1967ApJ...150..571G

THE ASTROPHYSICAL JOURNAL, Vol. 150, November 1967

DIFFERENTIAL ROTATION IN STARS

PETER GOLDREICH
California Institute of Technology
AND
GERALD SCHUBERT
University of California at Los Angeles
Received April 4, 1967; revised June 16, 1967

ABSTRACT

The stability of differential rotation in the radiative zones of stars is investigated. For sufficiently large χ/ν (χ is the thermal diffusivity and ν the kinematic viscosity), it is shown that a necessary condition for stability in regions of homogeneous chemical composition is that the angular momentum per unit mass be an increasing function of distance from the rotation axis. In cylindrical coordinates (ϖ, φ, z), this condition is given by $\partial(\varpi^2\Omega)/\partial\varpi \geq 0$ and $\partial\Omega/\partial z = 0$, where Ω is the angular velocity. The condition is also a sufficient one when applied to axisymmetric perturbations. The stable thermal stratification which exists in the radiative zone cannot prevent the instability since, in stars, the thermal diffusivity is much greater than the kinematic viscosity. The turbulent diffusion of angular momentum, which arises when the stability condition is violated, is so rapid that it would appear to preclude the fast rotation of the Sun's interior which has been proposed by Dicke * In the absence of the instability associated with thermal diffusion, i.e , if $x = 0$, Dicke's solar model is found to be stable.

Another means whereby angular momentum might be brought up from the solar interior is by the mechanism of spin-down associated with the formation of an Ekman boundary layer just below the solar convective envelope. The transport of angular momentum, either by spin-down or by turbulent diffusion, would result in the mixing of material below the convective zone of solar type stars if an external torque were applied to the stellar surface. Thus, the depletion of lithium and beryllium would be an inevitable consequence of the loss of a significant fraction of the star's initial angular momentum.

I. INTRODUCTION

Evidence that the interior of the Sun is in rapid rotation has recently been presented by Dicke (1967). He has interpreted his measurement of a solar oblateness of 5×10^{-5} as implying a 1.8-day rotation period for the solar radiative interior. This period is derived on the assumption that the slow surface rotation is confined to the convective zone. Of course, many other models are also consistent with Dicke's observation.

Our investigation concentrates on the stability of differential rotation in stellar interiors. The major portion of this paper is devoted to the derivation of linear stability criteria for axisymmetric disturbances. We include, and find to be of essential importance, the effects of radiation transfer and viscosity. Where instability is indicated we make estimates of the turbulent diffusion rates.

Dicke (1964) has claimed that his model of the differentially rotating Sun is stable. We do not agree. Indeed, most differentially rotating solar models which are consistent with Dicke's observation are found to be unstable.

II. THE DIFFUSION OF MOMENTUM AND HEAT

We shall be concerned with three types of viscosity: molecular, radiative, and turbulent. Expressions for the molecular and radiative dynamic viscosities are

$$\eta_m = \frac{0.4 m^{1/2} (kT)^{5/2}}{e^4 \ln \Lambda} \qquad \text{(Spitzer 1962)}, \tag{1}$$

* It should be noted that the analysis presented here does not account for the possible stabilizing effect of toroidal magnetic fields.

© American Astronomical Society • Provided by the NASA Astrophysics Data System

1967ApJ...150..571G

where

$$\Lambda = \frac{3}{2e^3}\left(\frac{m k^3 T^3}{\pi \rho}\right)^{1/2}$$

and

$$\eta_r = \frac{16\sigma T^4}{15 c^2 \kappa \rho} \quad \text{(Thomas 1930)}. \tag{2}$$

In equations (1) and (2), m is the proton mass, e the electronic charge in e.s.u., c the velocity of light, ρ the density, T the temperature, k the Boltzmann constant, σ the Stefan-Boltzmann constant, and κ the mass absorption coefficient. The quantity Λ is proportional to the cutoff distances for collisions. We have taken the cutoff to be at the Debye length. As written, equation (1) applies to a pure hydrogen plasma which is accurate enough for our purposes.

In all our numerical applications to the Sun, we shall use Weymann's (1957) solar model except that we shall take the bottom of the convective zone to lie at 0.7 $R_\odot$ (instead of the older value of 0.8 $R_\odot$) in accord with more modern calculations (Sears and Weymann 1965).

The ratio of molecular to radiative viscosity in the Sun varies from about 12 at the center to about 4 just below the convective zone. We shall always use the total viscosity, $\eta = \eta_m + \eta_r$, which we call the microscopic viscosity to distinguish it from the turbulent or macroscopic viscosity. The corresponding kinematic viscosity is denoted by $\nu = \eta/\rho$. The characteristic time for the diffusion of angular momentum (or protons) in the Sun due to microscopic viscosity alone is

$$T \simeq \frac{R_\odot^2}{\nu} = 1.6 \times 10^{13} \text{ years}, \tag{3}$$

where we have evaluated ν at a radius exterior to half of the Sun's mass. Clearly, microscopic viscosity cannot significantly alter differential rotation in the Sun.

The diffusion of heat in stars is much faster than the diffusion of momentum. For example, the half-life for diffusion of the Sun's thermal energy (which is essentially the Kelvin-Helmholtz contraction time) is about 3×10^7 years, almost six orders of magnitude shorter than the viscous diffusion time. The coefficient of thermal diffusivity in stellar interiors is given (when only radiation transfer is of importance) by

$$\chi = \frac{16\sigma T^3}{3\kappa \rho^2 c_v} \quad \text{(Schwarzschild 1958)}, \tag{4}$$

where c_v is the specific heat at constant volume. In terms of the adiabatic exponent $\gamma (\gamma = c_p/c_v)$ and the mean molecular weight μ

$$c_v = \frac{k}{(\gamma - 1)\mu}. \tag{5}$$

Just below the solar convective zone $\nu = 22$ cm^2 sec^{-1}, $\chi = 2.8 \times 10^7$ cm^2 sec^{-1} and the dimensionless ratio

$$\frac{\chi}{\nu} = 5.9 \times 10^6. \tag{6}$$

III. THE STABILITY OF DIFFERENTIALLY ROTATING STARS

Let us consider an arbitrary differentially rotating star. We assume that the initial motion is steady except for the slow changes produced by viscous diffusion and meridional circulation. For the time being, we shall restrict our attention to regions in the star where the gas is chemically homogeneous. The flow will be described in cylindrical co-

© American Astronomical Society • Provided by the NASA Astrophysics Data System

1967ApJ...150..571G

ordinates (ϖ,φ,z) with the z-axis in the direction of the rotation axis. On occasion, we shall transform to spherical coordinates (r,θ,φ). In addition to variables previously defined, we shall use the pressure p, velocity $\boldsymbol{v}$, angular velocity $\boldsymbol{\Omega}$, gravitational acceleration $\boldsymbol{g}$ (the positive direction is taken outward), entropy per unit mass s and the bulk viscosity ξ.

The equations describing the fluid motion are the momentum equation, the continuity equation, the heat equation, and the equation of state. We will assume that the gravitational force per unit mass always has the unperturbed value $\boldsymbol{g}$. Due to the small scale of the perturbations of interest the effects of their self-gravitation will be shown to be negligible. We also assume the perfect gas equation of state. The equations are (Landau and Lifshitz 1959)

$$\rho\left(\frac{\partial \boldsymbol{v}}{\partial t}+\boldsymbol{v}\cdot\nabla\boldsymbol{v}\right)=-\nabla p+\rho\boldsymbol{g}+\nabla\{(\xi-\tfrac{2}{3}\eta)\nabla\cdot\boldsymbol{v}\}+\nabla(\eta\,\mathrm{def}\,\boldsymbol{v}), \tag{7}$$

$$\frac{\partial\rho}{\partial t}+\nabla(\rho\boldsymbol{v})=0, \tag{8}$$

$$\rho T\left(\frac{\partial s}{\partial t}+\boldsymbol{v}\cdot\nabla s\right)=\nabla(\rho c_v\chi\nabla T)+(\xi-\tfrac{2}{3}\eta)(\nabla\cdot\boldsymbol{v})^2+\frac{\eta}{2}(\mathrm{def}\,\boldsymbol{v}):(\mathrm{def}\,\boldsymbol{v}), \tag{9}$$

$$p=\frac{\rho kT}{\mu}, \tag{10}$$

where the deformation tensor, def $\boldsymbol{v}$, is given by def $\boldsymbol{v}=\nabla\boldsymbol{v}+\boldsymbol{v}\nabla$. With the identity

$$ds=\frac{c_v dT}{T}-\frac{p}{\rho^2}\frac{d\rho}{T}\quad\text{(Landau and Lifshitz 1958)}, \tag{11}$$

equation (9) may be transformed into

$$c_v\rho T\left(\frac{\partial}{\partial t}+\boldsymbol{v}\cdot\nabla\right)\ln[T\rho^{-(\gamma-1)}]=\nabla(\rho c_v\chi\nabla T)+(\xi-\tfrac{2}{3}\eta)(\nabla\cdot\boldsymbol{v})^2+\frac{\eta}{2}(\mathrm{def}\,\boldsymbol{v}):(\mathrm{def}\,\boldsymbol{v}). \tag{12}$$

The unperturbed state (variables denoted by subscript 0) is steady and

$$v_{0\varpi}=v_{0z}=0,\qquad v_{0\varphi}=\varpi\Omega_0, \tag{13}$$

if we neglect the slow changes due to viscosity and meridional currents Then equation (7) implies

$$-\Omega_0{}^2\tilde{\boldsymbol{\omega}}=-\frac{\nabla p_0}{\rho_0}+\boldsymbol{g}, \tag{14}$$

and the continuity equation is identically satisfied Note that

$$\nabla\cdot\boldsymbol{v}_0=0,\qquad \boldsymbol{v}_0\cdot\nabla\rho_0=0. \tag{15}$$

Since $\boldsymbol{v}_0\cdot\nabla\ln[T_0\rho_0{}^{-(\gamma-1)}]=0$, the condition that the basic temperature field remain steady would require

$$\nabla(\rho_0\chi_0\nabla T_0)=0. \tag{16}$$

It is well known that this condition cannot in general be satisfied and that the non-vanishing of the divergence of the energy flux will drive meridional circulations (Edding-

© American Astronomical Society • Provided by the NASA Astrophysics Data System

1967ApJ...150..571G

ton 1929). However, the circulation currents are so slow that they have an insignificant effect on the instabilities which we shall be studying (we shall demonstrate this later). Thus, we shall neglect the meridional currents and pretend that only $v_{0\phi}$ is non-zero. In the same spirit, we have neglected the stresses which arise from the transport of angular momentum by the luminous flux (Jeans 1928). The effect of these stresses may be shown to be negligible, although the radiative viscosity (which we have retained) is of importance.

We wish to investigate the stability of an initially steady flow to axially symmetric disturbances. The most important (unstable) perturbations will be found to have wavelengths which are very much smaller than the stellar radius. It is therefore expedient to work with a simplified set of equations which approximate the complete equations in a small region of the star. In order to derive these equations, we first linearize all the variables with respect to the perturbations (denoted by a prime). Differentiation of a perturbation variable (with respect to a spatial coordinate) is of order λ^{-1}, where λ is the characteristic size of the perturbation. In approximating the differential operators acting on the perturbations we retain only the leading terms in powers of the small quantity λ/ϖ_0. The next step is to expand the unperturbed variables and their derivatives in Taylor series about (ϖ_0, z_0). Discarding terms of order $(\varpi - \varpi_0)/\varpi_0$ and $(z - z_0)/\varpi_0$ we obtain the desired approximate equations. The coefficients in the equations are independent of ϖ, φ, z and t, so that the perturbation variables may be expanded in plane waves of the form $\exp(qt + ik_\varpi \varpi + ik_z z)$. Setting $k_T^2 = k_\varpi^2 + k_z^2$ and dropping the subscript 0 on the unperturbed quantities we obtain

$$(q + \nu k_T^2) v'_\varpi - 2\Omega v'_\varphi = -ik_\varpi c^2 \frac{p'}{p} + c^2 \frac{\partial \ln p}{\partial \varpi} \frac{\rho'}{\rho} - \left(\frac{\xi}{\rho} + \frac{\nu}{3}\right) k_\varpi (k_\varpi v'_\varpi + k_z v'_z), \quad (17)$$

$$(q + \nu k_T^2) v'_\varphi + \left(2\Omega + \varpi \frac{\partial \Omega}{\partial \varpi}\right) v'_\varpi + \varpi \frac{\partial \Omega}{\partial z} v'_z = i \frac{\varpi}{\rho} \left(\frac{\partial \Omega}{\partial \varpi} k_\varpi + \frac{\partial \Omega}{\partial z} k_z\right) \eta', \quad (18)$$

$$(q + \nu k_T^2) v'_z = -ik_z c^2 \frac{p'}{p} + c^2 \frac{\partial \ln p}{\partial z} \frac{\rho'}{\rho} - \left(\frac{\xi}{\rho} + \frac{\nu}{3}\right) k_z (k_\varpi v'_\varpi + k_z v'_z), \quad (19)$$

$$q \frac{\rho'}{\rho} + \frac{\partial \ln \rho}{\partial \varpi} v'_\varpi + \frac{\partial \ln \rho}{\partial z} v'_z + i(k_\varpi v'_\varpi + k_z v'_z) = 0, \quad (20)$$

$$\begin{aligned} q \left(\frac{T'}{T} - (\gamma - 1)\frac{\rho'}{\rho}\right) &+ \frac{\partial \ln[T\rho^{-(\gamma-1)}]}{\partial \varpi} v'_\varpi + \frac{\partial \ln[T\rho^{-(\gamma-1)}]}{\partial z} v'_z \\ &= -k_T^2 \chi \frac{T'}{T} + \frac{2i\nu\varpi}{c_v T}\left(k_\varpi \frac{\partial \Omega}{\partial \varpi} + k_z \frac{\partial \Omega}{\partial z}\right) v'_\varphi \\ &+ i\chi \left(k_\varpi \frac{\partial \ln T}{\partial \varpi} + k_z \frac{\partial \ln T}{\partial z}\right)\left(\frac{\rho'}{\rho} + \frac{\chi'}{\chi}\right) \\ &+ \frac{\nu}{c_v T}\left[\left(\varpi \frac{\partial \Omega}{\partial \varpi}\right)^2 + \left(\varpi \frac{\partial \Omega}{\partial z}\right)^2\right] \frac{\eta'}{\eta}, \end{aligned} \quad (21)$$

$$\frac{p'}{p} = \frac{\rho'}{\rho} + \frac{T'}{T}. \quad (22)$$

The symbol $c^2 = p/\rho$ is the isothermal sound speed. All unperturbed quantities in equations (17)–(22) are evaluated at (ϖ_0, z_0). The perturbations in η and χ may be expressed in terms of the perturbations in ρ and T.

© American Astronomical Society • Provided by the NASA Astrophysics Data System

1967ApJ...150..571G

The dispersion relation which follows from equations (17)–(22) contains many branches other than the one of relevance to our investigation. For example, one branch describes sound waves damped by viscous and thermal diffusion. We are interested in rotational instabilities which arise when the angular momentum per unit mass decreases outward (from the rotation axis). By an appropriate scaling of the perturbation variables it is possible to shorten the algebra needed to derive the dispersion relation for these modes and to simultaneously discard the extraneous modes. Consider a mode with wave vector $\boldsymbol{k}_T$. The appropriate scaling (denoting a scaled variable by an asterisk) is

$$q=\Omega q^*\,,\qquad \boldsymbol{v}'=\frac{\Omega}{k_T}\,\boldsymbol{v}'^*\,,\qquad \frac{p'}{p}=\frac{\Omega^2}{c^2k_T{}^2}\,\frac{p'^*}{p}\,,$$
$$\frac{\rho'}{\rho}=\frac{\Omega^2}{c^2k_T{}^2}(k_TH)\frac{\rho'^*}{\rho}\,,\qquad \frac{T'}{T}=\frac{\Omega^2}{c^2k_T{}^2}(k_TH)\frac{T'^*}{T}\,, \tag{23}$$

where H is a typical scale height at (ϖ_0, z_0). Differentiation of an unperturbed variable is thus of order H^{-1}. After substituting equations (23) into equations (17)–(22) we order the terms by the following dimensionless parameters:

$$\frac{\Omega^2}{c^2k_T{}^2}\simeq\frac{1}{k_T{}^2H^2}\ll 1\,;\qquad \frac{\chi k_T{}^2}{\Omega}\sim\frac{\nu k_T{}^2}{\Omega}\sim\frac{\xi k_T{}^2}{\rho\Omega}\sim 1\,. \tag{24}$$

Thus, we assume that (for the wavelengths of interest) the sound frequency is much greater than the rotation frequency and that the wavelength, $2\pi/k_T$, is small compared to the scale height H. The thermal and viscous diffusion rates are considered to be of the same order as the rotation rate. Finally, $c/\Omega H$ is assigned order unity although it is sometimes much larger (but never much smaller because of the requirement of stability). The order of η' is determined by setting

$$\eta'=\frac{\partial\eta}{\partial T}T'+\frac{\partial\eta}{\partial\rho}\rho'\quad\text{so that}\quad O(\eta')=\max\left[O\left(\eta\frac{T'}{T}\right),\quad O\left(\eta\frac{\rho'}{\rho}\right)\right]. \tag{25}$$

The perturbation in χ is handled similarly. With this ordering, and retaining only the lowest order terms, equations (17)–(22) reduce to (the asterisks have been dropped from the scaled variables):

$$\left(q+\frac{\nu k_T{}^2}{\Omega}\right)v'_\varpi-2v'_\varphi=-i\,\frac{k_\varpi}{k_T}\,\frac{p'}{p}+H\,\frac{\partial\ln p}{\partial\varpi}\,\frac{\rho'}{\rho}\,, \tag{26}$$

$$\left(q+\frac{\nu k_T{}^2}{\Omega}\right)v'_\varphi+\left(2+\frac{\varpi}{\Omega}\frac{\partial\Omega}{\partial\varpi}\right)v'_\varpi+\frac{\varpi}{\Omega}\frac{\partial\Omega}{\partial z}\,v'_z=0\,, \tag{27}$$

$$\left(q+\frac{\nu k_T{}^2}{\Omega}\right)v'_z=-i\,\frac{k_z}{k_T}\,\frac{p'}{p}+H\,\frac{\partial\ln p}{\partial z}\,\frac{\rho'}{\rho}\,, \tag{28}$$

$$k_\varpi v'_\varpi+k_z v'_z=0\,, \tag{29}$$

$$q\left[\frac{T'}{T}-(\gamma-1)\frac{\rho'}{\rho}\right]+\frac{c^2}{\Omega^2H}\left\{\frac{\partial\ln[T\rho^{-(\gamma-1)}]}{\partial\varpi}\,v'_\varpi+\frac{\partial\ln[T\rho^{-(\gamma-1)}]}{\partial z}\,v'_z\right\}$$
$$=-\frac{\chi k_T{}^2}{\Omega}\frac{T'}{T}\,, \tag{30}$$

$$\frac{\rho'}{\rho}+\frac{T'}{T}=0\,. \tag{31}$$

© American Astronomical Society • Provided by the NASA Astrophysics Data System

1967ApJ...150..571G

Had we included the perturbation in the gravitational acceleration, we would have found, with the aid of Poisson's equation for the potential, that $i(k_\varpi/k_T^2)(4\pi G\rho H/c^2)\rho'/\rho$ would have been added to the right-hand side of equation (26) and $i(k_z/k_T^2)(4\pi G\rho H/c^2)\rho'/\rho$ to the right-hand side of equation (28). Here, G is the gravitational constant. Since $4\pi G\rho$ is $O(c^2/H^2)$, we see that these terms are $O(1/k_T H)$ smaller than those that were included. Thus we may safely neglect the perturbation in the gravitational acceleration.

Equations (26)–(31) yield the following dispersion relation:

$$\begin{aligned} & q^3+q^2\left(\frac{2\nu k_T^2}{\Omega}+\frac{\chi k_T^2}{\gamma\Omega}\right) \\ & +q\left\{\left(\frac{k_z}{k_T}\right)^2\frac{1}{\gamma\Omega^2}\left(g_\varpi+\Omega^2\varpi-\frac{k_\varpi}{k_z}g_z\right)\left[\frac{k_\varpi}{k_z}\frac{\partial\ln T\rho^{-(\gamma-1)}}{\partial z}-\frac{\partial\ln T\rho^{-(\gamma-1)}}{\partial\varpi}\right]\right. \\ & \left.+\frac{2}{\varpi\Omega}\left(\frac{k_z}{k_T}\right)^2\left[\frac{\partial}{\partial\varpi}(\varpi^2\Omega)-\frac{k_\varpi}{k_z}\varpi^2\frac{\partial\Omega}{\partial z}\right]+\frac{2}{\gamma}\left(\frac{\chi k_T^2}{\Omega}\right)\left(\frac{\nu k_T^2}{\Omega}\right)+\left(\frac{\nu k_T^2}{\Omega}\right)^2\right\} \\ & +\left(\frac{k_z}{k_T}\right)^2\frac{1}{\gamma\Omega^2}\left(\frac{\nu k_T^2}{\Omega}\right)\left(g_\varpi+\Omega^2\varpi-\frac{k_\varpi}{k_z}g_z\right)\left[\frac{k_\varpi}{k_z}\frac{\partial\ln T\rho^{-(\gamma-1)}}{\partial z}-\frac{\partial\ln T\rho^{-(\gamma-1)}}{\partial\varpi}\right] \\ & +\frac{2}{\gamma\varpi\Omega}\left(\frac{k_z}{k_T}\right)^2\left(\frac{\chi k_T^2}{\Omega}\right)\left[\frac{\partial}{\partial\varpi}(\varpi^2\Omega)-\frac{k_\varpi}{k_z}\varpi^2\frac{\partial\Omega}{\partial z}\right]+\frac{1}{\gamma}\left(\frac{\chi k_T^2}{\Omega}\right)\left(\frac{\nu k_T^2}{\Omega}\right)^2 \\ & =0\,. \end{aligned} \tag{32}$$

If the constant term in this cubic equation is negative, q has at least one positive real root and the motion is unstable. We are now led to the central theorem of this investigation, namely: *For sufficiently large* χ/ν, *a necessary condition for stability is that the angular momentum per unit mass be an increasing function of distance from the rotation axis.* The preceding condition is equivalent to the requirement that $\partial(\varpi^2\Omega)/\partial\varpi > 0$ and $\partial\Omega/\partial z = 0$. If we drop the qualification concerning χ/ν, this theorem appears identical to Rayleigh's theorem for incompressible, inviscid flows (Chandrasekhar 1961). However, for compressible flows the qualification is essential. In order to prove the theorem, we observe that for sufficiently large χ/ν the gravitational part of the constant term will become negligible relative to the rotational part and there will be instability if

$$\left[\frac{\partial}{\partial\varpi}(\varpi^2\Omega)+\frac{k_\varpi}{k_z}\frac{\partial}{\partial z}(\varpi^2\Omega)\right]+\frac{\varpi\Omega}{2}\left(1+\frac{k_\varpi^2}{k_z^2}\right)\left(\frac{\nu k_T^2}{\Omega}\right)^2<0\,. \tag{33}$$

Furthermore, for small enough k_T, only the rotational term in inequality (33) need be considered. Finally, unless both $\partial/\partial\varpi(\varpi^2\Omega) > 0$ and $\partial\Omega/\partial z = 0$, it is always possible to choose k_ϖ/k_z such that the rotational term is indeed negative. There are several interesting limiting forms of the dispersion relation which we now proceed to investigate.

In the inviscid, non-conducting limit ($\nu = 0$, $\chi = 0$), the dispersion relation reduces to

$$\begin{aligned} & q^2+\left(\frac{k_z}{k_T}\right)^2\frac{1}{\gamma\Omega^2}\left(g_\varpi+\Omega^2\varpi-\frac{k_\varpi}{k_z}g_z\right)\left[\frac{k_\varpi}{k_z}\frac{\partial\ln T\rho^{-(\gamma-1)}}{\partial z}-\frac{\partial\ln T\rho^{-(\gamma-1)}}{\partial\varpi}\right] \\ & +\frac{2}{\varpi\Omega}\left(\frac{k_z}{k_T}\right)^2\left[\frac{\partial}{\partial\varpi}(\varpi^2\Omega)-\frac{k_\varpi}{k_z}\frac{\partial}{\partial z}(\varpi^2\Omega)\right]=0\,. \end{aligned} \tag{34}$$

© American Astronomical Society • Provided by the NASA Astrophysics Data System

1967ApJ...150..571G

If we consider a cylindrically symmetric basic state (i.e., no z-dependence) and take the limit $\gamma \to \infty$ we obtain the dispersion relation for incompressible Couette flow (of course, boundary conditions are not included)

$$q^2+\frac{2}{\varpi\Omega}\left[\frac{\partial}{\partial\varpi}(\varpi^2\Omega)\right]\left(\frac{k_z}{k_T}\right)^2=0\,. \tag{35}$$

Equation (35) predicts stability or instability depending on whether $\partial/\partial\varpi(\varpi^2\Omega)$ is >0 or <0. This is Rayleigh's stability criterion for incompressible, inviscid, Couette flow (Chandrasekhar 1961).

Defining $\omega = q\Omega$, letting $\Omega \to 0$, and choosing a spherically symmetric basic state we arrive at

$$\omega^2+\frac{[(\gamma-1)n-1]}{\gamma}\frac{g_r}{r^2}\frac{d\ln T}{dr}\left(\frac{k_\theta}{k_T}\right)^2=0\,, \tag{36}$$

where we have introduced $n = d\ln\rho(r)/d\ln T(r)$. We observe that instability arises if $(\gamma-1)n<1$ (if $dT/dr<0$). This is Schwarzschild's (1906) criterion for convective instability under adiabatic perturbations.

Considerable simplification of equation (32) may be achieved before more detailed application is made to differentially rotating solar models. Even with an interior rotation period as short as one day,

$$\Omega^2\varpi \ll g_\varpi\,. \tag{37}$$

Furthermore, if we restrict attention to modes for which heat diffusion is important,

$$\frac{\chi k_T^2}{\gamma\Omega} \gg q\,. \tag{38}$$

Finally, $\chi/\nu > 10^5$ in the Sun. On the basis of equation (37), we may approximate the solar density distribution by a spherically symmetric one and reduce equation (32) to

$$\begin{aligned} q^2+q\Bigg\{&\frac{[(\gamma-1)n-1]}{\chi k_T^2\Omega}\frac{g_r}{r^2}\frac{d\ln T}{dr}\left(\frac{k_\theta}{k_T}\right)^2+\frac{2\nu k_T^2}{\Omega} \\ &+\frac{2\gamma}{\chi k_T^2}\left(\frac{k_r}{k_T}\cot\theta-\frac{k_\theta}{rk_T}\right)\left[\frac{k_r}{k_T}\frac{\partial(\Omega\sin^2\theta)}{\partial\theta}-\frac{k_\theta}{k_T}\frac{\sin^2\theta}{r^2}\frac{\partial(r^2\Omega)}{\partial r}\right]\Bigg\} \\ &+\frac{\nu}{\chi}\frac{[(\gamma-1)n-1]}{\Omega^2}\frac{g_r}{r^2}\frac{d\ln T}{dr}\left(\frac{k_\theta}{k_T}\right)^2+\left(\frac{\nu k_T^2}{\Omega}\right)^2 \\ &+\frac{2}{\Omega}\left(\frac{k_r}{k_T}\cot\theta-\frac{k_\theta}{rk_T}\right)\left[\frac{k_r}{k_T}\frac{\partial(\Omega\sin^2\theta)}{\partial\theta}-\frac{k_\theta}{k_T}\frac{\sin^2\theta}{r^2}\frac{\partial(r^2\Omega)}{\partial r}\right] \\ &=0\,. \end{aligned} \tag{39}$$

In a similar fashion we can rewrite equation (34) in spherical coordinates as

$$\begin{aligned} q^2+\Bigg\{&\frac{[(\gamma-1)n-1]}{\gamma\Omega^2}\frac{g_r}{r^2}\frac{d\ln T}{dr}\left(\frac{k_\theta}{k_T}\right)^2 \\ &+\frac{2}{\Omega}\left[\frac{k_r}{k_T}\frac{\partial(\Omega\sin^2\theta)}{\partial\theta}-\frac{\sin^2\theta}{r^2}\frac{k_\theta}{k_T}\frac{\partial(r^2\Omega)}{\partial r}\right]\left(\frac{k_r}{k_T}\cot\theta-\frac{k_\theta}{rk_T}\right)\Bigg\} \\ &=0\,. \end{aligned} \tag{40}$$

© American Astronomical Society • Provided by the NASA Astrophysics Data System

1967ApJ...150..571G

From equation (40) we see that for modes with $k_\theta = 0$, gravity does not provide a stabilizing influence. For these modes

$$q^2 + \frac{2 \cot \theta}{\Omega} \frac{\partial (\Omega \sin^2 \theta)}{\partial \theta} = 0 . \quad (41)$$

Instability of these modes arises whenever the angular momentum per unit mass does not increase outward (from the rotation axis) on a given spherical surface. Equation (41) describes modes whose wavelengths are long enough so that the effects of viscosity and radiation transfer may be neglected. This implies that

$$\frac{\nu k_T^2}{\Omega} < \frac{\chi k_T^2}{\Omega} \ll q \quad (42)$$

for these modes. When there is instability, the exponential growth time is on the order of a rotation period (i.e., $q = O[1]$), and all modes develop at the same rate. The wavelengths of the largest unstable modes will certainly approach, if not exceed, a scale height. Our local analysis breaks down at wavelengths approaching a scale height, so we cannot resolve the question of whether larger modes will be unstable. The large wavelengths and rapid growth rates of the instabilities which arise if $\cot \theta[\partial(\Omega \sin^2 \theta)/\partial\theta] < 0$ imply that this quantity can never be appreciably negative in a star.

Dicke (1964) has proposed a solar model in which the convective zone has been spun down by the solar-wind torque. Because the viscous diffusion of angular momentum is so slow, Dicke concluded that the radiative zone would retain its primordial angular velocity except for a thin transition region (about $\frac{1}{20} R_\odot$ thick) just below the convection zone. It is claimed that viscous stresses acting across the thin transition layer would maintain the slow rotation of the convective envelope against further braking by the solar wind.

On the basis of the application of the Richardson criterion for stably stratified shear flows to the transition layer, Dicke claims stability for his model. However, the Richardson criterion (as used by Dicke) is valid only for adiabatic, inviscid motions. Our criterion for adiabatic, inviscid motions also predicts stability for Dicke's model (at least to axisymmetric disturbances). This conclusion follows directly from equation (40) after we substitute the appropriate numerical values of $\gamma = \frac{5}{3}$, $\partial\Omega/\partial\theta = 0$, $\{[(\gamma - 1)n - 1]/\gamma\Omega^2\} g_r(d \ln T/dr) = 6.9 \times 10^2$ and $-1/r\Omega[d(r^2\Omega)/dr] = 10$. We have chosen to evaluate Ω at the inner boundary of the transition region since this is the most nearly unstable position. Equation (40) then reads:

$$q^2 + \left[\frac{(690 - 20 \sin^2 \theta)}{r^2} \left(\frac{k_\theta}{k_T}\right)^2 + \frac{16 \sin \theta \cos \theta}{r} \frac{k_\theta k_r}{k_T^2} + 4 \cos^2 \theta \left(\frac{k_r}{k_T}\right)^2\right] = 0 . \quad (43)$$

The term in brackets is easily seen to be positive for all values of θ and k_θ/rk_r. This proves that Dicke's model is stable to all axisymmetric, adiabatic, inviscid perturbations.

It is perhaps worth noting that our adiabatic, inviscid, stability criterion is not equivalent to the Richardson criterion. The latter criterion gives a sufficient (but not necessary) condition for the stability of stratified, plane-parallel flows (Miles 1961; Howard 1961). Our criterion is both a necessary and sufficient condition for the stability of differentially rotating flows to axisymmetric disturbances. As such, it gives a necessary (but not sufficient) condition for the stability of differentially rotating flows to arbitrary perturbations. The condition is definitely not a sufficient one since there are known examples of unstable flows for which all axisymmetric perturbations are stable (Howard and Gupta 1962).

We turn next to the application of the complete dispersion relation (eq. [39]) to

© American Astronomical Society • Provided by the NASA Astrophysics Data System

1967ApJ...150..571G

Dicke's model. We will restrict our attention to the equator plane ($\theta = \pi/2$). The numerical values listed before equation (43) and the value of $\chi/\nu = 5.9 \times 10^6$ (eq. [3]), justify neglecting the rotational part of the coefficient of q and the gravitational part of the constant term. Equation (39) becomes

$$q^2 + q \left\{ \frac{[(\gamma-1)n-1]}{\chi k_T^2 \Omega} \frac{g_r}{r^2} \frac{d \ln T}{dr} \left(\frac{k_\theta}{k_T}\right)^2 + \frac{2\nu k_T^2}{\Omega} \right\} + \left[\frac{2}{\Omega r^3} \frac{d(r^2\Omega)}{dr} \left(\frac{k_\theta}{k_T}\right)^2 + \left(\frac{\nu k_T^2}{\Omega}\right)^2 \right] = 0 . \tag{44}$$

In the transition zone, $1/\Omega r[d(r^2\Omega)/dr] < 0$, and we have instability for those modes which satisfy

$$\frac{2}{\Omega r^3} \frac{d}{dr}(r^2\Omega)\left(\frac{k_\theta}{k_T}\right)^2 + \left(\frac{\nu k_T^2}{\Omega}\right)^2 < 0 . \tag{45}$$

For a given total wave vector, $\boldsymbol{k}_T$, the fastest growing mode has $k_\theta = r k_T$. For these modes, the growth rate rises from 0 at

$$\frac{1}{k_{T1}} = \frac{(\nu/\Omega)^{1/2}}{\{(-2/\Omega r)[d(r^2\Omega)/dr]\}^{1/4}}, \tag{46}$$

to approximately

$$q_{1,2} = \left[\frac{-2}{\Omega r} \frac{d(r^2\Omega)}{dr}\right]^{1/2}, \tag{47}$$

a value it maintains almost up to

$$\frac{1}{k_{T2}} = \left[\frac{-2}{\Omega r} \frac{d(r^2\Omega)}{dr}\right]^{1/4} \left\{\frac{\chi\Omega}{[(\gamma-1)n-1]g_r(d \ln T/dr)}\right\}^{1/2} . \tag{48}$$

For $1/k_T \geq 1/k_{T2}$, q decreases as

$$q = q_{1,2}\left(\frac{k_T}{k_{T2}}\right)^2 . \tag{49}$$

In order to obtain some familiarity with the wavelengths and growth rates, we evaluate the expressions given in equations (46)–(48) at the inner edge of the shear layer in Dicke's model and find

$$\lambda_1 = \frac{2\pi}{k_{T1}} = 2.6 \times 10^3 \text{ cm}, \qquad \lambda_2 = \frac{2\pi}{k_{T2}} = 2.8 \times 10^5 \text{ cm}, \qquad P_{1,2} = \frac{1}{q_{1,2}\Omega} = 0.09 \text{ day}. \tag{50}$$

The unstable branch of the dispersion relation (with $k_\theta = r k_T$) is displayed in Figure 1.

Up to this point we have explicitly assumed uniform chemical composition in our analysis. We now relax this restriction and allow the mean molecular weight to be a function of position and time. The rate of diffusion of chemical species will be comparable to the viscous diffusion rate. In order to simplify the analysis, we shall neglect these phenomena. We will, however, retain the thermal diffusivity. This specialization is not a serious one, since, as we have already seen, molecular diffusion is only of significance at the smallest unstable scales. With the above restrictions, the equation for μ becomes

$$\frac{\partial \mu}{\partial t} + \boldsymbol{v} \cdot \nabla \mu = 0 . \tag{51}$$

© American Astronomical Society • Provided by the NASA Astrophysics Data System

1967ApJ...150..571G

The dispersion relation will be derived by using equations (7)–(10) (with ξ and η set equal to zero) and equation (51). The energy equation may be transformed as before by using the identity

$$ds = c_v d\left[\ln \frac{T\rho^{-(\gamma-1)}}{\mu}\right]. \tag{52}$$

The appropriate form of the energy equation is

$$c_v \rho \mu \left(\frac{T}{\mu}\right)\left(\frac{\partial}{\partial t}+v\cdot\nabla\right)\ln\left[\frac{T\rho^{-(\gamma-1)}}{\mu}\right] = \nabla\cdot\left[\rho c_v \chi \nabla\left(\frac{T}{\mu}\cdot\mu\right)\right]. \tag{53}$$

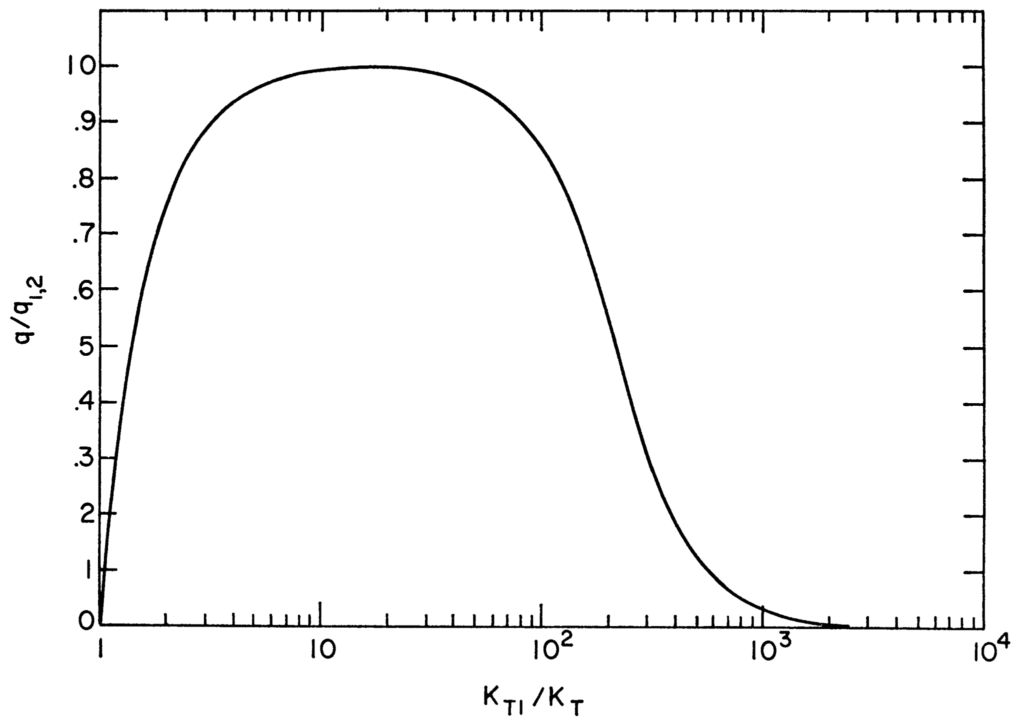

FIG. 1.—The unstable branch of the dispersion relation (eq. [44]). The parameter k_{T2}/k_{T1} has the value 180 (from Dicke's model).

The task of deriving the dispersion relation will be considerably simplified by regarding T/μ and μ as dependent variables rather than T and μ. The proper scaling of μ'/μ and $(T/\mu)'/(T/\mu)$ is given by

$$\frac{\mu'}{\mu}=\frac{1}{k_T H}\frac{\mu'^*}{\mu}, \qquad \frac{(T/\mu)'}{(T/\mu)}=\frac{\Omega^2}{c^2 k_T{}^2}(k_T H)\frac{(T/\mu)'^*}{(T/\mu)}. \tag{54}$$

It is clear that equations (26)–(29) (with ν set equal to zero) are unchanged. Equations (30) and (31) are to be replaced by

$$q\left[\frac{(T/\mu)'}{(T/\mu)}-(\gamma-1)\frac{\rho'}{\rho}\right]+\frac{c^2}{\Omega^2 H}\left\{\frac{\partial \ln[(T\rho^{-(\gamma-1)}/\mu)]}{\partial\varpi}v'_\varpi + \frac{\partial \ln[(T\rho^{-(\gamma-1)}/\mu)]}{\partial z}v'_z\right\} = -\frac{\chi k_T{}^2}{\Omega}\left[\frac{(T/\mu)'}{(T/\mu)}+\frac{\mu'}{\mu}\frac{c^2}{\Omega^2 H^2}\right], \tag{55}$$

$$\frac{\rho'}{\rho}+\frac{(T/\mu)'}{(T/\mu)}=0. \tag{56}$$

© American Astronomical Society • Provided by the NASA Astrophysics Data System

1967ApJ...150..571G

If we now augment equations (26)–(29) (with $\nu = 0$) and (55)–(56) by

$$q\left(\frac{\mu'}{\mu}\right)+Hv'_{\varpi}\frac{\partial \ln \mu}{\partial \varpi}+Hv'_{z}\frac{\partial \ln \mu}{\partial z}=0\,, \tag{57}$$

we have the complete description of the perturbation problem including the effects of varying composition. Note that in equations (55)–(57) the asterisk has been dropped from the scaled variables. The resulting dispersion relation is identical to equation (32) (with $\nu = 0$) except for the addition of

$$\frac{1}{\gamma\Omega^2}\left(\frac{\chi k_z^2}{\Omega}\right)\left(g_{\varpi}+\Omega^2\varpi-\frac{k_{\varpi}}{k_z}g_z\right)\left(\frac{\partial \ln \mu}{\partial \varpi}-\frac{k_{\varpi}}{k_z}\frac{\partial \ln \mu}{\partial z}\right)$$

to the constant term. We observe that the necessary condition for stability (with $\nu/\chi = 0$) now reads

$$\frac{2\Omega}{\varpi}\left[\frac{\partial(\varpi^2\Omega)}{\partial \varpi}-\frac{k_{\varpi}}{k_z}\varpi^2\frac{\partial\Omega}{\partial z}\right]+\left(g_{\varpi}+\Omega^2\varpi-\frac{k_{\varpi}}{k_z}g_z\right)\left(\frac{\partial \ln \mu}{\partial \varpi}-\frac{k_{\varpi}}{k_z}\frac{\partial \ln \mu}{\partial z}\right)>0\,. \tag{58}$$

If $\varpi\Omega^2 \ll g_{\varpi}$ and if the surfaces of constant μ are nearly spherical (as is the case in the Sun), then inequality (58) may be simplified to read

$$2\Omega\left[\frac{k_r}{k_T}\frac{\partial(\Omega\sin^2\theta)}{\partial\theta}-\frac{k_\theta\sin^2\theta}{r^2k_T}\frac{\partial(r^2\Omega)}{\partial r}\right]\left(\frac{k_r}{k_T}\cot\theta-\frac{k_\theta}{rk_T}\right)+\frac{k_\theta^2 g_r}{r^2k_T^2}\frac{d\ln\mu}{dr}>0\,. \tag{59}$$

We see that the stratification of mean molecular weight does not influence the stability of modes with $k_\theta = 0$. Instability of these modes still occurs whenever the angular momentum per unit mass does not increase outward on a given spherical shell. However, if the mean molecular weight increases with depth in the star (i.e., $d\ln\mu/dr < 0$), it does have a stabilizing influence on modes with $k_\theta \neq 0$. For example, suppose the rotation period is a function of r alone and increases from $\frac{1}{2}$ day to 25 days in a layer of thickness $0.05\,R_\odot$ centered at $r = 0.24\,R_\odot$. The most unstable part of the shear layer is at the bottom where the rotation period is $\frac{1}{2}$ day. At this position $g_r/\Omega^2 = -1.0 \times 10^{13}$ cm, $d\ln\mu/dr = -8.0 \times 10^{-12}$ cm^{-1} and $(-2/\Omega r)\,[d(r^2\Omega)/dr] = 4.4$. Substituting these numerical values into inequality (59) we arrive at

$$(80-4.4\sin^2\theta)\left(\frac{k_\theta}{rk_T}\right)^2+0.4(\sin\theta\cos\theta)\frac{k_\theta k_r}{rk_T^2}+4\cos^2\theta\left(\frac{k_r}{k_T}\right)^2>0\,, \tag{60}$$

which is readily seen to be satisfied by all values of θ and k_θ/rk_r. Thus, large differential angular velocities can be stabilized by the composition gradient whish has resulted from hydrogen burning at $r < 0.3\,R_\odot$.

IV. EXPLANATION OF THE DISPERSION RELATION

The physical basis which underlies equations (44)–(49) may be revealed by a heuristic derivation of those equations. The model is the one described at the beginning of § III.

a) Adiabatic, Inviscid Perturbations

Let us imagine an axially symmetric perturbation which consists of the interchange of two equatorial mass rings initially at ϖ_1 and ϖ_2 ($\varpi_2 - \varpi_1 = l \ll \varpi_1, \varpi_2$). The displacements are assumed to be at very subsonic speeds. Thus, during the exchange, the material in each ring will expand or contract adiabatically while maintaining pressure equilibrium with its surroundings. Denoting Lagrangian changes by the symbol Δ, we have

$$\frac{\Delta p}{p}=\gamma\frac{\Delta\rho}{\rho}\,, \tag{61}$$

© American Astronomical Society • Provided by the NASA Astrophysics Data System

1967ApJ...150..571G

for the material in each displaced ring. In order to satisfy the equation of continuity, ring 1 (initially at ϖ_1) must arrive at ϖ_2 with the same volume that ring 2 possessed when it was at ϖ_2 (and vice-versa for ring 2). Thus, the initial volumes (V_1, V_2) of the two rings are related by (ρ_1, ρ_2 are the initial densities in the rings):

$$\rho_1 V_1 = (\rho_1 + \Delta\rho_1) V_2 , \qquad \rho_2 V_2 = (\rho_2 + \Delta\rho_2) V_1 . \tag{62}$$

Setting $\Delta p_1 = -\Delta p_2 = l(\partial p/\partial\varpi)$, and defining $\partial V/\partial\varpi = (V_2 - V_1)/l$ we obtain

$$\frac{1}{V}\frac{\partial V}{\partial\varpi} = -\frac{1}{\gamma p}\frac{\partial p}{\partial\varpi} . \tag{63}$$

During the exchange the rings will conserve their individual angular momenta (since viscosity is neglected and the motion is axisymmetric). The change in kinetic energy (per unit volume of ring) is

$$\text{K.E.} = \frac{p^{1/\gamma}}{\varpi^3}\frac{\partial}{\partial\varpi}\left(\frac{\rho\varpi^4\Omega^2}{p^{1/\gamma}}\right) l^2 . \tag{64}$$

The change in potential energy (per unit volume of ring) is given by P.E. $= -g_\varpi(\rho_1 V_1 - \rho_2 V_2)l/V_1$, which yields

$$\text{P.E.} = -g_\varpi \rho \frac{\partial}{\partial\varpi}\left(\ln\frac{p^{1/\gamma}}{\rho}\right) l^2 . \tag{65}$$

We would expect that the necessary and sufficient condition for stability under axisymmetric perturbations would be

$$\text{K.E.} + \text{P.E.} > 0 . \tag{66}$$

Indeed, it may be shown rigorously that this is in fact the case (without recourse to a local analysis). We may rewrite inequality (66) (using eq. [10], [64], and [65]) as

$$-\frac{(g_\varpi + \varpi\Omega^2)}{\gamma\Omega^2}\frac{\partial \ln\{T\rho^{-(\gamma-1)}\}}{\partial\varpi} + \frac{2}{\Omega\varpi}\frac{\partial(\varpi^2\Omega)}{\partial\varpi} > 0 . \tag{67}$$

In this form, the heuristic stability criterion is seen to be identical with that obtained from the local analysis (eq. [34]). Note that in deriving the criterion (67) we implicitly assumed that the perturbation did not alter the background pressure field. Thus we set $\Delta p_1 = l(\partial p/\partial\varpi)$, where p is the unperturbed pressure. In the exact analysis (unpublished), we found that for the marginally stable disturbances the pressure perturbation vanished, thus assuring the validity of our heuristic result.

b) Modifications Due to Radiative Transfer and Viscosity

The heuristic stability criterion derived previously is based on the assumption that displaced fluid elements move adiabatically and inviscidly. Of course, this assumption must fail for sufficiently small elements because of the smoothing of temperature fluctuations by radiative transfer. For the moment, we shall continue to neglect viscous effects since the diffusion of momentum is so much slower than the diffusion of heat.

Once again we shall derive an expression for the change in energy due to the exchange of two fluid rings at different equatorial radii. Because we are concerned with the effects of radiative transfer, we must specify both the linear dimensions of the rings and the time scale of their interchange. In order to simplify the comparison with the results of our local analysis we shall describe the ring by a characteristic dimension k_T^{-1}. We shall also assume that the vertical height of the ring is much shorter than its radial extent. This is in keeping with our prior specialization to disturbances with $k_\theta = rk_T$ (cf. the

© American Astronomical Society • Provided by the NASA Astrophysics Data System

1967ApJ...150..571G

sentence following eq. [45]). For the characteristic time scale of interchange, τ_1, we cheat a bit and choose

$$\tau_1 = \left[-\frac{2\Omega}{\varpi} \frac{\partial (\varpi^2 \Omega)}{\partial \varpi} \right]^{-1/2} . \tag{68}$$

This choice is motivated by our knowledge of the growth rate of instabilities in the incompressible inviscid case (cf. eq. [35], with $k_z = k_T$). The rationale is that we certainly expect a compressible flow to contain all the modes available to its incompressible counterpart. Therefore, when it is energetically favorable, we would expect the compressible flow to develop instabilities on the time scale given by τ_1. The smoothing time for temperature fluctuations (relative to the unperturbed background) is given by

$$\tau_2 = (\chi k_T{}^2)^{-1} . \tag{69}$$

We are concerned with motions for which $\tau_1 \gg \tau_2$. If we denote the ratio τ_1/τ_2 by R, we can write

$$\text{K.E.} = \frac{\rho}{\varpi^3} \frac{\partial (\varpi^4 \Omega^2)}{\partial \varpi} l^2 , \tag{70}$$

to lowest order in R^{-1}. Equation (70) is obtained in the same way as equation (64), except that equation (63) must be replaced by $1/V(\partial V/\partial \varpi) = -1/\rho(\partial \rho/\partial \varpi)$, a result which follows if during the displacements, the rings are assumed to instantaneously adjust to the background temperature. Furthermore, we would expect the value of P.E. to be smaller by a factor of about R than the value given in equation (65). This follows because the temperature of the displaced ring will approximately have the value appropriate to its surroundings at a time τ_1/R in the past. A more accurate approximation yields a reduction of P.E. by a factor of R/γ. The extra factor of γ^{-1} arises because in addition to adjusting its temperature, a ring will do work as a result of pressure forces acting during its displacement. This factor may be deduced from equations (30) and (31) by those so inclined. Otherwise, it should just be regarded as an adjustment factor used to arrive at the correct result. In any case, we set

$$\text{P.E.} = -\frac{\gamma g_\varpi \rho}{R} \frac{\partial}{\partial \varpi} \left(\ln \frac{p^{1/\gamma}}{\rho} \right) l^2 . \tag{71}$$

Again, we assume that the necessary and sufficient condition for stability is P.E. + K.E. > 0. If we make use of equations (10) and (68)–(71), we can derive an expression for the largest unstable mode by setting P.E + K.E. = 0. We find

$$\frac{1}{k_{T2}} = \left[\frac{-2}{\Omega \varpi} \frac{\partial (\varpi^2 \Omega)}{\partial \varpi} \right]^{1/4} \left[\frac{\chi \Omega}{-\gamma g_\varpi (\partial/\partial \varpi)(\ln p^{1/\gamma}/\rho)} \right]^{1/2} , \tag{72}$$

which is identical with equation (48) if we change ϖ to r and introduce the quantity n.

The instability we have been studying requires that the angular momentum in each ring be conserved during its displacement. Hence, disturbances of sufficiently small scale will be stable, because for them, the viscous diffusion time $(\nu k_T{}^2)^{-1}$ will be shorter than τ_1. The critical scale, below which there is stability, is derived by equating τ_1 and $(\nu k_T{}^2)^{-1}$. It is

$$\frac{1}{k_{T1}} = \left\{ \frac{(\nu/\Omega)^2}{(-2/\Omega \varpi)[\partial (\varpi^2 \Omega)/\partial \varpi]} \right\}^{1/4} , \tag{73}$$

an expression given previously by equation (46) (with r in place of ϖ).

According to the present heuristic analysis, there is a maximum unstable scale given by equation (72). The existence of a maximum scale results from the assumed value of

© American Astronomical Society • Provided by the NASA Astrophysics Data System

1967ApJ...150..571G

the interchange time τ_1. However, our local analysis has shown that disturbances with $(k_T)^{-1} > (k_{T2})^{-1}$ are also unstable because these modes have characteristic growth times larger than our assumed value of τ_1 by a factor $(k_{T2}/k_T)^2$ (cf. eq. [49]). We did not anticipate these more slowly growing instabilities when we assumed τ_1 to be given by equation (68).

We would be remiss if we failed to mention a thermohaline instability which resembles in many respects the rotational instabilities that we have been analyzing. Imagine the situation in which a layer of warm salty water lies on top of cold fresh water. Even if the mean density field is stable (i.e., the density increases downward) this physical system may be unstable. The instability arises from the fact that the molecular diffusivity of heat is much greater than the diffusivity of salt. As first pointed out by Stern (1960); "If a parcel of small radius is given a vertical displacement it will lose its temperature excess much more rapidly than its salinity excess and the resulting buoyant force may be sufficient to maintain a free convection despite the fact that the density field increases in the direction of gravity." If we substitute angular momentum for salinity in this last sentence, we have a description of the rotational instabilities. The connection between the shear and thermohaline instabilities was realized by Lieber and Rintel (1962). It is worth noting that in laboratory experiments the modes that develop are those for which the linear analysis predicts the maximum growth rate. The unstable disturbances are in the form of long thin columns, commonly called salt fingers. The small horizontal dimension is required in order to allow rapid heat transfer between the salt finger and its surroundings. However, a non-zero lower bound on the horizontal dimension of the fastest growing mode is set by viscosity which impedes the instability.

There is an important lesson to be drawn from this thermohaline instability. Over much of the world's oceans, conditions (i.e., the temperature and salinity variation with depth) appear to be favorable for the growth of salt fingers with the subsequent formation of an isohaline layer. Nevertheless, the instability does not arise. It is now believed that externally driven motions disrupt the salt fingers before they have a chance to grow (Veronis 1965). We might expect this to happen if the motions transported fluid over distances larger than the horizontal extent of a salt finger in the time it requires to develop.

We might inquire whether a similar mechanism could act to suppress the growth of rotational instabilities in stars. The prime candidate would appear to be meridional currents which are driven by the non-vanishing divergence of the luminous flux. However, in order to prevent the growth of instabilities with $k_T = k_{T2}$, the meridional circulation speed would have to exceed

$$\Omega\left[\frac{-2}{\Omega r}\frac{d(r^2\Omega)}{dr}\right]^{3/4}\left\{\frac{\chi\Omega}{[(\gamma-1)n-1]g_r(d\ln T/dr)}\right\}^{1/2}.$$

At the bottom of the transition layer in Dicke's model, this speed $(\Omega q_{1,2}k_{T2}^{-1})$ is 13 cm/sec. (cf. eq. [50]), which is several orders of magnitude faster than the speed of the meridional currents (Schwarzschild 1958).

V. TURBULENT DIFFUSION IN DIFFERENTIALLY ROTATING STARS

We have completed our investigation of the linear stability of differentially rotating stars and now go on to study the motions in the unstable regions of these stars. Our main concern is to obtain some estimate of the turbulent diffusion rates. For simplicity, we shall limit most of our discussion to the equatorial plane of a nearly spherical star. All our numerical applications will be made to differentially rotating solar models. Unless otherwise stated, we assume constant chemical composition.

We found previously (cf. eqs. [46]-[49]) that wherever the angular momentum per unit mass is decreasing outward, all axisymmetric perturbations for which $k_T < k_{T1}$

© American Astronomical Society • Provided by the NASA Astrophysics Data System

1967ApJ...150..571G

are unstable. These are the disturbances that are able to exchange heat with their surroundings while retaining their angular momentum. We would expect disturbances (or eddies) of all scales $> k_{T1}^{-1}$ to be present in the fully developed turbulent flow. It is misleading to picture a disturbance with wave number k_T as being of size k_T^{-1}. Only its shortest dimension need be of this order. Its other dimensions may be much larger. Indeed, for a given value of k_T, we have seen that the fastest growing linear disturbance has $k_\theta = rk_T$ and hence $k_r = 0$. This is understandable, since only one dimension of an eddy need be small in order to permit a rapid exchange of heat with its surroundings. (Recall the shape of the salt fingers described at the end of the preceding section.) The longer dimensions of the eddies will undoubtedly approach, if not exceed, the local scale height. As long as one dimension of an eddy $\simeq k_{T2}^{-1}$, its circulation period will be of order $(q_{1,2}\Omega)^{-1}$. This time scale follows from the arguments based on energetic considerations (presented in § IV). Such eddies will produce very rapid diffusion in the unstable regions of a star. Complete mixing of the Sun should not take very much longer than $(R_\odot/H)^2$ rotation periods or approximately 10 years (using the surface rotation period of 25 days). It would thus appear to be impossible for the Sun to maintain regions where the angular momentum per unit mass decreased significantly outward. The sole exception to this is the possibility of maintaining large shears in regions where there is a stabilizing gradient of mean molecular weight. We have seen that the μ gradient produced by the transmutation of hydrogen into helium in the core of the Sun could stabilize large shears (cf. ineq. [59]). However, we shall argue against this possibility in § VI.

In addition to transporting momentum, turbulence will also transport energy. In the radiative interior, the turbulent transfer of energy will be directed toward the center of the Sun. This arises because the ascending material in the turbulent flow will, on the average, be cooler than its surroundings, whereas the descending material will be hotter. The requisite net outward transport of energy in the Sun will be maintained by a slight increase in the temperature gradient, which in turn will increase the outward radiative flux.

VI. "SPIN DOWN" AND LITHIUM DEPLETION IN STARS

How is the interior of a star affected by an external torque applied to its surface? This question is an important one since, based on current estimates of the angular momentum lost to the solar wind (Dicke 1964; Brandt 1966; Weber and Davis 1967), the Sun's angular momentum has a half-life of about 5×10^9 years (if we assume uniform rotation throughout).

Dicke has concluded that rotational braking only occurs in the convective envelope of the Sun since molecular viscosity cannot transmit the solar-wind torque to the radiative interior. We have shown that the differentially rotating transition layer in Dicke's model is unstable. Furthermore, the rapidity of turbulent diffusion in unstable regions makes it impossible for the angular momentum to significantly decrease outward in the Sun (except in the inhomogeneous inner core).[1]

There is yet another mechanism (besides molecular and turbulent diffusion) by which angular momentum may be brought out from the radiative interior of stars. This mechanism, which we refer to as "spin-down," has recently been studied both theoretically and in the laboratory (Greenspan and Howard 1962). Imagine a glass sphere (radius a) filled with liquid. The system is in uniform rotation with angular velocity Ω. If the spin rate is decreased slightly, a viscous layer (the Ekman layer) of thickness $\sim(\nu/\Omega)^{1/2}$ forms (within a few rotation periods) at the inner surface of the sphere. Within this Ekman layer there is a net transport of fluid toward the poles of the sphere. The interior flow is essentially inviscid. Each cylindrical shell of fluid conserves its angular momentum and expands radially, thus decreasing its rate of rotation. In order to conserve mass, there is an efflux of fluid from the boundary layer into the interior region. Fluid from the interior

[1] See footnote to Abstract.

© American Astronomical Society • Provided by the NASA Astrophysics Data System

1967ApJ...150..571G

enters the boundary layer in the neighborhood of the equator. After a characteristic time of order

$$T=\frac{a}{\sqrt{(\nu\Omega)}}, \tag{74}$$

the interior fluid will have adjusted its rotation rate to match that of the container. If the container's rotation rate is changed by $\epsilon\Omega$ ($\epsilon \ll 1$), then the ring of interior fluid initially at a distance ϖ from the rotation axis will be displaced an additional distance $\epsilon\varpi/2$. If the container's angular velocity is steadily decreasing, the rotation rate at the center of the sphere will follow behind the surface value with a time lag of approximately T (for $T \ll \Omega/[d\Omega/dt]$).

It is a great extrapolation to apply the results derived for rigidly contained spheres of liquid of uniform density to the interior of the Sun. The lack of a rigid container (for the Sun) is not very important since turbulent viscosity in the convective envelope will transmit the solar-wind torque to the top of the radiative zone. The stresses acting there may cause an Ekman layer to form, although penetrative convection would certainly complicate its structure. The essential question is whether an Ekman layer could circulate fluid against the stable thermal stratification in the radiative interior of the Sun. We cannot give a definite answer to this question at present. Certainly, radiation transfer will at least partially alleviate the burden of pumping fluid against the stable stratification. Barcilon and Pedlosky (1967) have recently investigated the problem of flow in a rotating, stratified, Boussinesq liquid. The liquid was confined by a cylindrical vessel whose axis coincided with the rotation axis. The stratification and the gravitational acceleration were in the direction of the axis of rotation. They assumed that the Froude number and the Rossby number were both much less than the Ekman number, and found that the motion is similar to that of a homogeneous fluid only if the ratio of the spin-down time $R_\odot/(\nu\Omega)^{1/2}$ to the cooling time $(R_\odot^2/\chi)$ is much greater than the ratio of the gravitational force to the centrifugal force. Otherwise, the Ekman layers disappear and spin-down is replaced by viscous diffusion. If these results are applicable to the Sun, it appears that laminar spin-down will not work there. Had Barcilon and Pedlosky not made the approximation concerning the Froude number, they would have found meridional circulations to be present. For the Sun the meridional currents would be more important than viscous diffusion in transferring angular momentum. Also, the inclusion of the non-linear terms in the equations of motion may make it possible for smaller elements of fluid, for which radiative transfer is much faster, to be drawn into and pushed out of the boundary layer. In any case, spin-down would not work for the inner core of the Sun because of its stabilizing gradient of mean molecular weight. If spin-down does occur, an additional Ekman layer would form at the core surface and a detached shear layer would be present along the cylindrical surface which is tangent to the core and parallel to the rotation axis.[2]

If we calculate the spin-down time for the Sun, using the kinematic microscopic viscosity behind the convective zone and the surface angular velocity, we find that

$$T = 2 \times 10^5 \text{ years},$$

a result which could have been larger had we accounted for the variation of density with radial position. This time is dependent upon the Ekman layer remaining laminar. If it becomes turbulent, T is reduced.

The transfer of angular momentum, by either turbulent viscosity or the phenomenon of spin-down, involves the mixing of material behind the convective zone. An inevitable consequence of the mixing will be the depletion of lithium and beryllium in stellar at-

[2] *Note added in proof:* Since this paper was written, the phenomenon of solar spin-down has been discussed by L N. Howard, D. W Moore, and E. A Spiegel (1967, *Nature*, **214**, 1297).

© American Astronomical Society • Provided by the NASA Astrophysics Data System

1967ApJ...150..571G

mospheres. The mixing would decrease the ratios of Li^6/Li^7 and Li^7/Be^9 since Li^6, Li^7, and Be^9 are destroyed in consecutively smaller fractions of the star. Presumably, the Li^6/Li^7 ratio would not decrease as fast as the Li^7/Be^9 ratio since the critical temperature for the destruction of Li^7 (2.4×10^6 ° K) is closer to that for Li^6 (2.0×10^6 ° K) than to that for Be^9 (3.2×10^6 ° K).

We see, contrary to current beliefs, that material may be mixed in the radiative interiors of stars (except where there is a strong stabilizing gradient of mean molecular weight). Thus it is not necessary to appeal to penetrative convection or unknown sources of opacity (which would deepen the convective envelope) in order to understand how Li and Be might be depleted in main-sequence stars of solar type (Sears and Weymann 1965; Herbig 1965; Conti and Danziger 1966). From the presence of beryllium (and possibly lithium) in the solar photosphere, we may infer that the solar angular momentum has not decreased drastically (by orders of magnitude) since the Sun arrived on the main sequence (unless Li and Be have been formed since that time). Thus, it appears unlikely that the inner core of the Sun is in fact rotating much more rapidly than the surface even though such a rotation could be stable.[3]

We are grateful to Drs. F. H. Busse, A. P. Ingersoll and W. V. R. Malkus for many helpful discussions. One of us, Peter Goldreich, is indebted to NASA for support through Nsg (T-37).

REFERENCES

Barcilon, V., and Pedlosky, J. 1967, *J. Fluid Mech* , **29**, 1.
Brandt, J. C. 1966, *Ap J.*, **144**, 1221.
Chandrasekhar, S. 1961, *Hydrodynamic and Hydromagnetic Stability* (London: Oxford University Press).
Conti, P. S., and Danziger, I J. 1966, *Ap J.*, **146**, 383.
Dicke, R. H 1964, *Nature*, **202**, 432.
———. 1967, *Phys. Rev. Letters* (in press).
Eddington, A S 1929, *M N R A S* , **90**, 54.
Greenspan, H P., and Howard, L. N. 1963, *J. Fluid Mech* , **17**, 385.
Herbig, G. M. 1965, *Ap. J* , **141**, 588.
Howard, L N. 1961, *J. Fluid Mech* , **10**, 509.
Howard, L N., and Gupta, A. S. 1962, *J. Fluid Mech.*, **14**, 463.
Jeans, J. H. 1928, *Astronomy and Cosmogony* (Cambridge: Cambridge University Press).
Landau, L. D , and Lifshitz, E M. 1959, *Fluid Mechanics* (London: Pergamon Press).
——— 1958, *Statistical Physics* (London: Pergamon Press).
Lieber, P., and Rintel, L 1962, *Trans A G.U.*, **43**, 430
Miles, J. W. 1961, *J. Fluid Mech* , **10**, 496.
Schwarzschild, K. 1906, *Gesellschaft der Wiss., Göttingen Nach Math -Phys Kl* , **1**, 41
Schwarzschild, M. 1958, *Structure and Evolution of the Stars* (Princeton, N J : Princeton University Press).
Sears, R. L., and Weymann, R. 1965, *Ap J* , **142**, 174
Spitzer, L , Jr. 1962, *Physics of Fully Ionized Gases* (2d ed ; New York: Interscience).
Stern, M. E. 1960, *Tellus*, **12**, 172.
Thomas, L H. 1930, *Quart. J. Math* , **1**, 239.
Veronis, G. 1965, *J. Marine Res* , **23**, 1.
Weber, E. J., and Davis, L , Jr 1967, *Ap J* , **148**, 217.
Weymann, R 1957, *Ap J.*, **126**, 208

[3] See footnote to Abstract.

Copyright 1967 The University of Chicago Printed in U S.A.

© American Astronomical Society • Provided by the NASA Astrophysics Data System

1967ApJ...150..571G

© American Astronomical Society • Provided by the NASA Astrophysics Data System