and that, even in *H* I regions, the grains are somewhat dielectric rather than pure metallic in character, having high albedo. Absorption of light in *H* II regions, providing color excess without polarization, would dilute the polarization greatly on the Spitzer and Tukey theory.

In an attempt to work with the small aligning torques characteristic of less specialized paramagnetic and diamagnetic substances and to deal with temperatures that might run as high as 10,000° K—a value appropriate for diffuse emission nebulae or *H* II regions—Davis and Greenstein[14] originally proposed that the kinetic energy of rotation could be very much less than the mean energy of the hydrogen atoms if, during the long intervals between collisions, the grains were braked almost to zero angular velocity by small torques produced by hysteresis or similar effects. It can be objected, however, that no magnetic hysteresis, in the strict meaning of the term, can occur, since either the ferromagnetic domains will be too small and isolated or the iron atoms will be so diluted by other atoms that there will be no ferromagnetism present. Although there might well be some braking action due to the magnetic analogue of dielectric loss, it would not leave the particles motionless but, rather, spinning rapidly about the magnetic lines of force. Such considerations lead to a second basic method[14] of aligning rigid bodies, a method in which small nonconservative torques are used to damp out nutations and to cause a noncircular precession that will carry the dynamic axis to the desired position. A familiar example is provided by a rapidly spinning top. By far the largest torque is produced by gravity, but this produces only a circular precession, not an alignment in which the top hangs down vertically from its pivot. The very small frictional torques at the pivot damp out the wobbles or nutations and then cause the axis to rise slowly against gravity until the axis is vertical and the top "sleeps." We shall find that, if there is a galactic magnetic field, very small torques should act on the dust grains in such a way that a significant, or indeed an almost complete, alignment might be expected, even though the kinetic energy of rotation is but little affected.

3.3. *The torque due to paramagnetic absorption.*—A consideration of the various torques that might act on a grain of dust rotating in a magnetic field whose strength is of the order of 10^{-5} gauss leads one to conclude (cf. § 8) that the torques due to eddy currents and the Rowland effect are probably completely negligible, the important torque being due to paramagnetic absorption. Paramagnetic absorption is the magnetic analogue of dielectric loss and may be thought of as due to internal friction that hinders the alignment of the atomic dipoles by an applied field. Thus, when a grain rotates with the vector

[11] *Science*, **109**, 461, 1949.

[12] *Ap. J.*, **114**, 187, 1951.

[13] *A.J.*, **54**, 195, 1949.

[14] *Phys. Rev.*, **75**, 1605, 1949; **78**, 84, 1950; *A.J.*, **55**, 71, 1950.

angular velocity $\boldsymbol{\omega}$, the magnetization, $\boldsymbol{M}$, is dragged along with the material and, as shown in Figure 1, *a*, is not parallel to the field,[15] $\boldsymbol{B}$.

The analysis of § 6.1 shows that the magnetization is, except for a term that can be ignored,

$$\boldsymbol{M} = \chi''\omega^{-1}(\boldsymbol{\omega}\times\boldsymbol{B}), \tag{6}$$

where χ'' is the imaginary part of the complex susceptibility and measures the angle through which $\boldsymbol{M}$ is dragged away from $\boldsymbol{B}$. If V is the volume of the grain, its magnetic moment is $V\boldsymbol{M}$, and the torque on this dipole in a magnetic field is

$$\boldsymbol{L} = V\boldsymbol{M}\times\boldsymbol{B} = V\chi''\omega^{-1}(\boldsymbol{\omega}\times\boldsymbol{B})\times\boldsymbol{B} \tag{7}$$

in Gaussian units. Although these expressions are derived only for grains that rotate about an axis fixed in the grain, we assume that they hold approximately in the actual

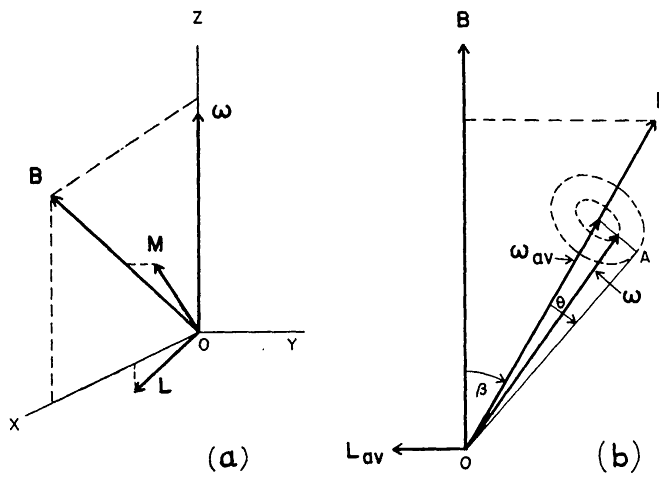

FIG. 1.—*a*, schematic representation of the magnetization, $\boldsymbol{M}$, and torque, $\boldsymbol{L}$, produced by rotation with angular velocity $\boldsymbol{\omega}$ in a magnetic field, $\boldsymbol{B}$. (Note that each dotted line locating a vector is parallel to one of the co-ordinates axes.) *b*, the nutational motion of the grain and the average torque.

nutational motion. We suppose (cf. § 4) that the dust grains consist mainly of crystals of such substances as ice, methane, and ammonia with various heavier atoms present as diffuse impurities. In particular, about one atom in one hundred should be iron or a similar atom having a large magnetic moment. Hence the grain will be paramagnetic. Both theory and experiment indicate tentatively (cf. § 6.2) that χ'' should be of the order of

$$\chi'' = 2.5\times 10^{-12}\frac{\omega}{T_g}, \tag{8}$$

where T_g, the internal temperature of the grain, may be expected to be about 10° K.

3.4. *The motion produced by paramagnetic absorption.*—As a preliminary to determining the effect of this torque on the general nutational motion of the grain, we must first describe the motion of an axially symmetric grain[16] when no torque acts. We shall treat

[15] We describe the magnetic field in terms of $\boldsymbol{B}$, the magnetic flux density, measured in gauss rather than in terms of $\boldsymbol{H}$, the magnetic field intensity, measured in oersteds. This follows the current strong trend in physics and is based on the undesirability of taking as primary a quantity defined in terms of nonexistent magnetic poles. In this connection it should be noted (*Nature*, **135**, 419, 1935) that at London in 1934 the International Union of Pure and Applied Physics adopted the oersted as the unit of $\boldsymbol{H}$. Since we do not use $\boldsymbol{H}$ for the magnetic field, we are free to use it for the angular momentum in accordance with a fairly prevalent custom in classical mechanics.

[16] L. Page, *Introduction to Theoretical Physics* (2d ed.; New York: D. Van Nostrand Co., 1933), pp. 133–137.

the grains of dust as spheroids, in order to simplify the mathematical complexities of working with grains of arbitrary shapes. This should not affect our understanding of the phenomena or the order of magnitude of our numerical estimates. Here $\boldsymbol{H}$, the angular momentum vector, is constant, remaining fixed in space, and, as indicated in Figure 1, *b*, the axis of symmetry OA moves on a circular axis about $\boldsymbol{H}$, the grain spinning about OA. The angular velocity vector, $\boldsymbol{\omega}$, is obtained as the sum of the rotations about $\boldsymbol{H}$ and OA. Therefore, $\boldsymbol{\omega}$ moves both in space and through the grain, always lying in the plane of $\boldsymbol{H}$ and OA. We denote this type of motion as "nutational motion," and by "one nutation" we mean one rotation of OA about $\boldsymbol{H}$ as determined by a nonrotating observer. We use the term "precession" to denote a very slow motion of $\boldsymbol{H}$ due to small external torques. The orientation is given by the angles β and θ, where β is the angle between $\boldsymbol{B}$ and $\boldsymbol{H}$ and θ is the angle between $\boldsymbol{H}$ and the axis of symmetry. When there is no torque, these angles remain constant.

We can now consider the cumulative effects of a very small torque that produces only an infinitesimal modification of the motion during a single nutation. Thus, letting R be the kinetic energy of rotation, we can take the usual equations of motion,

$$\frac{dH}{dt}=L=V\chi''\omega^{-1}(\boldsymbol{\omega}\times\boldsymbol{B})\times\boldsymbol{B}\,, \tag{9}$$

$$\frac{dR}{dt}=L.\boldsymbol{\omega}=-V\chi''\omega^{-1}(\boldsymbol{\omega}\times\boldsymbol{B})^{2}\,, \tag{10}$$

where the second form in each case follows from equation (7), and average over a nutation, using the unperturbed value for $\boldsymbol{\omega}$. When we average the torque as given by equation (7), $\boldsymbol{\omega}_{\mathrm{av}}$ is parallel to $\boldsymbol{H}$ and hence $\boldsymbol{L}_{\mathrm{av}}$ lies in the plane of $\boldsymbol{H}$ and $\boldsymbol{B}$ and is normal to $\boldsymbol{B}$. Therefore, we see by equation (9) that $\boldsymbol{H}$ tends to approach $\boldsymbol{B}$, its tip moving along the dotted line of Figure 1, *b*, so that the component of $\boldsymbol{H}$ parallel to $\boldsymbol{B}$ remains constant and the component normal to $\boldsymbol{B}$ vanishes. Thus β decreases. To see that, for prolate spheroidal grains, θ increases, we note that the right-hand side of equation (10) is negative; hence R decreases, except when $\boldsymbol{\omega}$ is parallel to $\boldsymbol{B}$. Therefore, if such a torque acted long enough, it would make $\boldsymbol{H}$ parallel to $\boldsymbol{B}$ and would make R as small as possible, subject to the condition that the component of $\boldsymbol{H}$ along $\boldsymbol{B}$ remain fixed. To achieve this requires that the grain rotate about its shortest axis or, more precisely, the shortest principal axis of the ellipsoid of inertia. (This may be seen from the Poinsot construction, since there $[2R]^{1/2}/H$ is the distance from the center of the inertia ellipsoid to the invariable plane. See also § 7.1.) Thus the state toward which the torque urges the grains is one in which they spin about $\boldsymbol{B}$ with their long axes normal to $\boldsymbol{B}$, with $\boldsymbol{\omega}$ parallel to $\boldsymbol{H}$ and $\boldsymbol{B}$ so that there is no further decrease in R. The angle β would be zero and θ would be $\pi/2$ for prolate grains. If we start with the randomly oriented equipartition distribution, then the exact calculations of §§ 6 and 7 show that for reasonably eccentric grains the average rate at which the angles approach these end values is of the order of $0.1\chi''B^2/\omega_e a^2$, which is about 10^{-13} rad/sec if $B = 10^{-5}$ gauss and $a = 10^{-5}$ cm.

3.5. *The distribution parameter F.*—This tendency toward orientation due to paramagnetic relaxation is countered by the tendency toward the randomly oriented equipartition distribution due to the collisions of the atoms with the dust grains. The amount by which any particular distribution varies from the equipartition distribution can be measured by a quantity F whose proper definition is given in § 5.2 but which here we visualize in simplified terms. We picture the randomly oriented equipartition distribution as one in which one-third of the grains have their axes of symmetry in each of three mutually perpendicular directions, the directions chosen being the direction OZ of the magnetic field $\boldsymbol{B}$, a line OY normal to $\boldsymbol{B}$ in the plane containing $\boldsymbol{B}$ and the light-beam, and a line OX normal to this plane (cf. Fig. 4). In a nonisotropic distribution we picture the fraction F of the grains as having been rotated from OZ to OX and OY; thus $\frac{1}{3} - F$ of the grains have their axes of symmetry parallel to $\boldsymbol{B}$ and $\frac{1}{3} + \frac{1}{2}F$ have theirs in each of the perpendicular directions. For example, if the torque due to paramagnetic relaxation has oriented all grains with their axis of symmetry normal to $\boldsymbol{B}$, then F is $\frac{1}{3}$. Both the true definition of F and the simple picture lead to the same expression for the polarization in the case of grains that are small compared to the wave length; hence in this case no error is introduced by the use of the simple description.

3.6. *The kinetic theory of the grains.*—An adequate treatment of the statistical mechanics of a spheroidal dust grain bombarded by atoms and acted upon by a weak torque of a very general character, one not derivable from a potential, will not be attempted here. Instead, we shall give an elementary, inexact treatment from which only estimates of order of magnitude can be hoped for. We start by considering the rate at which an arbitrary distribution would tend toward the equipartition distribution if nothing but collisions affected the motion of the grains. Proceeding by analogy with the equations governing other more completely analyzed systems,[17] we write for the rate at which F is changed

$$\left.\frac{dF}{d\tau}\right|_{\text{coll}} = -\frac{F}{\tau}, \tag{11}$$

where τ is the relaxation time. This equation is certainly valid if we regard it as defining τ as a function of the distribution; our approximation lies in using under all circumstances that same crude estimate of τ given below. If we now assume that, in addition to collisions, certain weak torques act on the grain, we compute in § 7 the rate at which these torques, acting alone, would affect the distribution of orientations and hence the value of F. This rate of change, which, of course, depends on the distribution considered, can be denoted by

$$\left.\frac{dF}{dt}\right|_{\text{torque}}$$

and can be calculated exactly for any specified distribution from the expressions for the average rates of change of θ and β given in § 7.1. If both the torques and the collisions act, the system will approach a steady state, in which the sum of the two rates of change of F must be zero. Hence

$$F = \tau \left.\frac{dF}{dt}\right|_{\text{torque}}. \tag{12}$$

In § 7.2 we calculate F approximately from this equation by using on the right-hand side the equipartition distribution, which is known, rather than the correct, but unknown, steady-state distribution. This approximation should be satisfactory when the torques are so small that the two distributions do not differ greatly; it becomes completely invalid as F approaches $\frac{1}{3}$ or $-\frac{2}{3}$, since then the right-hand side is zero. Equation (12) shows that τ can be pictured as the time available for the applied torques to alter the distribution before collisions stop further drift away from the equipartition distribution.

We must now estimate τ. Because our simple approach could be in error by one or two powers of 10, we shall ignore some numerical factors of the order of unity. Letting m_d denote the mass of a dust grain, its average moment of inertia is $\bar{I} = 0.4a^2 m_d$, and its equipartition angular momentum is, by equation (5),

$$H = \bar{I}\omega_e = (0.8\, R_e a^2 m_d)^{1/2}\,. \tag{13}$$

Since $2a/3$ is the average radius of a collision and $v_H = (2R_e/m_H)^{1/2}$ is the root-mean-

[17] Cf. J. H. Jeans, *The Dynamical Theory of Gases* (1st ed.; Cambridge: At the University Press, 1904), p. 294.

square velocity of the hydrogen atoms (m_H is their mass), the average angular momentum imparted to a grain by a hydrogen atom that sticks to the grain is

$$\Delta H = \tfrac{2}{3} a m_H v_H = \left(\frac{m_H}{m_d}\right)^{1/2} (2R_e \tfrac{4}{9} a^2 m_d)^{1/2} \approx \left(\frac{m_H}{m_d}\right)^{1/2} H . \tag{14}$$

During the collision, which initially changes the angular velocity but not the orientation of the grain, the vector $\boldsymbol{H}$, shown in Figure 1, *b*, swings through an angle of the order of $(m_H/m_d)^{1/2}$ radians while $\boldsymbol{B}$ and OA remain fixed. Thus the average change of β and θ in a collision is of that order. Since the collisions are random, the root-mean-square change in angle produced by n collisions is $(nm_H/m_d)^{1/2}$, and the number of collisions required to produce a change of one radian is $n = m_d/m_H$. We shall take τ to be the average time required for this number of collisions. Now, for grains containing many molecules, the average time between the impacts of the hydrogen on a particular grain is

$$t_{H,d} = \frac{1}{\pi a^2 v_H n_H} = \frac{2.02 \times 10^{-5}}{a^2 n_H T^{1/2}} \text{ sec}, \tag{15}$$

$$= 2.02 \times 10^{3} \text{ sec in both } H\,\text{I and } H\,\text{II} .$$

Thus we shall use, as the best available estimate of the relaxation time,

$$\tau = \frac{t_{H,d}\, m_d}{m_H} = \frac{5.05 \times 10^{19} \rho_g a}{n_H T^{1/2}}, \tag{16}$$

$$= 5.05 \times 10^{12} \text{ sec in both } H\,\text{I and } H\,\text{II} .$$

Next consider a criterion that will indicate whether a suggested source of torque is large enough to be worth considering in detail. If a torque is to change the orientation significantly in the time τ, it must change the angular momentum by an amount that is not very small compared to the total angular momentum and hence must not be small compared to

$$L_R = \frac{H}{\tau} = \frac{5.19 \times 10^{-28} a^{3/2} n_H T}{\rho_g^{1/2}} . \tag{17}$$

Another possibility is that the retarding torque could be great enough to reduce the angular velocity imparted to a nonrotating grain in a single collision to zero in the time $t_{H,d}$ between collisions. Then any directional effect whatever would align the grains. But this torque is greater than L_R by a factor of about $(m_d/m_H)^{1/2}$; hence we do not consider it further.[18]

3.7. *The approximate value of F.*—On the basis of this argument we conclude (cf. §§ 7.2 and 7.3) that for prolate spheroidal grains of reasonable eccentricity and for small deviations from the equipartition distribution

$$F = F_l \quad \text{if} \quad |F_l| \ll \tfrac{1}{3} , \tag{18}$$

[18] We are indebted to Dr. Spitzer for pointing out a correction to τ. He has shown (*Ap. J.*, **93**, 369, 1941) that in an H II region more electrons than protons strike an uncharged grain because of the higher velocity of the former. Thus an electrostatic potential is built up that attracts protons and makes the collision probabilities equal. The effective cross-section of the grain for protons and the average angular momentum imparted per collision are increased by a factor of about 3.5. Therefore, we should decrease $t_{H,d}$ and τ and increase L_R in H II regions, with a consequent increase by a factor of $(3.5)^{3/4} = 2.6$ in the magnetic fields required by our mechanism.

where we have introduced, as a parameter measuring the relative importance of collisions and paramagnetic relaxation,

$$F_l = \tau \left.\frac{dF}{dt}\right|_{\text{torque}}^{\text{equipartition}} = \frac{1.6\times 10^{10}\chi'' a^{3/2}\rho_g^{1/2}B^2}{Tn_H} = \frac{6.3\times 10^6 B^2}{aT^{1/2}n_H T_g}\,. \tag{19}$$

$$F_l = 6.3\times 10^8 B^2 \text{ in both } H\,\text{I and } H\,\text{II regions}\,.$$

When F_l is large, the deviation from the equipartition distribution is large and F approaches $\frac{1}{3}$, as discussed above; thus

$$F = \tfrac{1}{3} \qquad \text{if} \qquad |F_l| \gg \tfrac{1}{3}\,. \tag{20}$$

For intermediate values of F_l we should get a satisfactory rough approximation by using equation (18) when $F_l < \frac{1}{3}$ and equation (20) when $F_l > \frac{1}{3}$, although any reasonable interpolation formula should be an improvement.

3.8. *The approximate polarization expected.*—We now turn to a consideration of the effect on a beam of light of dust grains oriented in this way. We wish to compare the weakening due to absorption and scattering of a beam whose plane of vibration contains $\boldsymbol{B}$ with that of a beam whose plane of vibration is turned through 90° and with that of unpolarized light; we are not interested in making absolute calculations of any of these quantities. It has been established that nearly spherical dust grains, mainly dielectric in composition, can produce the observed interstellar reddening and absorption with very moderate space density. It is generally assumed that a wide range of grain sizes exists, with the maximum contribution to the extinction produced by sizes slightly less than a wave length. Exact theoretical results and extensive computations exist for dielectric and metallic spherical grains. At present, exact computations for spheroidal grains much smaller than the wave length can be based on the work of R. Gans;[19] the theory of infinite cylinders of any size has been developed by P. Wellman[20] and H. C. van de Hulst.[21] We know of no usable theoretical or experimental data on the effects of spheroidal grains whose diameter is comparable to the wave length. Accordingly, we shall, of necessity, use the expressions for very small grains in the hope that this will give us the order of magnitude of the expected effect. We obviously cannot hope thus to predict the correct wave-length dependence of the polarization; a consideration either of the results of Wellman and of van de Hulst for large cylinders or of the fact that very large grains will produce only shadows with no polarization indicates that the ratio of polarization to total absorption deduced from small grains is likely to be too large.

In § 5.1 detailed calculations for a small prolate spheroidal grain show that it weakens a beam of light so polarized that its electric vector is parallel to the long axis of the grain more than light polarized at right angles, in the ratio σ_A/σ_T. The values obtained for this ratio are given in Table 2; they range from 1.33 for dielectric grains having a ratio of axes $x = \frac{5}{3}$, to 1.95 for dielectric grains having $x = 5.02$, and to 4.08 for metallic grains having $x = 5.02$. The effect of a small grain inclined at any specified angle to the light is easily expressed in terms of σ_A/σ_T, a simplification of the theory that does not hold for the larger grains. Hence we can compute (cf. § 5.3) the effect of a distribution of small grains whose orientation is specified by F. The most interesting result is that, if the grains are all of the same size, the ratio of the polarization to the total absorption is

$$\frac{p}{A_{pg}} = 2.07F\cos^2\nu\,\frac{(\sigma_A/\sigma_T)-1}{(\sigma_A/\sigma_T)+2}, \tag{21}$$

[19] *Ann. d. Phys.*, **37**, 881, 1912.

[20] *Zs. f. Ap.*, **14**, 195, 1937.

[21] *Ap. J.*, **112**, 1, 1950.

where $(\pi/2) - \nu$ is the angle between the magnetic field and the beam of light.

3.9. *Interpretation of results.*—According to equation (2), the left side of equation (21) must be at least 0.02 in some regions. Suppose that the grains are completely aligned, with their long axes normal to $\boldsymbol{B}$ so that $F = \frac{1}{3}$ and take $\nu = 0$, then $\sigma_A/\sigma_T = 1.09$. The values of the ratio quoted above and in Table 2 show that this condition is easily met by grains of quite moderate eccentricity. If the grains have considerable eccentricity, there is a fairly large factor of safety that could be used to allow for the fact that σ_A/σ_T is considerably less for large grains than for small. Alternatively, many of the grains might be spherical or of a size that was not easily aligned. It follows from equations (19) and (20) that F will be of the order of $\frac{1}{3}$, provided that $B \gtrsim 2.3 \times 10^{-5}$ gauss for our standard illustrative conditions or $B \gtrsim 4.0 \times 10^{-5}$ gauss for $a = 3 \times 10^{-5}$ cm, other conditions being unchanged. If the effect of charge in H II regions is considered,[18] this becomes $B \gtrsim 10^{-4}$ gauss. With lower temperatures or lower hydrogen density or smaller particles, smaller fields or a smaller value of χ'' would suffice. On the other hand, if we wish to assume that the mean σ_A/σ_T of all grains is, say, 1.33, then $F = \frac{1}{9}$ and B would be 2.3×10^{-5} gauss if $a = 3 \times 10^{-5}$ cm. Thus the proposed mechanism seems adequate to explain quantitatively the observed polarization with fields somewhere in the range from 10^{-4} to 10^{-5} gauss.

Throughout the above discussion we have, for simplicity, assumed that the ellipsoids are prolate. If they are oblate, neither the plane of polarization nor the order of magnitude of the effect is changed. Equation (21) holds, but both $(\sigma_A/\sigma_T) - 1$ and F are negative, the sign of equation (19) being changed. The limiting value for complete alignment is $F = -\frac{2}{3}$, so that the limits in equations (18) and (20) require a corresponding change.

Without undertaking an extensive interpretation of the observations in terms of this theory, we may draw a few quite speculative inferences as to the probable nature of the magnetic field. The first is that the observed uniformity in the planes of vibration mentioned in § 1 (ii) implies a corresponding uniformity, on the basis of this theory, in the direction of the magnetic field averaged in space and time. The average is to be taken over a time of the order of τ, which is about 1.7×10^5 years, since the orientation of the grains will not follow shorter-period fluctuations in the field. The light observed often travels 1000 psc and must traverse several independent complexes of dust and gas. The detailed structure of the field cannot deviate from point to point too wildly from the average, or the ratio of polarization to reddening would be smaller than that observed. Thus the very irregular magnetic field proposed by E. Fermi[22] in his theory of the origin of cosmic rays seems less plausible than a nearly uniform field. Perhaps the observed motion of gas clouds is to be accounted for by magneto-hydrodynamic waves,[23] in which the lines of force do not deviate far from their original direction, the gas moving with the lines of force like beads on a taut vibrating string rather than at random.

The second point is that, in our theory, $\boldsymbol{B}$ is normal to the long axis of the grains and to the conventional plane of polarization (not the plane of vibration) of the light and cannot lie too close to the line of sight. Thus one can get a suggestion of the path of the lines of force from Hall and Mikesell's[6] Figures 4–7, showing the planes of polarization, and from the fact that the lines must be continuous. The suggestion seems to be that for galactic longitudes 80°–120° or perhaps even to 170° the lines of force lie in the plane of the galaxy, possibly along a spiral arm of dust and gas. This inference is consistent with the remark of Hall and Mikesell[6] that in Cygnus, where they observe a relatively small and randomly oriented polarization, one may be looking along a spiral arm. For in this case the component of $\boldsymbol{B}$ normal to the line of sight would be small and randomly oriented. A theory of the nature of the interstellar magnetic field proposed by Schlüter

[22] *Phys. Rev.*, 75, 1169, 1949.

[23] H. Alfvén, *Cosmical Electrodynamics* (Oxford: Clarendon Press, 1950), pp. 76–88.

and Biermann[24] indicates that the magnetic field will affect the nature of turbulent motions. The smaller eddies will be rapidly damped out. The larger eddies will presumably still show the effect of the anisotropic driving force, i.e., galactic rotation; hence the lines of force would be in the galactic plane. A field in which the lines of $\boldsymbol{B}$ make random small-scale whirls in the galactic plane would produce a polarization of the type observed if our theory is correct. The magnitude of the polarization for a given average field strength would be half that produced by a uniform field normal to the line of sight.

The tentative value of 10^{-4} to 10^{-5} gauss for B in regions of maximum polarization suggested by our theory does not seem unreasonable if we consider the energies available. The corresponding energy densities are $E_B = B^2/8\pi \approx 4 \times 10^{-10}$ to 4×10^{-12} erg/cm³ or 4×10^{-31} to 4×10^{-33} gm/cm³. This is so much less than the mass density of interstellar matter, $\rho \approx 10^{-24}$ gm/cm³, that it produces no gravitational effects. The larger value of E_B is of the order of the density of kinetic energy due to galactic rotation, $E_R = \frac{1}{2}\rho v_{\text{rot}}^2 \approx 8 \times 10^{-10}$ erg/cm³ if $v_{\text{rot}} \approx 3 \times 10^7$ cm/sec; thus the mechanism that set the galaxy rotating would have enough energy to produce the magnetic field. If there is any galactic magnetic field at all, it follows from the arguments of Alfvén, as discussed by Fermi,[22] that it will be increased by the peculiar motion of the interstellar gas clouds until $E_B \approx E_p = \frac{1}{2}\rho \bar{v}_p^2 \approx 3 \times 10^{-12}$ erg/cm³ if $\bar{v}_p \approx 2 \times 10^6$ cm/sec. The scale of irregularity in $\boldsymbol{B}$ will be that of the peculiar motions of the gas clouds. If the magnetic field is to have the uniformity we ascribe to it on the basis of the observations described in § 1 (ii), then it must dominate the peculiar motions, requiring that $E_B \gg E_p$. This supports a value of B of the order of 10^{-4} gauss for the regions where polarization is observed.

4. COMPOSITION, SIZE, AND PHYSICAL PROPERTIES OF THE DUST GRAINS

Special theories exist for the growth of interstellar grains out of the interstellar gas. It is generally assumed that the abundance of the elements in space is the same as on the surface of the stars and that the particles grow by successive random captures of the interstellar atoms on a small nucleus. Hydrogen and helium are overwhelmingly abundant, but, even at the low temperatures of the solid grains, they would evaporate rather than remain in the solid, unless chemically bound. From Harrison Brown's compilation of astrophysical and terrestrial abundance data[25] we adopt in Table 1 the abundances of

TABLE 1

ADOPTED ABUNDANCES OF THE ELEMENTS IN SOLID GRAINS

Element	Atomic Weight	Stellar Abundance by Number	Weight Bound in Grain	Percentage by Weight	Percentage by Number of Atoms
H	1	1600	6.0	14	72
C	12	0.36	4.3	10	4
N	14	0.73	10.3	23	9
O	16	1.00	16.0	36	12
Mg	24	0.040	0.96	2	0.5
Al	27	0.004	0.11	0.3	0.05
Si	28	0.046	1.27	3	0.6
S	32	0.016	0.51	1	0.2
Fe	56	0.083	4.7	11	1.0
Ni	58	0.006	0.34	0.8	0.07

[24] *Zs. f. Naturforsch.*, **5a**, 237, 1950.

[25] *Rev. Mod. Phys.*, **21**, 625, 1949.

interest here. In the fourth column we estimate the contribution to the weight of the grain of elements bound chemically. Except for hydrogen, this is obtained from the abundance times the weight; for hydrogen it is given by the number that could be chemically bound, mainly as H_2O, NH_3, and CH_4, etc. Only 0.4 per cent of the original hydrogen is thus permanently bound. If H_2 is formed catalytically in the particle, a small part may be absorbed and retained in the solid lattice and on the surface. This amount, however, would be less than the number of other molecules in the solid, since H, H_2, and He tend to evaporate. In the next to the last column is given the final abundance by weight. We find that most of the mass is in the form of compounds based on C, N, O, and H (H_2O, NH_3, CH_4) and that a wide variety of compounds of lower abundance could be formed involving Fe, Mg, Al, Si, and S. We shall not consider grains in which the Fe is so concentrated that the grains are ferromagnetic, a case considered by Spitzer and Tukey.[12] Whether more complex compounds will, in fact, occur depends on the evolution of the particle. If the solid is bombarded with heavy elements at thermal energies in an H II region, collisions involving several electron-volts occur. Such heavy atoms entering the solid may have sufficient energy of activation to produce chemical combinations. Further, the continuous local heating by proton and perhaps by cosmic-ray bombardment favors chemical combination and migration of ions to begin the process of crystal formation. It is quite possible that moderately complex molecules are formed and that the grains show the irregular, but generally nonspherical, character of terrestrial solids. The major difference from rocks, of course, is the preponderance of the frozen solid state of ordinarily gaseous compounds of hydrogen. If only simple compounds are formed, approximately 4 per cent would contain Fe. Whether or not the shape of grains formed by random collision would be spherical cannot be stated in the presence of these complex mixtures. In the growth of pure substances—say, solidified gases—it is very likely that semicrystalline structures would occur and that the final grains would be appreciably elongated. Another serious question would be whether the grains are compact solids of moderate density and normal indices of refraction or whether they would grow as open, spongelike flakes (similar to snow or some industrial metallic smoke or dust) which would have very low density and low index of refraction. Assuming the grain to be compact in order to provide sufficient mechanical strength to withstand the rapid rotation, we find from Table 1 that it is a dielectric of low index of refraction—say, 1.2 to 1.4—of low density, ρ_g, which we take as unity, and with a small imaginary component in the index of refraction, perhaps of the order of 0.05, due to iron and its compounds.

Spherical dust grains of radius less than 10^{-2} cm have sufficiently large mass-absorption coefficients to account for the interstellar absorption; a size somewhat less than a wave length of light will also produce the correct space reddening. The more recent theoretical work[26, 27, 28] has emphasized the importance of a wide frequency distribution of sizes, with numbers increasing rapidly as the size decreases. The proper choice of the typical grain size depends on the nature of the frequency distribution and may differ in the theory of interstellar polarization from the choice in the theory of reddening. H. C. van de Hulst[28] has developed a theory of the frequency distribution and applied it to the observed interstellar reddening. The number of dielectric grains ($m = \frac{4}{3}$) varies as $f(a/a_1)da/a_1$; his Table 10 and solution 15 of his Table 14 contain the details. The value of $f(a/a_1) = 0.064$ with $a = a_1 = 4 \times 10^{-5}$ cm, the value of $f(a/a_1) \approx 1$ for $a < 2 \times 10^{-5}$ cm. The mean radius would be very small, but the size most effective in producing interstellar extinction is near 4×10^{-5} cm. The small spheroid theory of Gans does not hold if $\alpha = 2\pi a/\lambda > 1$. We show below that the extinction by the grains may be metallic in character if $\alpha < 1$. If this is confirmed, the required modification of the frequency distribution will probably give a most effective size different from a_1 and between 10^{-5} and

[26] J. L. Greenstein, *Harvard Circ.*, No. 422, 1938.

[27] J. L. Greenstein, *Ap. J.*, **104**, 403, 1946.

[28] H. C. van de Hulst, *Rech. Astr. Obs. Utrecht*, Vol. **11**, Part II, 1949.

3×10^{-5} cm. However, we shall see that there are many serious problems in the transmission coefficient of partially oriented spheroids if $\alpha > 1$; since we shall not now try to explain the wave-length dependence of the polarization, it will be simplest if we limit ourselves to the orientation problem for typical particles with $a = 10^{-5}$ cm, and use the order-of-magnitude polarization predicted by the electric-dipole scattering of pure dielectric spheroids. Van de Hulst[21] has found a possible fit to both the wave-length dependence of polarization and the absorption for infinite cylinders of radius 4×10^{-5} cm.

One comment may be made on the basis of Table 1, which predicts a small but finite complex index of refraction, due to the 4 per cent abundance of iron by number of molecules. The complex index of refraction can then be roughly estimated as

$$m = n - ik = 1.4 - 0.05i\ . \tag{22}$$

Van de Hulst[28] has shown that the phase shift of a ray through the center of a sphere is

$$\rho^* = 2\alpha(n-1) - 2\alpha(n-1)\, i \tan\beta\ , \tag{23}$$

$$\tan\beta = \frac{k}{n-1}\ . \tag{24}$$

The complex part of ρ^* represents a decay of the wave and permits an evaluation of the true absorption within the particle. For our case β is 7°; if one examines his Figure 5, it is clear that the grain has a small but finite absorption, as well as scattering; more importantly, the high maximum found in the extinction coefficient near $2\alpha(m - 1) = 4$ and the minimum near $2\alpha(m - 1) = 8$ both nearly disappear because of the disappearance of resonance effects. Furthermore, if $\alpha < 1$, it is possible that the metallic absorption by the particle will be important. Greenstein[26] showed that the ratio of the absorption K to the scattering S by a grain, if $k < n$ and $\alpha < 1$, is

$$\frac{K}{S} = \frac{9}{\alpha^3}\, \frac{nk}{(n^2-1)^2}\ . \tag{25}$$

If we adopt the 4 per cent iron abundance, with the refractive index $m = 1.4 - 0.05i$, we find that K is of the order of, or larger than, S when α is less than 0.88. Our approximation in equation (25) is not valid for so large an α, but it is clear that metallic absorption becomes important near this point; for visible radiation, this size corresponds to 8×10^{-6} cm. We must then expect particles at the maximum of the frequency function to be effectively metallic absorbers, with an extinction coefficient nearly proportional to $1/\lambda$, while those of larger size act as dielectric scattering agents; since the latter have $\alpha > 1$, we expect a wave-length dependence also near $1/\lambda$ or slower. Thus, rather than the $1/\lambda^4$ dielectric scattering of very small dielectric spheres, the present theory essentially predicts the $1/\lambda$ law over a wide range of wave length. These complexities will have to be taken into account when an attempt is made to explain the small wave-length dependence of the polarization.

5. THE POLARIZATION PRODUCED BY ORIENTED DUST GRAINS

The discussion of the previous sections indicates that we should consider in detail the polarization that would be produced by nonspherical, partially oriented grains of dust. Rather than consider mixtures of grains of arbitrary and varying shapes and chemical composition, we shall consider a model in which all grains are spheroids of the same eccentricity and composition. The resulting expressions for the polarization would be needed in almost any theory in which the polarization is produced by oriented dust grains.

5.1. *The scattering and absorption of a single spheroidal particle.*—R. Gans[19] has treated the absorption and scattering of light by small dielectric spheroids. His procedure is to compute the induced dipole moment on the assumption that the electric field is uniform over the particle and then to compute the absorption and scattering from this. Thus it is essential that the method be applied only to particles small compared to the wave length and of a composition such that the magnetic component of the electromagnetic wave is unimportant. Define σ_A to be the extinction cross-section in square centimeters of a single spheroid whose axis of symmetry is parallel to the electric vector in the light; similarly, define σ_T to be the extinction cross-section for a spheroid whose transverse axis is parallel to the electric vector, i.e., whose axis of symmetry is normal to the electric vector. The procedure used to compute σ_A and σ_T may be outlined as follows: Let $2a_A$ be the length of the axis of symmetry, $2a_T$ that of the transverse axis, and $x = a_A/a_T$ the ratio of the axes. Then the so-called "depolarization factors" are, in terms of x rather than the more conventional eccentricity e, used by Gans:

$$P = \frac{4\pi}{x^2 - 1}\left[\frac{x}{(x^2-1)^{1/2}} \cosh^{-1} x - 1\right] \text{for prolate spheroids}, \tag{26}$$

$$P = \frac{4\pi}{1 - x^2}\left[1 - \frac{x}{(1-x^2)^{1/2}} \cos^{-1} x\right] \text{for oblate spheroids}, \tag{27}$$

$$P' = 2\pi - \tfrac{1}{2}P . \tag{28}$$

Let m be the complex index of refraction and V the volume of the particle. Two cases are to be considered. In the first, the dielectric case, $m = n$ is real, and the process is one of pure scattering. Here

$$\sigma_A = \frac{128\pi^5 a^6}{3\lambda^4}\left[\frac{m^2-1}{3 + (m^2-1)(3P/4\pi)}\right]^2 = \frac{128\pi^5 a^6}{3\lambda^4} G_A , \tag{29}$$

where G_A has been introduced as a convenient abbreviation defined by equation (29). In the second, the metallic case, m is complex. If its imaginary part is large, the process is one of pure absorption with negligible scattering, and

$$\sigma_A = \frac{8\pi^2 a^3}{\lambda} Im \left[\frac{1 - m^2}{3 + (m^2-1)(3P/4\pi)}\right] = \frac{8\pi^2 a^3}{\lambda} H_A . \tag{30}$$

The quantities σ_T, G_T, and H_T are obtained by replacing P by P'. Equation (29) shows an essential similarity to Rayleigh scattering with the cross-section proportional to a^6/λ^4; equation (30) shows metallic absorption with a cross-section proportional to a^3/λ, if we neglect the variation of n with λ. In either case polarization exists if $P \neq P'$. Table 2 contains the detailed results of some computations, with final results for oblate spheroids. The ratio σ_A/σ_T is given, together with γ the ratio of the moments of inertia, Γ, and $100\Gamma\rho_g Va^2/I$, quantities required below in the theory of the orientation of the spheroids. These dynamic parameters are computed by means of equations (91)–(94). Note that the index of refraction adopted for the metallic case involves a large value of the absorptivity, k. A trial computation for $m = 2^{1/2}(1 - 0.5i)$ gave about half the polarization. Table 2 shows that completely aligned small particles give appreciably different extinction coefficients in two planes of polarization when the deviation from a sphere is quite small. A ratio of axes of 1.091 gives about 2.7 per cent polarization, p, for the dielectric and 7.2 per cent polarization for the metallic particle if the absorption is 1 mag. This is further discussed in § 3.9.

The variation of σ_A and σ_T with wave length should properly include the variation of the index of refraction with wave length. A trial computation made with $x = 1.25$ showed that a large change of refractive index, $\delta n = 0.01$, changed $(\sigma_A/\sigma_T) - 1$ by 3 per cent. This is small compared to the corresponding change in the $1/\lambda^4$ or $1/\lambda$ factor in each σ. Since, in the dipole approximation for small grains, $(\sigma_A/\sigma_T) - 1$ measures p/A and p/E_1, where p is the polarization, A the absorption, and E_1 the color excess, this might lead one to expect that these ratios would be essentially independent of the wave length for any reasonable variation in the index of refraction. Since we know that A varies roughly as $1/\lambda$, an observable change in p could be expected from blue to infrared. This is not in agreement with Hiltner's observations,[3] thus proving that the dipole-scattering approximation is insufficient. Spheroids with $2\pi a/\lambda$ near unity would have a smaller

TABLE 2

EXTINCTION CROSS-SECTIONS AND INERTIAL PROPERTIES OF SMALL GRAINS

	Prolate Spheroids					
e	0	0.4	0.8	0.9	0.95	0.98
$a_A/a_T = x$	1	1.091	1.667	2.294	3.203	5.025
G_A for $n=2^{1/2}$	0.06250	0.06487	0.07590	0.08416	0.09175	0.09973
G_T for $n=2^{1/2}$	0.06250	0.06140	0.05707	0.05471	0.05285	0.05126
σ_A/σ_T for $n=2^{1/2}$	1.000	1.058	1.328	1.538	1.738	1.945
H_A for $m=2^{1/2}(1-i)$	0.6000	0.6612	1.0105	1.2352	1.3739	1.4163
H_T for $m=2^{1/2}(1-i)$	0.6000	0.5714	0.4640	0.4135	0.3764	0.3465
σ_A/σ_T for $m=2^{1/2}(1-i)$	1.000	1.158	2.18	2.99	3.65	4.08
γ	1	1.095	1.889	3.132	5.628	13.126
$100\,\Gamma$	0	0.303	1.816	1.888	1.800	1.358
$100\,\Gamma\rho_g\,Va^2/I$	0	0.804	6.38	8.16	9.76	9.97
	Oblate Spheroids					
e	0	0.4	0.8	0.9	0.95	0.98
x	1	0.916	0.600	0.436	0.312	0.199
σ_A/σ_T for $n=2^{1/2}$	1	0.952	0.732	0.605	0.505	0.398
σ_A/σ_T for $m=2^{1/2}(1-i)$	1	0.870	0.422	0.258	0.169	0.105
γ	1	0.920	0.680	0.595	0.549	0.520
$-100\,\Gamma$	0	0.311	1.71	2.55	3.09	3.48
$-100\,\Gamma\rho_g\,Va^2/I$	0	0.732	3.02	3.66	3.56	2.96

polarization but would also have a smaller change of polarization with wave length, as van de Hulst has shown for semi-infinite cylinders.[21] It is difficult to put these arguments on a quantitative basis at present, both because of the lack of theoretical computations for spheroids near a wave length in size and because then it is much more difficult to treat obliquely inclined grains.

Consider, now, a spheroid whose axis of symmetry makes the angle α (not to be confused with $2\pi a/\lambda$) with the electric vector of the radiation; let $\sigma(\alpha)$ be its extinction coefficient. In the approximation in which $\sigma(\alpha)$ is computed from the dipole moment induced by a quasi-static field that is uniform over the grain, one can resolve the electric field into components along and normal to the axis of symmetry and easily find that

$$\sigma(\alpha) = \sigma_A \cos^2 \alpha + \sigma_T \sin^2 \alpha = \sigma_T + (\sigma_A - \sigma_T)\cos^2 \alpha \,. \tag{31}$$

If $a \gtrsim \lambda$, this derivation breaks down; indeed, even the definition of $\sigma(\alpha)$ breaks down, since the angle between the direction of propagation and the axis of symmetry must be considered. In this case a qualitative understanding of the situation can be obtained from a consideration of the three extinction coefficients, σ_E, σ_B, and σ_S, where the subscript indicates that the axis of symmetry is parallel, respectively, to $\boldsymbol{E}$, the electric vector of the wave; to $\boldsymbol{B}$, the magnetic vector; and to $\boldsymbol{S}$, the propagation vector. The complexity of the situation is apparent when one realizes that, if $a \ll \lambda$, $\sigma_E = \sigma_A$, $\sigma_B = \sigma_S = \sigma_T$, while, if $a \gg \lambda$, σ is measured by the area of the shadow and diffraction pattern, and $\sigma_E = \sigma_B \neq \sigma_S$. A schematic representation of the situation for prolate spheroids is shown in Figure 2, with the assumption that the index of refraction has a small complex part. We shall see below that, if all the grains are aligned in one of the three directions, the total absorption is proportional to $\frac{1}{3}(\sigma_E + \sigma_B + \sigma_S)$ but that the polarization is proportional to $\sigma_E - \sigma_B$. Thus we see from the figure that A and p have the same dependence on λ for $a \ll \lambda$, that p is zero and A is independent of λ for $a \gg \lambda$, and that there is a range near $\lambda = 2\pi a$ in which p is independent of λ because the σ_E and σ_B curves are parallel, while A varies with λ because all three curves are rising.

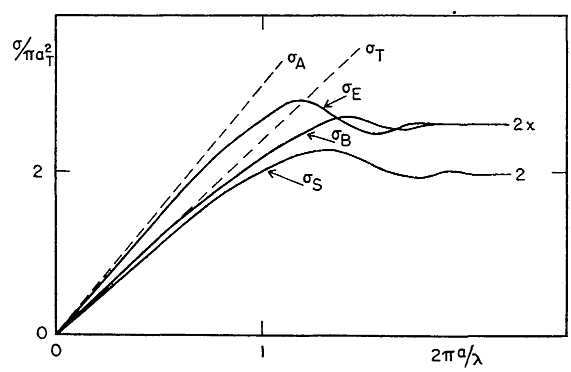

FIG. 2.—A schematic representation of the extinction cross-sections of a prolate spheroid; each cross-section, σ, is expressed in units of πa_T^2; $x = a_A/a_T$.

Returning to the case in which $\lambda \gg a$, we next ask: What is the average effect of a spinning grain upon which no torques act? Then $\boldsymbol{H}$, the angular momentum of the grain, will remain fixed in space and the axis of symmetry, OA, will nutate around it, making the constant angle θ with $\boldsymbol{H}$ and moving at a constant rate around $\boldsymbol{H}$ in the circular cone indicated by the dotted circle in Figure 3. We wish to average over all positions of OA on this cone. Before we can do this, we must make more definite the situation contemplated. We suppose that $\boldsymbol{B}$ is a uniform field that lays down a unique direction in space and that exerts small torques on the grains. We shall later take it to be a magnetic field, but for the present its only purpose is to specify a unique direction. We consider a beam of light for which $\boldsymbol{S}$, the direction of propagation, makes the angle $(\pi/2) + \nu$ with $\boldsymbol{B}$. To locate $\boldsymbol{H}$, the angular momentum, we give β, its polar angle, and ϕ, its azimuthal angle measured from the plane of $\boldsymbol{B}$ and $\boldsymbol{S}$. To locate OA, the axis of symmetry, we use the polar and azimuthal angles θ and ψ, respectively, with respect to $\boldsymbol{H}$ as pole. Then, if no torques act on an axially symmetric grain, ϕ, β, and θ remain constant while ψ increases at a uniform rate. By "one nutation" we shall mean one revolution of A about $\boldsymbol{H}$, i.e., a motion in which ψ increases by 2π. In order to get the average effect of a grain, we average over a nutation. Also, by the symmetry of the situation, even if the presence of $\boldsymbol{B}$ alters the distribution of orientations of OA over β and θ and produces a precession in which ϕ varies slowly, all values of ϕ are equally likely. Hence we shall average over ϕ and ψ.

We see from equation (31) that the quantity to be averaged is the square of the cosine of the angle between the axis of symmetry and $\boldsymbol{E}$, the electric vector in the plane-polarized light. Hence we must resolve the light into two plane-polarized components, one of intensity I_π, with its $\boldsymbol{E} = \boldsymbol{E}_\pi$ parallel to the plane of $\boldsymbol{B}$ and $\boldsymbol{S}$; the other of intensity I_σ, with its $\boldsymbol{E} = \boldsymbol{E}_\sigma$ perpendicular to $\boldsymbol{B}$, as shown in Figure 4. Let α_π denote the angle between $\boldsymbol{E}_\pi$ and OA, α_σ the angle between $\boldsymbol{E}_\sigma$ and OA. Hence, by equation (31), the average energy removed from I_π by the grain is

$$I_\pi [\sigma_T + (\sigma_A - \sigma_T) \overline{\cos^2 \alpha_\pi}] = I_\pi \sigma_\pi (\beta, \theta, \nu, a), \tag{32}$$

where the bar indicates the average over ϕ and ψ and the equation defines σ_π. Likewise the average energy removed from I_σ is

$$I_\sigma [\sigma_T + (\sigma_A - \sigma_T) \overline{\cos^2 \alpha_\sigma}] = I_\sigma \sigma_\sigma (\beta, \theta, \nu, a), \tag{33}$$

where σ_σ is a coefficient giving the mean effect of a single grain on a light-wave whose plane of vibration is normal to $\boldsymbol{B}$.

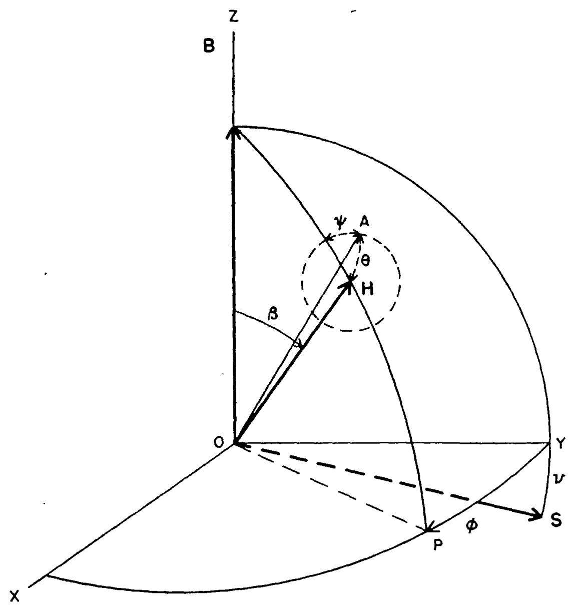

FIG. 3.—The angles defining the orientation

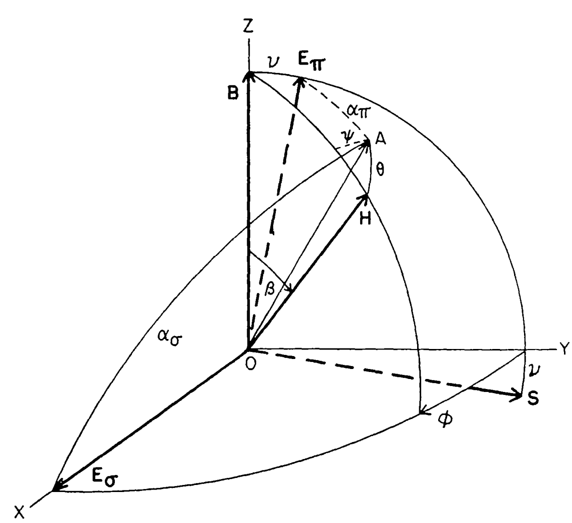

FIG. 4.—The orientation of the electric vector of the radiation for the two planes of polarization

By a straightforward application of analytic geometry and spherical trigonometry, the averages in equations (32) and (33) can be evaluated, giving (cf. Appendix 1)

$$\sigma_\sigma = \sigma_T + (\sigma_A - \sigma_T)\tfrac{1}{2}(1 - \cos^2\beta\cos^2\theta - \tfrac{1}{2}\sin^2\beta\sin^2\theta)\,, \tag{34}$$

$$\sigma_\pi = [\sigma_T + (\sigma_A - \sigma_T)(\cos^2\beta\cos^2\theta + \tfrac{1}{2}\sin^2\beta\sin^2\theta)]\cos^2\nu + \sigma_\sigma\sin^2\nu\,. \tag{35}$$

5.2. *The effect of many grains.*—Let y be the distance measured along the light-ray in centimeters. Let

$$f(\beta,\ \theta,\ a)\,\rho(a)\,n_d(y)\,d\beta\ d\theta\ da \tag{36}$$

be the number of grains of dust per cubic centimeter with mean radius a in the range from a to $a + da$, and so oriented that β lies between β and $\beta + d\beta$ and θ lies in the range from θ to $\theta + d\theta$. In equation (36), $n_d(y)$ represents the number of grains of all sizes and orientations per cubic centimeter; $\rho(a)da$ the fraction of these lying in the indicated size range; and $f(\beta,\ \theta,\ a)d\beta d\theta$ the fraction having a specified orientation.

If β exerts no torque on the grains, then $f(\beta, \theta, a)$ will in time take on the equipartition value, denoted by $f_e(\beta,\ \theta)$, because of the collisions between the grains and hydrogen atoms and ions. The equipartition distribution function, obtained from the Maxwell-Boltzmann distribution when momenta conjugate to our co-ordinates are used (cf. Appendix 2), is found to be

$$f_e(\beta,\ \theta) = \tfrac{1}{2}\gamma^{1/2}\sin\beta\ \sin\theta\,(\gamma\cos^2\theta + \sin^2\theta)^{-3/2}\,, \tag{37}$$

where I is the moment of inertia of the grain about its axis of symmetry and γI is its moment of inertia about a transverse axis. The quantity a is not included as an argument of f_e, since the latter is independent of a and I. If, now, $\boldsymbol{B}$ does exert a small torque on the grains, the resulting disturbance in the angular distribution can be specified by the function $f_1(\beta,\ \theta,\ a)$, where

$$f(\beta,\ \theta,\ a) = f_e(\beta,\ \theta) + f_1(\beta,\ \theta,\ a)\,. \tag{38}$$

Next we wish to calculate the extinction of a beam of light by grains distributed according to equation (38), if a single grain produces the extinction described in § 5.1. Define N by

$$dN = n_d(y)\,dy\,, \tag{39}$$

and measure distance along the beam by N rather than by y. The light-loss of a beam polarized with its plane of vibration parallel to $\boldsymbol{B}$, so that its intensity is I_π, is

$$dI_\pi = -S_\pi I_\pi dN\,, \tag{40}$$

where

$$S_\pi = \int_0^\infty \int_0^{\pi/2} \int_0^\pi \sigma_\pi(\beta,\ \theta, \nu,\ a)\,f(\beta,\ \theta,\ a)\,\rho(a)\,d\beta d\theta da\,. \tag{41}$$

The integration over θ extends from 0 to $\pi/2$ only, since we adopt the convention that A is always on that end of the axis of symmetry which makes an acute angle with $\boldsymbol{H}$. Likewise, for a beam polarized so that its electric vector is normal to $\boldsymbol{B}$,

$$dI_\sigma = -S_\sigma I_\sigma dN\,, \tag{42}$$

with

$$S_\sigma = \int_0^\infty \int_0^{\pi/2} \int_0^\pi \sigma_\sigma(\beta,\ \theta, \nu,\ a)\,f(\beta,\ \theta,\ a)\,\rho(a)\,d\beta d\theta da\,. \tag{43}$$

It may be seen that S_π and S_σ depend only on ν, σ_A, σ_T, and the distribution function.

When equations (34) and (35) are substituted in the above integrals, rather formidable-appearing expressions are obtained. However, the integrals over the angles can be separated from the integrations over a and the results expressed in terms of three functions, defined as follows:

$$F(a) = -\int_0^{\pi/2}\int_0^{\pi}(\cos^2\beta\cos^2\theta + \tfrac{1}{2}\sin^2\beta\sin^2\theta)\, f_1(\beta,\theta,a)\, d\beta d\theta\,, \tag{44}$$

$$S_t = \frac{1}{3}\int_0^{\infty}(2\sigma_T + \sigma_A)\,\rho(a)\,da\,, \tag{45}$$

$$S_p = \frac{3}{2}\int_0^{\infty}(\sigma_A - \sigma_T)F(a)\,\rho(a)\,da\,. \tag{46}$$

The physical meaning of these functions will be developed below, where it will be found that F provides a simple description of the distribution, while $S_t N$ is the optical depth for the beam as a whole, $S_\pi N$ is the optical depth for the component polarized parallel to $\boldsymbol{B}$, $S_\sigma N$ is the optical depth for the component polarized normal to $\boldsymbol{B}$, and $S_p N\cos^2\nu$ is the "optical depth" for the polarization. A straightforward reduction of equations (41) and (43) gives (cf. Appendix 3):

$$S_\pi = S_t - S_p(\cos^2\nu - \tfrac{1}{3})\,, \tag{47}$$

$$S_\sigma = S_t + \tfrac{1}{3}S_p\,. \tag{48}$$

We shall show that, in the most plausible model, $S_p > 0$; hence $S_\pi < S_\sigma$.

Consider, first, the approximate physical picture developed in § 3.5, where F, the distribution integral, was not defined by equation (44) but rather as a characteristic of a distribution in which all the particles have their axes of symmetry parallel either to the X, the Y, or the Z axis. If f_1 is computed for this distribution, we find that F as defined by equation (44) is equal to F as used in § 3.5. Thus, throughout our discussion the distribution integral represents either the value of equation (44) or the fraction of the grains that have been turned from the $\boldsymbol{B}$ direction to the two orthogonal directions. Clearly, the maximum possible value for F is $\frac{1}{3}$ and the minimum, attained when all the grains are parallel to $\boldsymbol{B}$, is $-\frac{2}{3}$.

5.3. *The polarization, total absorption, and color excess produced by dust.*—If S is a constant, the solution of

$$dI = -SIdN \tag{49}$$

is

$$\Delta m = -(\tfrac{5}{2})\log_{10}\frac{I}{I_0} = 1.086SN\,, \tag{50}$$

where Δm is the absorption measured in stellar magnitudes and SN is the optical depth.

Let us assume for the present that, for some particular star, $\boldsymbol{B}$ is uniform over all parts of the light-path from the star to the earth where there is an appreciable density of dust, and that $\rho(a)$ and f are constants in these regions. Thus S_π and S_σ are independent of N, and equations (40) and (42) have the form of equation (49). In each of the two planes of polarization the absorptions are

$$\Delta m_\pi = 1.086S_\pi N\,, \qquad \Delta m_\sigma = 1.086S_\sigma N\,. \tag{51}$$

If $S_\pi < S_\sigma$, then $I_\pi > I_\sigma$, and I_π is the maximum intensity observed as a plane polarizer is rotated, while I_σ is the minimum intensity. When the polarizer is rotated from the position in which the most light is transmitted to that in which the least is transmitted, the observed magnitude change is

$$\Delta m_p = -(\tfrac{5}{2})\log_{10}\frac{I_\sigma}{I_\pi} = \Delta m_\sigma - \Delta m_\pi = 1.086N(S_\sigma - S_\pi)\,,$$
$$\Delta m_p = 1.086NS_p\cos^2\nu \tag{52}$$

where we have used equations (47) and (48). The quantity Δm_p is that used by Hiltner. On the other hand, the degree of polarization is obtained from the classical definition

$$p=\frac{I_\pi - I_\sigma}{I_\pi + I_\sigma}=\frac{1-\exp(-S_p N \cos^2\nu)}{1+\exp(-S_p N \cos^2\nu)} \doteq \tfrac{1}{2} S_p N \cos^2\nu = 0.4605 \Delta m_p , \tag{53}$$

provided that $S_p N \ll 1$. We shall use p, defined by equation (53) exclusively, in order to follow the notation in Hall and Mikesell's catalog,[6] although from a theoretical and observational point of view the quantity Δm_p is more satisfactory.

The total intensity of light as observed when there are no polarizers in the photometer is $I = I_\pi + I_\sigma$, while if there were no dust it would be $2I_0$. Hence the absorption due to dust is

$$\begin{aligned} \Delta m_T &= -\left(\tfrac{5}{2}\right)\log_{10}\left[\frac{I_\pi + I_\sigma}{2I_0}\right], \\ &= -1.086 \ln \left\{ \exp\left[-\tfrac{1}{2}(S_\pi + S_\sigma) N\right] \frac{\exp\left[\tfrac{1}{2}(S_\sigma - S_\pi) N\right] + \exp\left[-\tfrac{1}{2}(S_\sigma - S_\pi) N\right]}{2} \right\} \\ &\doteq 1.086 \left[S_t - S_p\left(\tfrac{1}{2}\cos^2\nu - \tfrac{1}{3}\right)\right] N \doteq 1.086 S_t N = A_{pg} , \end{aligned} \tag{54}$$

since $S_p N \ll 1$, $S_p \ll S_t$. The color excess E_1 is about $\Delta m_T/9$, hence

$$E_1 = 0.121 N S_t \tag{55}$$

and

$$\frac{p}{E_1} = 4.14 \frac{S_p}{S_t} \cos^2\nu , \tag{56}$$

a result showing that, on the basis of the assumed homogeneity of $\boldsymbol{B}$ and $\rho(a)$, p/E_1 is independent of N and hence of E_1.

It is easy to obtain a rough evaluation of the right-hand side of equation (56) by assuming in equations (45) and (46) that all the grains are of the same size. Hence,

$$\frac{p}{E_1} = 18.6 F \cos^2\nu \, \frac{(\sigma_A/\sigma_T) - 1}{(\sigma_A/\sigma_T) + 2} . \tag{57}$$

Equation (21) for p/A_{pg} is obtained on the same assumptions. From this we see that the experimental value of 0.18 for the maximum observed value of p/E_1 could be explained by taking $\nu = 0$, $F = \frac{1}{3}$ (i.e., all grains lined up normal to $\boldsymbol{B}$), and $\sigma_A/\sigma_T = 1.086$, or by taking $\nu = 0$, $F = \frac{1}{15}$ (i.e., one-fifth of the grains lined up) and $\sigma_A/\sigma_T = 1.52$. Reference to Table 2 shows that the first value of σ_A/σ_T corresponds to very moderate eccentricities. The second is such that we would not expect much smaller values of F to lead to an attractive theory unless the dust grains are assumed to have relatively large elongations (axes in ratio 2:1 or more). It must be emphasized that this treatment is useful in dealing with rough orders of magnitude. Since it assumes that particles of only one size are present and that this size is small compared to the wave length, it will give an incorrect dependence of the interstellar absorption on wave length; thus little weight is to be attached to the fact that it also gives an incorrect dependence on wave length of p and of p/E_1.

6. THE TORQUE DUE TO PARAMAGNETIC ABSORPTION

We now turn to a consideration of the processes which might lead to a partial orientation of the grains of dust, starting with what appears to us to be the most likely mechanism, paramagnetic absorption. We develop an expression for the torque to be expected and, in the next section, for the effect of this torque on the distribution function. Later we consider briefly other mechanisms which might be suspected of producing orientation.

6.1. *The nature of the phenomena.*—The magnetic properties of many substances are such that if the substance is put in a magnetic field $\boldsymbol{B}$, whose magnitude varies sinusoidally, so that

$$\boldsymbol{B} = \boldsymbol{e} B_0 \cos(\omega t + \delta) , \tag{58}$$

where $\boldsymbol{e}$ is a fixed unit vector, then the magnetization is [29]

$$\boldsymbol{M} = \boldsymbol{e} B_0 [\chi' \cos(\omega t + \delta) + \chi'' \sin(\omega t + \delta)] . \tag{59}$$

Here χ' and χ'' are the real and imaginary parts of the complex susceptibility; their numerical values as functions of ω will be considered below. When $\omega = 0$, then $\chi'' = 0$ and $\chi' = \chi_0$, the ordinary static susceptibility. The amount of heat generated by the alternating magnetic field is determined by the value of χ''. This absorption of energy from a changing magnetic field is called "paramagnetic absorption."

Next suppose the magnetic field, $\boldsymbol{B}$, to be constant and the grain to rotate with the constant angular velocity ω about an axis fixed in the grain and in space. Let $OXYZ$ be axes fixed in space and $Oxyz$ be rotating axes fixed in the grain, taking both OZ and Oz along ω. Let $\boldsymbol{e}_X$, $\boldsymbol{e}_Y$, and $\boldsymbol{e}_Z$ be the unit vectors in the stationary system and $\boldsymbol{i}, \boldsymbol{j}$, and $\boldsymbol{k}$ be those in the rotating system. Choose the origin of time so that

$$\boldsymbol{e}_X = \boldsymbol{i} \cos \omega t - \boldsymbol{j} \sin \omega t , \qquad \boldsymbol{e}_Y = \boldsymbol{i} \sin \omega t + \boldsymbol{j} \cos \omega t , \qquad \boldsymbol{e}_Z = \boldsymbol{k} = \frac{\boldsymbol{\omega}}{\omega} . \tag{60}$$

No generality is lost if OX is oriented as shown in Figure 1, *a* and

$$\boldsymbol{B} = B_X \boldsymbol{e}_X + B_Z \boldsymbol{e}_Z . \tag{61}$$

Then equations (60) show that, from the point of view of an observer rotating with the particle,

$$\boldsymbol{B} = B_Z \boldsymbol{k} + B_X \left[\boldsymbol{i} \cos \omega t + \boldsymbol{j} \cos \left(\omega t + \frac{\pi}{2} \right) \right] . \tag{62}$$

Hence he sees three fields, each described by equation (58), superposed, and, if the magnetization can be obtained by superposition, it is given by equation (59) as

$$\begin{aligned} \boldsymbol{M} &= \chi_0 B_Z \boldsymbol{k} + B_X [\boldsymbol{i} (\chi' \cos \omega t + \chi'' \sin \omega t) + \boldsymbol{j} (-\chi' \sin \omega t + \chi'' \cos \omega t)] \\ &= \chi_0 B_Z \boldsymbol{e}_Z + \chi' B_X \boldsymbol{e}_X + \chi'' B_X \boldsymbol{e}_Y . \end{aligned} \tag{63}$$

The physical interpretation of this is that the component of $\boldsymbol{M}$ along $\boldsymbol{\omega}$ is the same as though there were no rotation, the component at right angles to $\boldsymbol{\omega}$ is reduced in the ratio χ'/χ_0 and is dragged along by the rotation through the relatively small angle $\tan^{-1}(\chi''/\chi')$.

Since $V\boldsymbol{M}$ is the magnetic moment of a grain of volume V, the torque due to magnetization is given by equations (63) and (61) as

$$\boldsymbol{L} = V\boldsymbol{M} \times \boldsymbol{B} = V(\chi_0 - \chi') \omega^{-2} (\boldsymbol{B} \cdot \boldsymbol{\omega}) (\boldsymbol{\omega} \times \boldsymbol{B}) + V \chi'' \omega^{-1} (\boldsymbol{\omega} \times \boldsymbol{B}) \times \boldsymbol{B} , \tag{64}$$

where the last expression is arranged to formulate the result in a way independent of the co-ordinate axes. We shall see in later sections that, since the first term is perpendicular to $\boldsymbol{\omega}$, it does not affect the rotational energy; although it produces a precession (i.e., changes ϕ in the notation of Fig. 3), it has no significant effect on the distribution of orientations, because $VB^2(\chi_0 - \chi') \ll kT$. For similar reasons we neglect an additional

[29] C. J. Gorter, *Paramagnetic Relaxation* (New York: Elsevier Pub. Co., 1947), p. 20.

very small torque, of the order of $\chi_0^2 B^2 V$, which tends to rotate the long axis of the grain into parallelism with $\boldsymbol{B}$.[30] Hence the term in χ'' produces the entire effect by its cumulative action. These points are touched on again in § 8.1.

We shall use equation (64) for $\boldsymbol{L}$ not only when $\boldsymbol{\omega}$ is fixed in the body but also when the body is nutating. This seems a reasonable assumption which greatly simplifies the subsequent analysis as compared to that necessary with slightly more accurate assumptions; but it must be remembered that, actually, $\boldsymbol{\omega}$ moves through the body relatively rapidly, unless the body is greatly elongated. We shall assume that the magnitude of $\boldsymbol{\omega}$ is given by ω_e from equation (5).

6.2. *The magnitude of* χ''.—The magnitude of the effect produced by paramagnetic absorption depends on the magnitude of χ''. We now estimate theoretically the value of χ'' for a material of the type discussed in § 4, that is, one in which each atom of a strongly magnetic element such as *Fe, Ni, Cr, Gd,* etc., is diluted by about 100 diamagnetic atoms. No significant error should be introduced if all the magnetic atoms are considered to be *Fe;* on the basis of Table 1 we shall regard *Fe* as comprising 12 per cent of the mass of the grain. We also show that experiment tends to confirm the theoretical estimate of χ''. Our treatment has been guided by suggestions from Gorter and Van Vleck and depends upon results given in Gorter's *Paramagnetic Relaxation*.[29] Unfortunately, it seems to be neither possible to find the required result in a directly quotable form nor desirable to give here a treatment sufficiently extensive to stand independently. Hence we shall combine a number of Gorter's formulas, leaving to him the discussion of their significance and the definition of some of the quantities that do not occur in our final result.

The case of interest to us seems to be governed by Gorter's equation (82) (p. 97), which should hold for frequencies of the order of ω_e,

$$\chi'' = \frac{\chi_0 \omega}{\omega_0}\left(\frac{\pi}{2}\right)^{1/2} \exp\left(-\frac{\omega^2}{2\omega_0^2}\right), \tag{65a}$$

where, with $2\pi\nu = \omega$, from his equation (80) and the related definitions we get

$$h\,\frac{\omega_0}{2\pi} = 2\beta^2\left[8S(S+1)\sum_{p\neq q} r_{pq}^{-6}\right]^{1/2}. \tag{65b}$$

Here $\beta = eh/4\pi mc = 0.927 \times 10^{-20}$ erg/gauss is the Bohr magneton, S is the spin quantum number of the magnetic ions, and for Σr_{pq}^{-6} we may take $7.2 n_c^2$, where n_c = number of *Fe* ions per cc. This value is the one given by Gorter on page 13 for a face-centered cubic lattice; the small difference in the coefficient of n_c for other arrangements is not significant for our purposes. In Table III (p. 15) Gorter gives 5.92 as the value of $2[S(S+1)]^{1/2}$ for Fe^{+++}; other magnetic ions have comparable, although usually somewhat smaller, values. Hence, if $n = 10^{-24} n_c$ = number of *Fe* ions per cubic angstrom, we have

$$\omega_0 = 3.66 \times 10^{12} n\,. \tag{66}$$

If Curie's law holds, we can use the expressions on page 6 of Gorter to evaluate χ_0. If T_g is the internal temperature of the grain, then we have

$$\chi_0 = \frac{n_c g^2 J(J+1)\beta^2}{3kT_g} = \frac{7.28 n}{T_g}, \tag{67}$$

where in obtaining the second form we used data given above and the value $g[J(J+1)]^{1/2} = 2[S(S+1)]^{1/2} = 5.92$ taken from Gorter (p. 15, Table III) and from the discussion on page 18.

[30] W. R. Smythe, *Static and Dynamic Electricity* (2d ed.; New York: McGraw-Hill Book Co., Inc., 1950), p. 421.

Before substituting equations (66) and (67) in equation (65a), we can reduce the latter to an even more convenient form for estimating the order of magnitude of χ'' by noting that, if $\omega/\omega_0 < 0.7$, the exponential may be taken to be unity. Thus we have

$$\chi'' = 2.5 \times 10^{-12} \frac{\omega}{T_g}, \qquad (68)$$

provided that

$$10^{-2} > n > 6 \times 10^{-21} \left(\frac{T}{a^5 \rho_g}\right)^{1/2} = 1.9 \times 10^{-7} \text{ in } H\,\text{I} = 1.9 \times 10^{-6} \text{ in } H\,\text{II}. \qquad (69)$$

These limitations on the validity of equation (68) arise as follows. The lower limit on n arises from the condition that $\omega/\omega_0 < 0.7$ and follows from equations (5) and (66). The upper limit is required, according to Gorter (private communication), in order not to enter the range where Curie's law breaks down because the iron atoms get so close together that the interactions are strong. Material whose density is ρ_g and which contains 12 per cent Fe by weight must have $0.12 \times 10^{-24}\ \rho_g$ gm of Fe atoms, each weighing 9.37×10^{-23} gm in each cubic angstrom. Hence $n = 1.3 \times 10^{-3}\ \rho_g$, and we see that the composition adopted in Table 1 easily satisfies condition (69) for all except very small grains ($a \ll 10^{-6}$).

It will be noted that χ'' as given by equation (68) is independent of the number of Fe ions per cubic angstrom as long as condition (69) is satisfied, that is, over a wide range of concentrations. This surprising and fortunate result means that our conclusions do not depend in a sensitive way on the amount of magnetic material in the grains. Let us consider the origin of this result. In χ_0 we find a term proportional to n giving the expected proportionality of the effect to the amount of Fe. But in equation (65) we note that the farther the resonance at ω_0 is from the frequency of interest, $\omega = \omega_e$, the smaller is the effect of each ion. Also we would expect the resonance frequency to be higher when the "coupling" is stronger, that is, when the density of interacting ions is greater, a result in agreement with equation (66). Combining the two tendencies, the result appears plausible.

Since the theoretical derivation of equation (68) is based on many assumptions and simplifications, it is most desirable that its predictions be compared with experiment. Gorter's discussion on pages 67–68 shows that the general dependence of χ''/χ_0 on T and ω is correctly given by equation (68) for frequencies up to 10^6 sec^{-1} and temperatures of the order of 70° K and that the dependence on n is roughly correct. The numerical value of this ratio for $FeNH_4(SO_4)_2 \cdot 12H_2O$ deduced from his Table XV, which is based on experiment, is 65 per cent of the value deduced from our equations (66) and (68); values for a number of other substances show that our result is a quite reasonable estimate of the order of magnitude of χ''.

7. THE EFFECT OF A TORQUE ON THE DISTRIBUTION INTEGRAL

We now wish to consider the effect of a torque so small that the change in angular momentum during a single nutation is negligible and only cumulative effects are significant. Thus we can describe the motion as in § 5.1, using Figure 3, with the modification that the angles β and θ are regarded as changing very slowly rather than being constant. To simplify our notation, we shall work out their rates of change for the torque considered in the previous section, although the modification for any other small torque is straightforward. In the latter parts of this section we consider the effect of these changes on the distribution integral.

7.1. *The mean rates of change of β and θ.*—We must first develop a somewhat more extensive description of the motion when there is no torque. Consider the plane through the angular momentum, $\boldsymbol{H}$, and the axis of symmetry, $\boldsymbol{e}_A$, shown in Figure 5, *a*. Because of the symmetry, the angular velocity $\boldsymbol{\omega}$ must lie in this plane. Let θ, θ_H, and θ_ω be the angles indicated where θ is as defined above, and let $\boldsymbol{e}_A$ and $\boldsymbol{e}_T$ be the unit vectors in this plane along and normal to the axis of symmetry, respectively. Since we have defined I to be the moment of inertia about $\boldsymbol{e}_A$ and γI to be that about $\boldsymbol{e}_T$ and since these are principal axes of inertia, we can write the angular velocity and angular momentum as

$$\boldsymbol{\omega} = \boldsymbol{e}_A \omega \cos\theta_\omega + \boldsymbol{e}_T \omega \sin\theta_\omega , \tag{70}$$

$$\boldsymbol{H} = \boldsymbol{e}_A H \cos\theta + \boldsymbol{e}_T H \sin\theta \tag{71}$$

$$= \boldsymbol{e}_A I\omega \cos\theta_\omega + \boldsymbol{e}_T I\gamma\omega \sin\theta_\omega . \tag{72}$$

From equations (71) and (72) it follows that

$$\cos\theta_\omega = \frac{H}{I\omega}\cos\theta , \quad \sin\theta_\omega = \frac{H}{I\gamma\omega}\sin\theta . \tag{73}$$

Since $\theta_H = \theta - \theta_\omega$, it then follows that

$$\begin{aligned} \cos\theta_H &= \frac{H}{I\gamma\omega}(\gamma\cos^2\theta + \sin^2\theta) , \\ \sin\theta_H &= \frac{H}{I\gamma\omega}(\gamma - 1)\sin\theta\cos\theta . \end{aligned} \tag{74}$$

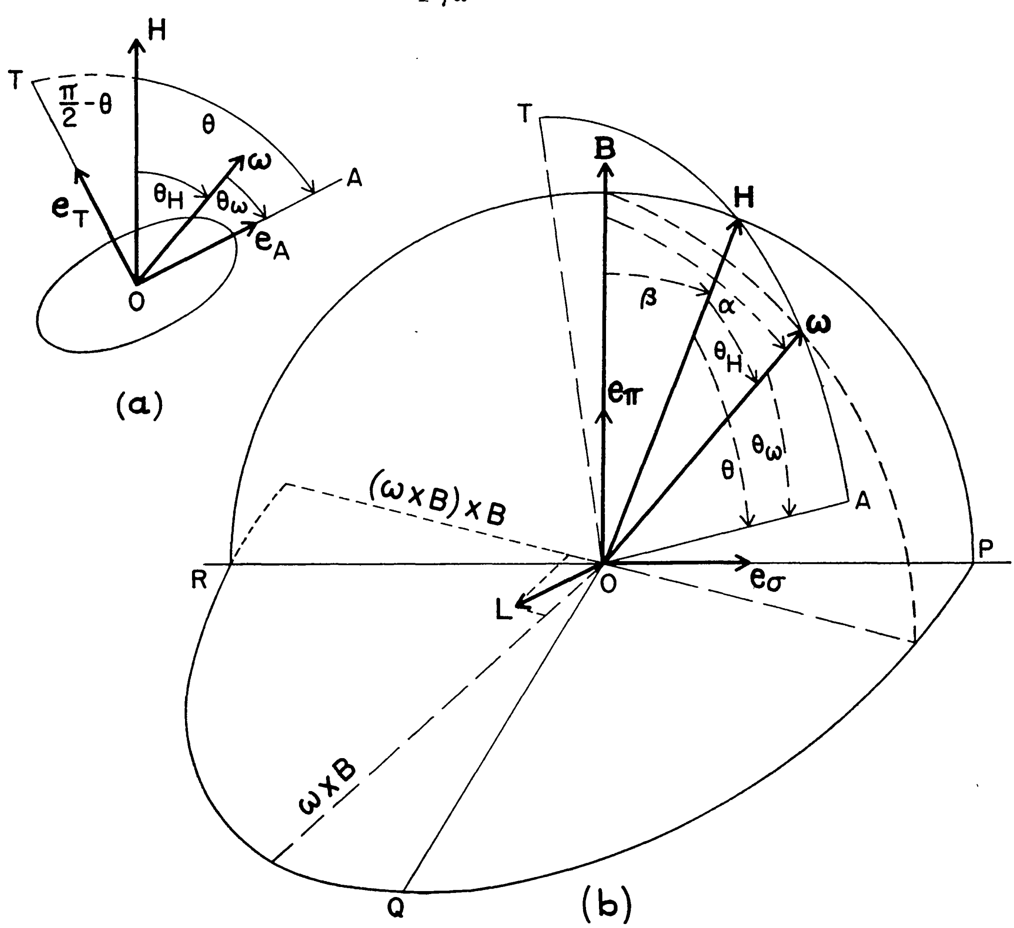

FIG. 5.—*a*, the angles between $\boldsymbol{H}$, $\boldsymbol{\omega}$, and the axis of symmetry, *A*. *b*, the relations between $\boldsymbol{\omega}$, $\boldsymbol{H}$, $\boldsymbol{B}$, and $\boldsymbol{L}$. The angles are $\angle HOA = \theta$; $\angle BOH = \beta$; $\angle BO\omega = \alpha$; $\angle BHA = \psi$; $\angle HO\omega = \theta_H$; $\angle \omega OA = \theta_\omega$.

The kinetic energy of rotation is

$$R = \tfrac{1}{2}\left[I(\omega\cos\theta_\omega)^2 + I\gamma(\omega\sin\theta_\omega)^2\right], \tag{75}$$

so that, by equations (75) and (73),

$$\frac{2I\gamma R}{H^2} = \gamma\cos^2\theta + \sin^2\theta\,. \tag{76}$$

Further, by equations (74) and (76)

$$\frac{2R}{H\omega} = \cos\theta_H\,. \tag{77}$$

As our basic equations of motion we take

$$\dot{\boldsymbol{H}} = \boldsymbol{L} = V(\chi_0 - \chi')\,\omega^{-2}(\boldsymbol{B}\cdot\boldsymbol{\omega})(\boldsymbol{\omega}\times\boldsymbol{B}) + V\frac{\chi''}{\omega}(\boldsymbol{\omega}\times\boldsymbol{B})\times\boldsymbol{B}\,, \tag{78}$$

$$\dot{R} = \boldsymbol{\omega}\cdot\boldsymbol{L} = -V\frac{\chi''}{\omega}\left[\omega^2B^2 - (\boldsymbol{\omega}\cdot\boldsymbol{B})^2\right], \tag{79}$$

where the dot indicates differentiation with respect to time and where we have used equation (64). Next we average these over one nutation to get $[\dot{\boldsymbol{H}}]_{av}$ and $[\dot{R}]_{av}$. We see from Figure 3 that a knowledge of the former of these gives the average rate at which β is changing. Equation (76) allows us to obtain the average rate at which θ changes from $[\dot{\boldsymbol{H}}]_{av}$ and $[\dot{R}]_{av}$.

In order to carry out the averaging, consider Figure 5, *b*, which is the same as Figure 3 except that $\boldsymbol{\omega}$ and the various vectors needed to determine $\boldsymbol{L}$ have been added and the system has been rotated about $\boldsymbol{B}$. In the nutational motion, the plane containing O, T, $\boldsymbol{H}$, $\boldsymbol{\omega}$, and OA, i.e., the plane of Figure 5, *a*, rotates with constant angular velocity $\dot{\psi}$ about $\boldsymbol{H}$, the angles θ, θ_ω, and θ_H remaining constant. If, as $\boldsymbol{\omega}$ moves in a circle about $\boldsymbol{H}$, we first average over positions that are mirror images in the plane BOH, so that the values of α are the same, we see that the average over a nutation of $\boldsymbol{\omega}\times\boldsymbol{B}$ lies along OQ, i.e., along the vector $\boldsymbol{H}\times\boldsymbol{B}$, and that the average of $(\boldsymbol{\omega}\times\boldsymbol{B})\times\boldsymbol{B}$ lies along OR, i.e., along the vector $-\boldsymbol{e}_\sigma$ which lies in the plane of $\boldsymbol{B}$ and $\boldsymbol{H}$. Hence the first term on the right-hand side of equation (78) causes $\boldsymbol{H}$ to precess around $\boldsymbol{B}$ but produces no change in β or in H, the magnitude of $\boldsymbol{H}$. Thus, although $(\chi_0 - \chi')$ contributes to the torque, it does not affect the distribution of orientations, and we shall drop it from our treatment. The average of the last term of equation (78) is

$$\begin{aligned} V\frac{\chi''}{\omega}\left[(\boldsymbol{\omega}\times\boldsymbol{B})\times\boldsymbol{B}\right]_{av} &= V\frac{\chi''}{\omega}\left([\boldsymbol{\omega}]_{av}\times\boldsymbol{B}\right)\times\boldsymbol{B} \\ &= -V\frac{\chi''}{\omega}\,\omega B^2\sin\beta\cos\theta_H\,\boldsymbol{e}_\sigma \end{aligned} \tag{80}$$

because $[\boldsymbol{\omega}]_{av}$ is directed along $\boldsymbol{H}$ and has the magnitude $\omega\cos\theta_H$. Hence $\boldsymbol{H}$ tends to spiral toward $\boldsymbol{B}$, its projection on $\boldsymbol{B}$ remaining constant.

If $\boldsymbol{H}$ is resolved into components H_π and H_σ along $\boldsymbol{B}$ and OP, respectively, we see from equations (79) and (80) that

$$\begin{aligned} &[\dot{H}_\pi]_{av} = 0\,, \\ &\left[\frac{dH_\sigma}{dt}\right]_{av} = \left[\frac{d(H_\pi\tan\beta)}{dt}\right]_{av} = -V\frac{\chi''}{\omega}\,\omega B^2\sin\beta\cos\theta_H\,. \end{aligned} \tag{81}$$

It follows from this and equation (74) that the average rate of change of β is

$$[\dot{\beta}]_{\text{av}} = -\frac{\chi''}{\omega}\frac{VB^2}{I\gamma}\sin\beta\cos\beta\,(\gamma\cos^2\theta+\sin^2\theta)\,. \tag{82}$$

Next turn to the calculation of the average rate of change of θ. We shall need the expression

$$\left[\frac{dH^2}{dt}\right]_{\text{av}} = \left[\frac{d}{dt}(H_\pi^2+H_\sigma^2)\right]_{\text{av}} = -2\,V\frac{\chi''}{\omega}\,\omega B^2 H\sin^2\beta\cos\theta_H\,. \tag{83}$$

Also average equation (79):

$$\begin{aligned}[\dot{R}]_{\text{av}} &= -V\frac{\chi''}{\omega}\,\omega^2 B^2\,(1-[\cos^2\alpha]_{\text{av}}) \\ &= -V\frac{\chi''}{\omega}\,\omega^2 B^2\,(1-\cos^2\theta_H\cos^2\beta-\tfrac{1}{2}\sin^2\theta_H\sin^2\beta)\,,\end{aligned} \tag{84}$$

with $[\cos^2 \alpha]_{\text{av}}$ from equation (109) suitably modified. Now differentiate equation (76), solve for $\dot{\theta}$, and average. Reduce the resulting expression with the aid of equations (77) and (74):

$$\begin{aligned}[\dot{\theta}]_{\text{av}} &= -\frac{I\gamma R}{H^2(\gamma-1)\sin\theta\cos\theta}\left\{\frac{[\dot{R}]_{\text{av}}}{R}-\frac{1}{H^2}\left[\frac{dH^2}{dt}\right]_{\text{av}}\right\} \\ &= \frac{I\gamma\,V\chi''B^2\omega}{H^2(\gamma-1)\sin\theta\cos\theta}\sin^2\theta_H\,(1-\tfrac{1}{2}\sin^2\beta) \\ &= \frac{\chi''}{\omega}\frac{VB^2}{I\gamma}(\gamma-1)\sin\theta\cos\theta\,(1-\tfrac{1}{2}\sin^2\beta)\,.\end{aligned} \tag{85}$$

It might appear that the above treatment somewhere omits an important consideration, since it seems to imply that a torque due to the first term on the right-hand side of equation (78) or due to a permanent magnetization of the grain along its axis of symmetry produces only a precession and does not affect the distribution of orientations. It is indeed true that such torques affect the distribution, the effect being easily computed from the Maxwell-Boltzmann distribution when, as is usual, the torque is derivable from a potential energy. But the effect on the distribution is insignificant unless this potential energy is of the same order as R, since the cumulative effect is null.

7.2. *The distribution integral when the relaxation time is short.*—Consider the case in which the collisions of atoms and ions with the dust grains are so frequent that the distribution remains relatively near the equipartition value, the effect of the torques being to produce a small perturbation. Evaluate the distribution integral with equation (12). The problem then is to calculate the rate at which F changes when β and θ for each grain are changing as given by equations (82) and (85). Regard each grain as characterized by a representative point in β-θ space, the density of the representative points being just $f(\beta, \theta, a)$ as defined in § 5.2. These points move with the velocity

$$\boldsymbol{v} = [\dot{\beta}]_{\text{av}}\boldsymbol{e}_\beta + [\dot{\theta}]_{\text{av}}\boldsymbol{e}_\theta\,, \tag{86}$$

where $\boldsymbol{e}_\beta$ and $\boldsymbol{e}_\theta$ are unit vectors along the co-ordinate axes. Since the representative points never vanish, the usual equation of continuity gives

$$\frac{\partial f}{\partial t} = -\operatorname{div}(\boldsymbol{v}f)\,. \tag{87}$$

It follows that if we evaluate the derivative as the system is passing through the equipartition distribution,

$$\frac{\partial f_1}{\partial t} = -\frac{\partial}{\partial \beta}\{[\dot{\beta}]_{\text{av}} f_e\} - \frac{\partial}{\partial \theta}\{[\dot{\theta}]_{\text{av}} f_e\}. \tag{88}$$

To form the derivative with respect to t of F, as defined by equation (44), we need only differentiate f_1 under the integral sign, using equation (88). Hence, from equation (12),

$$F(a) \doteq F_l(a) \equiv \tau \int_0^{\pi/2} \int_0^{\pi} (\cos^2\beta \cos^2\theta + \tfrac{1}{2}\sin^2\beta \sin^2\theta) \left(\frac{\partial}{\partial \beta}\left\{[\dot{\beta}]_{\text{av}} f_e\right\} + \frac{\partial}{\partial \theta}\{[\dot{\theta}]_{\text{av}} f_e\}\right) d\beta\, d\theta, \tag{89}$$

where we introduce F_l for the value of the integral in any case. $F = F_l$ only for the linear range where the perturbation from the equipartition distribution is small, i.e., when $|F_l| \ll \frac{1}{3}$. We can now substitute from equations (37), (82), and (85), reducing equation (89) by an integration of elementary functions (cf. Appendix 4) to

$$F \doteq F_l = \frac{\Gamma(\gamma)\, V\chi'' B^2 \tau}{I\omega_e}, \tag{90}$$

where $\Gamma(\gamma)$ is an abbreviation for a constant that depends only on the eccentricity of the spheroid, being

$$\begin{aligned} \Gamma(\gamma) &= \frac{2}{15\gamma^{1/2}(\gamma-1)^{1/2}}\left[\frac{2\gamma+1}{\gamma-1}\sinh^{-1}(\gamma-1)^{1/2} - 3\left(\frac{\gamma}{\gamma-1}\right)^{1/2}\right] \quad \text{if } \gamma > 1 \\ &= \frac{-2}{15\gamma^{1/2}(1-\gamma)^{1/2}}\left[\frac{2\gamma+1}{1-\gamma}\sin^{-1}(1-\gamma)^{1/2} - 3\left(\frac{\gamma}{1-\gamma}\right)^{1/2}\right] \quad \text{if } 0 < \gamma < 1 \\ &= \frac{8(\gamma-1)}{225\gamma}\left[1 - \tfrac{1}{7}(\gamma-1) + \ldots\right] \quad \text{if} \quad |\gamma - 1| \ll 1, \end{aligned} \tag{91}$$

where in the last form the term neglected is of order $(\gamma - 1)^3$.

In Table 2 we have tabulated for a variety of grain shapes the values of various quantities of interest. If a_A is the radius of the grain along the axis of symmetry, a_T is the radius in the transverse direction, and a is the mean radius, used throughout our paper, the quantities tabulated are obtained from equation (91) and

$$V = \tfrac{4}{3}\pi a_A a_T^2 = \tfrac{4}{3}\pi a^3, \tag{92}$$

$$\gamma = \frac{I\gamma}{I} = \frac{\rho_g V(a_A^2 + a_T^2)/5}{\rho_g V(a_T^2 + a_T^2)/5} = \frac{1}{2}\left(\frac{a_A^2}{a_T^2} + 1\right), \tag{93}$$

$$\frac{100\Gamma \rho_g V a^2}{I} = 250\left(\frac{a_A}{a_T}\right)^{2/3}\Gamma. \tag{94}$$

Consulting Table 2, we see that over the plausible range of (a_A/a_T) we can take

$$100\Gamma\frac{V}{I} = \pm\frac{5}{a^2\rho_g} \tag{95}$$

without introducing errors as great as those due to our lack of knowledge of τ.

By substitution of equations (95), (68), and (16) we can now reduce equation (90) to

$$F_l = \pm \frac{6.31 \times 10^6 B^2}{a T^{1/2} n_H T_g} \tag{96}$$

$$= \pm\ 6.31 \times 10^8 B^2 \text{ in both } H\,\text{I and } H\,\text{II}\,,$$

the sign being + for prolate and − for oblate spheroids.

7.3. *The distribution integral when the relaxation time is very long.*—If we compare equation (90) with equations (82) and (85), we see that, to within a factor of the order of unity, F_l is just τ times the average rate of change of the orientation angles, provided that Γ is introduced to allow for averaging over all orientations. Thus, when $|F_l| \ll \frac{1}{3}$, the angles change but little during the relaxation time, which is to be regarded as the time available for the applied torque to alter the orientation before the collisions stop further drift away from equipartition. On the other hand, when $|F_l| \gg \frac{1}{3}$, the relaxation time is so long that the collisions do not affect the distribution significantly and the torque orients the grains to the maximum degree possible. It then follows from equation (82) that β approaches 0 if $\beta < \pi/2$ initially and β approaches π if $\beta > \pi/2$. That is, the angular momenta all line up either parallel or antiparallel with the magnetic field. Recalling that $0 \leqslant \theta \leqslant \pi/2$, we see from equation (85) that for prolate spheroids, where $\gamma > 1$, θ approaches $\pi/2$, while for oblate spheroids, where $\gamma < 1$, θ approaches 0. In either case the long dimension of the spheroid becomes normal to $\boldsymbol{H}$ and $\boldsymbol{B}$. Thus, by the discussion at the end of § 5.2, $F = \frac{1}{3}$ for prolate spheroids and $F = -\frac{2}{3}$ for oblate spheroids when $|F_l| \gg \frac{1}{3}$.

8. OTHER METHODS OF PRODUCING POLARIZATION

8.1. *Torques due to permanent magnetization.*—When the orientation of the grains can be specified directly without investigating $\boldsymbol{H}$, the angular momentum, it is convenient to set $\beta = 0$ in all our formulae and to regard θ as being the angle between $\boldsymbol{B}$ and the axis of symmetry. The resulting expressions enable one to treat situations, such as that considered by Spitzer and Tukey,[12] where the grains are assumed to have a permanent magnetization $\boldsymbol{M}$ along the axis of symmetry. Then the angular distribution function, f, defined by the expression (36) with β and $d\beta$ omitted, is easily found from the Maxwell-Boltzmann distribution law, and evaluation of equation (44) then gives

$$F(a) = -\left\{\frac{2}{3} + 2g^2 - \frac{2}{g}\coth g\right\}, \tag{97}$$

where $g = VMB/kT$. Our $F(a)$ is $-\frac{2}{3}M(g)$ in the notation of Spitzer and Tukey. Substitution in the equations of § 5.3 then gives an expression for p that is in agreement with their equation (26). The fact that F is less than zero means that the magnetic field required to produce a given orientation of the grains by this mechanism is at right angles to that required when paramagnetic relaxation, or any dissipative mechanism, produces the orientation. The fact that the limiting value here is $-\frac{2}{3}$, while for the case of § 3 it is $\frac{1}{3}$, means that a field strong enough to give complete alignment requires here a smaller value of σ_A/σ_T.

If $VMB/kT \ll 1$, the distribution is not much affected, and the polarization is very small. Similarly one can show that the torque described by the first term of equation (64) produces no significant effect. In a mixture of such a torque and a nonconservative torque of the type treated in § 7, one can treat the case in which the latter torque produces only a small perturbation merely by replacing the f_e of § 7 by the Maxwell-Boltzmann equipartition distribution for the conservative torque.

8.2. *Eddy currents.*—Among the possible torques that might act on a body rotating in a magnetic field is that due to eddy currents in a conductor. We may estimate the order of magnitude of this torque by assuming the grain of dust to be a sphere of conductivity σ e.m.u. and permeability κ. We find that u, the ratio of the diameter of the grain to the skin depth of the eddy currents, is

$$u = 2a(2\pi\omega\kappa\sigma)^{1/2} = 6.3 \times 10^{-4}(\kappa\sigma)^{1/2}\left(\frac{T}{a\rho_g}\right)^{1/4}, \tag{98}$$

where we have used equation (5) for ω, the angular velocity of the grain. Thus u is small compared to unity for all plausible values of the parameters. Rotation in a uniform field is equivalent to the action on a stationary sphere of two alternating fields at right angles and 90° out of phase. Hence the torque is twice the power absorbed in an alternating field divided by the angular velocity. We use the expression for the power given by W. R. Smythe,[31] converting it to e.m.u. and keeping only the lowest-order terms in u. The result is

$$L_E = \frac{6\pi a^5 \omega \kappa^2 \sigma B^2}{45 + 30(\kappa - 1) + 5(\kappa - 1)^2}. \tag{99}$$

Such a torque will not affect the motion significantly if $L_E/L_R \ll 1$. If we substitute from equation (17) for L_R, and from equation (5) for ω, and take $\kappa = 1$, we get

$$\begin{aligned} \frac{L_E}{L_R} &= 1.3 \times 10^{19} \frac{aB^2\sigma}{n_H T^{1/2}} \\ &= 1.3 \times 10^{12}\sigma B^2, \text{ in } H\,\text{I and } H\,\text{II regions}. \end{aligned} \tag{100}$$

As κ approaches infinity, the ratio approaches a value nine times as great. Now for pure iron at a temperature of 5° K, σ is of the order of 0.02 e.m.u. Impurities would diminish the conductivity very greatly, and, if the grains have the composition assumed in § 4, σ will be smaller by at least 10^7. Thus, if B is of the order of 10^{-4} gauss, eddy currents produce negligible effects unless the grains are assumed to be very good metallic conductors. If the grains were pure iron, eddy-current effects would probably significantly modify the type of behavior discussed by Spitzer and Tukey.

8.3. *The Rowland Effect.*—If the dust grains bear electrostatic charges, then, as follows from Rowland's famous experiment, torques will act on them as they rotate in a magnetic field.[32] If the potential of the grain is of the order of a few volts, then it is easy to show that for this reason the grains have a magnetic moment per unit volume of the order of 10^{-4} e.m.u. For magnetic fields of the order of 10^{-5} gauss, the resulting torque on the grains is much larger than L_R as given by equation (17) and hence is a torque whose effects should be examined with some care. On the other hand, its direction is such that it might be expected to produce mainly precession without appreciable alteration of the distribution of orientations. Indeed, we shall now show that the distribution of orientations is completely unchanged by the Rowland effect.

Our procedure is to obtain a Hamiltonian for the system; then the distribution of orientations in the steady state is obtained from the basic theorems of statistical mechanics. We shall consider only the rotational motion of the grain, since the inclusion of translatory degrees of freedom, as is apparently suggested by Hiltner,[2] should not affect the orientation. Assume that all effects due to motion of the charge relative to the grain have been accounted for in the treatment of eddy currents and hence assume that the charges are fixed with respect to the grain.

The Lagrangian for a rigid body is obtained by adding together the well-known

[31] *Static and Dynamic Electricity* (2d ed.; New York: McGraw-Hill Book Co., Inc., 1950), p. 400.

[32] The authors are indebted to Dr. Olin C. Wilson for the suggestion that this torque should be investigated. Some such torque was also suggested by Hiltner (*Ap. J.*, **109**, 471, 1949).

Lagrangians of the charged particles into which the body can be divided. Thus we get, for the rigid grain of dust,

$$\mathfrak{L} = \tfrac{1}{2}\boldsymbol{\omega}\cdot\boldsymbol{I}\cdot\boldsymbol{\omega} + \tfrac{1}{2}\boldsymbol{B}\cdot\boldsymbol{J}\cdot\boldsymbol{\omega}\,, \tag{101}$$

where $\boldsymbol{\omega}$ is regarded as expressed in terms of Eulerian angles and their derivatives; $\boldsymbol{I}$ is the familiar tensor of inertia defined in terms of integrals over the volume of certain functions of position multiplied by the mass density; and $\boldsymbol{J}$ is a tensor defined in exactly the same way except that the charge density replaces the mass density.

Let us denote the Eulerian angles used to specify the orientation of the grain by q_i, $i = 1, 2, 3$. Then $\boldsymbol{\omega}$ is linear in $\dot{q}_i$, although a somewhat complicated function of q_i. Thus equation (101) may be written as

$$\mathfrak{L} = \tfrac{1}{2}A_{ij}\dot{q}_i\dot{q}_j + C_i\dot{q}_i\,, \qquad A_{ij} = A_{ji}\,, \tag{102}$$

where a single index can have any of the values 1, 2, 3 while we sum over any index that occurs twice in a single term—i.e., the usual summation convention. The quantity A_{ij} comes from the first term of equation (101) and does not involve B, the magnetic field, while C_i comes from the second term and is zero when $B = 0$. From equation (102) we obtain the generalized momenta,

$$p_i = \frac{\partial\mathfrak{L}}{\partial\dot{q}_i} = A_{ij}\dot{q}_j + C_i\,, \tag{103}$$

and the Hamiltonian

$$\mathfrak{H} = p_k\dot{q}_k - \mathfrak{L} = \tfrac{1}{2}A_{ij}\dot{q}_i\dot{q}_j\,, \tag{104}$$

where $\mathfrak{H}$ is to be expressed in terms of q_i and p_i by means of equation (103).

The usual statistical mechanical procedures now give (cf. Appendix 2), as the number of grains whose representative points lie in a region $\delta\tau$ of phase space,

$$\delta N = c_1 \underset{\delta\tau}{\int\!\!\int\!\!\int\!\!\int\!\!\int\!\!\int} \exp\left\{\frac{-\mathfrak{H}}{kT}\right\} dq_1 dq_2 dq_3 dp_1 dp_2 dp_3\,, \tag{105}$$

where c_1 is a normalization constant. The regions $\delta\tau$ that interest us are those in which the q_i, i.e., the orientation, are confined to a narrow range while the p_i range over all possible values. We now change variables in equation (105) from p_i to $v_i = \dot{q}_i$. The appropriate expression for $\mathfrak{H}$ is already available in equation (104); we note that it is independent of B. The change of variables does not affect our description of $\delta\tau$. The Jacobian of the transformation is found from equation (103) to be the determinant of the matrix A_{ij}, and this is independent of B. Thus the expression for the angular distribution has been formulated in such a way that nowhere does the magnetic field enter; hence the distribution must be completely independent of the field.

We are deeply indebted to both Dr. W. A. Hiltner and Dr. John S. Hall for the observational material made available to us before publication. Dr. J. H. Van Vleck and Dr. C. J. Gorter have been most helpful in the discussion of paramagnetic relaxation. We are particularly grateful to Dr. Lyman Spitzer for discussions of all phases of the problem and for his many helpful suggestions on reading our manuscript.

APPENDIXES

Appendix 1. *Derivation of equations (34) and (35).*—We wish to compute $\overline{\cos^2\alpha_\sigma}$ and $\overline{\cos^2\alpha_\pi}$, where the angles are shown in Figure 4 and the average is over ϕ and ψ. Denote the direction cosines of OA by $\cos\alpha_X$, $\cos\alpha_Y$, and $\cos\alpha_Z$. Then, since the direction cosines of E_π are 0: sin ν: cos ν,

$$\cos\alpha_\pi = \sin\nu\cos\alpha_Y + \cos\nu\cos\alpha_Z\,, \tag{106}$$

$$\cos \alpha_\sigma = \cos \alpha_X \; . \tag{107}$$

By spherical trigonometry

$$\cos \alpha_Z = \cos \beta \cos\theta + \sin \beta \sin\theta \cos \psi \; . \tag{108}$$

Now, when we average over ψ, we readily obtain

$$\overline{\cos^2 \alpha_Z} = \cos^2 \beta \cos^2\theta + \tfrac{1}{2} \sin^2 \beta \sin^2\theta \; ; \tag{109}$$

and this result is unchanged if we next average over ϕ. When we average first over ϕ and then over ψ, we see by symmetry that

$$\overline{\cos^2 \alpha_X} = \overline{\cos^2 \alpha_Y} = \tfrac{1}{2} (1 - \overline{\cos^2 \alpha_Z}) \, . \tag{110}$$

Therefore,

$$\overline{\cos^2 \alpha_\sigma} = \tfrac{1}{2} (1 - \cos^2 \beta \cos^2\theta - \tfrac{1}{2} \sin^2 \beta \sin^2\theta) \, , \tag{111}$$

which gives equation (34) on substitution in equation (33).

When we square equation (106) and average over ϕ, we note that $\cos \alpha_Y \cos \alpha_Z$ averages to zero. Then, substituting from equations (109) and (110), we get

$$\overline{\cos^2 \alpha_\pi} = \tfrac{1}{2} \sin^2 \nu (1 - \cos^2 \beta \cos^2\theta - \tfrac{1}{2} \sin^2 \beta \sin^2\theta) + \cos^2 \nu (\cos^2 \beta \cos^2\theta + \tfrac{1}{2} \sin^2 \beta \sin^2\theta) \, , \tag{112}$$

which gives equation (35) on substitution in equation (32).

Appendix 2. *The equipartition distribution.*—We wish to derive equation (37), the expression for the equipartition distribution, which is the distribution when no torques act on the grain except during collisions with atoms. In this case we have a Maxwell-Boltzmann distribution, and the usual statistical mechanical procedures give

$$\delta N = c_1 \int\!\!\int\!\!\int_{\delta\tau}\!\!\int\!\!\int\!\!\int \exp \frac{-R}{kT} \, d\lambda d\psi d\phi d p_\lambda d p_\psi d p_\phi \tag{113}$$

as the number of particles whose representative points lie in a region $\delta\tau$ of phase space, where c_1 is a normalization constant; R is the kinetic energy of rotation of a grain; T is the temperature of the gas molecules whose collisions produce the equipartition distribution; λ, ψ, and ϕ are angles defined below that specify the orientation of the grain; and p_λ, p_ψ, and p_ϕ are the conjugate momenta needed to complete the phase space. Hence our problem is to define suitable co-ordinates and then select a region $\delta\tau$ in phase space such that δN gives us $f_e(\beta, \theta)$.

As shown in Figure 6, let $OXYZ$ be a system of nonrotating co-ordinates with O at the center of mass and OZ directed along $\boldsymbol{B}$. Let $Oxyz$ be a system of axes fixed in the particle with Oz along the axis of symmetry, and let λ, ϕ, and ψ be the usual Eulerian angles, λ replacing θ, which is used for the angle between $\boldsymbol{H}$ and the axis of symmetry. Note that temporarily we shall not use ψ with the same meaning as in § 5. The moments of inertia of the spheroid are $I_{xx} = I_{yy} = \gamma I$, $I_{zz} = I$. The kinetic energy of rotation is

$$R = \tfrac{1}{2} I \, [\gamma (\dot\lambda^2 + \dot\psi^2 \sin^2 \lambda) + (\dot\phi + \dot\psi \cos \lambda)^2] \, . \tag{114}$$

The conjugate momenta, p_λ, p_ψ, and p_ϕ, are defined in the usual way by $p_\lambda = \partial R / \partial \dot\lambda$, etc. Any standard treatment of the dynamics of rigid bodies gives the expression for the angular momentum, $\boldsymbol{H}$, referred to axes fixed in the spheroid in terms of the Eulerian angles and their rates of change. The expressions for the momenta and the abbreviation

$$\pi_\psi = \frac{p_\psi - p_\phi \cos\lambda}{\sin\lambda} \tag{115}$$

then enable us to write

$$R = \tfrac{1}{2}(I\gamma)^{-1}(p_\lambda^2 + \pi_\psi^2 + \gamma p_\phi^2), \tag{116}$$

$$H^2 = p_\lambda^2 + \pi_\psi^2 + p_\phi^2, \tag{117}$$

$$\cos\theta = \frac{\boldsymbol{H}\cdot\boldsymbol{e}_z}{H} = \frac{p_\phi}{H}, \tag{118}$$

where $\boldsymbol{e}_z$ is a unit vector along OA.

The dependence of f_e on β is easily found. In our present discussion, $\boldsymbol{B}$ is regarded as a

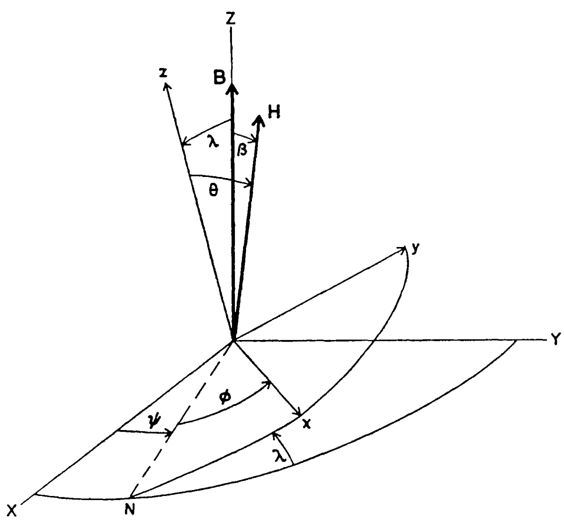

FIG. 6.—The Eulerian angles

vector that specifies a unique direction in space but that does not affect the orientation of the grain. Hence $\boldsymbol{H}$ is oriented at random over a sphere and f_e must be proportional to $\frac{1}{2}\sin\beta d\beta$. Thus we need only select $\delta\tau$ so as to give the dependence on θ; that is, to include those values of the canonical co-ordinates and momenta for which θ lies in the range $d\theta$. Now that we have equation (113), we no longer need to use canonical variables but can change to the set λ, ψ, ϕ, p_λ, π_ψ, p_ϕ. The Jacobian of the transformation is $\sin\lambda$. We note that R and θ do not depend on λ, ψ, and ϕ; hence we can carry out the integration over these variables and write equation (113) in the form

$$\delta N = c_2 \iiint_{\delta\tau} \exp\frac{-R}{kT}\, dp_\lambda d\pi_\psi dp_\phi, \tag{119}$$

where c_2 is one of a series of normalization coefficients, c_n, whose evaluation we shall postpone. Equations (117) and (118) show that we can regard

$$p_\lambda = H\sin\theta\cos\alpha, \qquad \pi_\psi = H\sin\theta\sin\alpha, \qquad p_\phi = H\cos\theta \tag{120}$$

as being Cartesian co-ordinates in a space whose spherical polar co-ordinates are H, θ, and α. In our complete phase space we see that p_λ, p_ψ, and p_ϕ, and hence π_ψ, run from

$-\infty$ to $+\infty$. Thus H runs from 0 to ∞, θ from 0 to π, and a from 0 to 2π. In the region $\delta\tau$ of interest, the only change to be made is that θ is restricted to the range $d\theta$. If we express R and $dp_\lambda\, d\pi_\psi\, dp_\phi = H^2 \sin\theta\, dH\, d\theta\, da$ in terms of the new variables, equation (119) becomes

$$\delta N = c_2 \int_0^\infty \int_\theta^{\theta+d\theta} \int_0^{2\pi} \exp\left[-\frac{H^2(\sin^2\theta+\gamma\cos^2\theta)}{2I\gamma kT}\right] H \sin\theta\, dH\, d\theta\, da\,, \quad (121)$$

$$= c_3 \sin\theta\,(\gamma\cos^2\theta+\sin^2\theta)^{-3/2} d\theta\,.$$

If we now multiply the left side of equation (121) by $\frac{1}{2}\sin\beta d\beta$, to give the dependence on β found above, and evaluate c_3 by integration over all θ and β, we get equation (37). In this normalization we return to the convention, used everywhere except in the present section, that θ lies between 0 and $\pi/2$.

Appendix 3. *The derivation of equations* (44)–(48).—For purposes of derivation it is convenient to regard S_t and S_p as being defined by equations (47) and (48) rather than by equations (45) and (46).

Using equations (41), (43), (34), and (35), one finds that

$$S_t = \frac{S_\pi + (2\cos^2\nu - \sin^2\nu)S_\sigma}{3\cos^2\nu} = \int_0^\infty \int_0^{\pi/2} \int_0^\pi \frac{\sigma_A + 2\sigma_T}{3} f(\beta,\theta,a)\,\rho(a)\, d\beta\, d\theta\, da\,, \quad (122)$$

$$S_p = \frac{S_\sigma - S_\pi}{\cos^2\nu} = \int_0^\infty \int_0^{\pi/2} \int_0^\pi (\sigma_A - \sigma_T)\left(\tfrac{1}{2} - \tfrac{3}{2}\cos^2\beta\cos^2\theta - \tfrac{3}{4}\sin^2\beta\sin^2\theta\right) \times [f_e(\beta,\theta) + f_1(\beta,\theta,a)]\,\rho(a)\, d\beta\, d\theta\, da\,. \quad (123)$$

If we next consider the definitions of f, f_e, and f_1 in § 5.2, it is evident that

$$\int_0^{\pi/2} \int_0^\pi f(\beta,\theta,a)\, d\beta\, d\theta = \int_0^{\pi/2} \int_0^\pi f_e(\beta,\theta)\, d\beta\, d\theta = 1\,, \quad (124)$$

$$\int_0^{\pi/2} \int_0^\pi f_1(\beta,\theta)\, d\beta\, d\theta = 0\,. \quad (125)$$

Now $f_e = \sin\beta[f_e/\sin\beta]$, where from equation (37) $f_e/\sin\beta$ is independent of β. Hence, by equation (124),

$$\frac{1}{2}\int_0^{\pi/2} \int_0^\pi \left[\frac{f_e(\beta,\theta)}{\sin\beta}\right] \sin\beta\, d\beta\, d\theta = \int_0^{\pi/2} \left[\frac{f_e(\beta,\theta)}{\sin\beta}\right] d\theta = \tfrac{1}{2}\,. \quad (126)$$

Therefore,

$$\int_0^{\pi/2} \int_0^\pi \left(\tfrac{3}{2}\cos^2\beta\cos^2\theta + \tfrac{3}{4}\sin^2\beta\sin^2\theta\right) \left[\frac{f_e(\beta,\theta)}{\sin\beta}\right] \sin\beta\, d\beta\, d\theta = \int_0^{\pi/2} (\cos^2\theta + \sin^2\theta) \left[\frac{f_e(\beta,\theta)}{\sin\beta}\right] d\theta = \tfrac{1}{2}\,. \quad (127)$$

It follows from equation (124) and the fact that σ_A and σ_T are independent of β and θ that equation (122) is equivalent to equation (45). It follows from equations (124), (125), and (127) that equation (123) is equivalent to equations (46) and (44).

Appendix 4. *The evaluation of the parameter F.*—Substitution in equation (89) shows that equation (90) holds, provided that

$$\Gamma(\gamma)=\frac{1}{2\gamma^{1/2}}\int_0^{\pi/2}\int_0^{\pi}(\cos^2\beta\cos^2\theta+\tfrac{1}{2}\sin^2\beta\sin^2\theta)\times\left[\frac{\partial(-\sin^2\beta\cos\beta)}{\partial\beta}\times\frac{\sin\theta}{(\gamma\cos^2\theta+\sin^2\theta)^{1/2}}+\frac{\partial}{\partial\theta}\left\{\frac{\sin^2\theta\cos\theta}{(\gamma\cos^2\theta+\sin^2\theta)^{3/2}}\right\}(\gamma-1)\sin\beta(1-\tfrac{1}{2}\sin^2\beta)\right]d\beta d\theta. \tag{128}$$

The differentiations and integrations over β are easily carried out, and the differentiation with respect to θ can be eliminated by integrating by parts. This gives

$$\Gamma(\gamma)=\frac{2}{15\gamma^{1/2}}\int_0^{\pi/2}\frac{(\gamma\cos^2\theta+\sin^2\theta)(1-3\cos^2\theta)+(\gamma-1)\sin^2\theta\cos^2\theta}{(\gamma\cos^2\theta+\sin^2\theta)^{3/2}}\sin\theta d\theta. \tag{129}$$

Let $x=(\gamma-1)^{1/2}\cos\theta$, to get

$$\Gamma(\gamma)=\frac{2}{15\gamma^{1/2}(\gamma-1)^{3/2}}\int_0^{(\gamma-1)^{1/2}}\frac{(x^2+1)(\gamma-1-3x^2)+x^2(\gamma-1-x^2)}{(x^2+1)^{3/2}}dx. \tag{130}$$

This gives equation (91).