ICARUS 69, 157–175 (1987)

On the Physics of Resonant Disk–Satellite Interaction

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Received June 6, 1986; revised September 15, 1986

Within the framework of a single derivation, we study the transfer of angular momentum in a disk subjected to a linear perturbation at Lindblad resonance, whenever the physics include friction, nonstationarity, or self-gravitation, pressure, and viscosity. Each of the above physical processes can be described by one parameter which indicates the main physics at work and the resonance width. We show that dissipation or waves are not formally necessary for a torque to appear, but only for the problem to remain stationary and/or linear. In this framework, the torque exerted at an isolated resonance is independent of the particular physics at work. We consider applications to numerical simulations, the impulse approximation, planetesimal accretion, and edges and gaps in planetary rings. © 1987 Academic Press, Inc.

I. INTRODUCTION

Resonance effects between a disk and a satellite orbiting a massive body can dramatically shape the structure and evolution of the disk. Such effects are found in different objects such as galaxies perturbed by a central bar, accretion disks in close binary systems, planetary rings perturbed by satellites, or the asteroid belt subjected to Jupiter's influence. These systems share common dynamics, even though their physics and spatial scales are very different.

The aim of this paper is to present in a general way the exchange of angular momentum (torque) between the "satellite" and the disk in a typical case of resonances, namely the Lindblad resonances which occur when the mean motion of the disk particles and that of the satellite are in the ratio $m/(m \mp 1)$, where m is a positive integer.

Our starting point lies in some previous studies of Lindblad resonances for various astrophysical objects and in the connection of such problems to plasma physics. An intriguing point is that the expression of the torque always bears the same form in spite of the very different physics at work (Goldreich and Tremaine, 1982). To quote a few examples, we can think of galactic dynamics (Lynden-Bell and Kalnajs, 1972), accretion disks (Lin and Papaloizou, 1979), planetary rings (Goldreich and Tremaine, 1979), and numerical simulation (Sicardy, 1985). Also, the sign of this torque is always the same, so that the satellite always repels the disk. The torque per unit of surface density does *not* depend on the physical parameters of the disk, but only on the mass of the satellite, the mass of the central body, and the order m of the resonance (cf. Eq. (16) of this paper).

Greenberg (1983) proposed to answer that puzzle by assuming that the particles are subjected to friction effects which suppress the singularity at exact resonance. As the damping decreases, the perturbed motion of the particle increases but the width of the resonance shrinks, so that the two effects compensate and the torque is independent of the damping coefficient.

However, other mechanisms such as density waves (Goldreich and Tremaine, 1979) can create a torque without explicit or implicit dissipation. We show in this paper that a wide class of physical processes can actually displace the resonance frequency in the complex plane, so that the torque density function (torque per unit radius) has

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0019-1035/87 $3.00

a complex pole. We will characterize the distance of the pole (i.e., the resonance) to the real frequency axis by a dimensionless coefficient α. The torque, computed through contour integration, is then independent of α, i.e., of the numerical value of the physical parameters describing the disk. If α is small, i.e., the coupling between the particles is weak, then the perturbed motion remains large and the resonance is narrow. If α is large (strong coupling), the perturbed motion is reduced but the resonance broadens. Thus $1/\alpha$ will characterize the perturbed motion of the particles and α will set the width of the resonance. The sign of the torque is directly linked to that of α: we will see that this sign is actually a consequence of dissipation of energy, or of causality effects when no dissipation is present.

Thus, the key point in the resonant perturbation is that the disk is equivalent to a continuous network of oscillators, one of which is in resonance with the satellite. This problem is well known in other contexts (cf. Meyer-Vernet, 1984). A classical result in "circuit theory" is that a continuous network of *undamped* oscillators exhibits *resistive* properties if the excitation frequency lies within the continuous range of resonant frequencies of the oscillators. This is not a paradox, because in the absence of explicit losses in the circuit, the problem is not stationary: introducing small resistances serves only to achieve a stationary problem (by dissipating the energy) but does not change the total resistance (cf. Buneman, 1961).

Many similar problems are encountered in plasma physics, like the resonant absorption in an inhomogeneous plasma (Crawford and Harker, 1972), or the linear Landau damping. As is well known in the latter case, the *linearized* Vlasov–Poisson equations yield plasma wave damping, though the system is apparently lossless. The classical solution to this paradox is the same as above. Indeed, when the explicit dissipation is insufficient to prevent the system from becoming nonlinear, the damping reverses and finally disappears (O'Neil, 1965). This analogy between the resonant transfer of angular momentum in disks and Landau damping has been used by Lynden-Bell and Kalnajs (1972).

This paper is organized as follows: in Section II, the torque is derived and validity conditions are set. In Section III, we examine the effect of friction and nonstationarity. The general case when pressure, viscosity, and self-gravity are present simultaneously is studied in Section IV. Some numerical applications are made in Section V and the conclusions are drawn in Section VI.

II. DERIVATION OF THE TORQUE

(*1*) *The Equations of the Problem*

We present here a simplified version of the torque derivation first studied by Lynden-Bell and Kalnajs (1972) in the context of galactic disks, and subsequently by Goldreich and Tremaine (1979) for planetary rings. We consider the classical infinitesimally thin, pressureless, and inviscid fluid disk, whose evolution is described by the momentum and continuity equations

$$\left(\frac{\partial}{\partial t} + \mathbf{U} \cdot \boldsymbol{\nabla}\right) \mathbf{U} = -\boldsymbol{\nabla}(\phi_P + \psi_S)$$

$$\frac{\partial}{\partial t}(\Sigma) + \boldsymbol{\nabla} \cdot (\Sigma \mathbf{U}) = 0 \tag{1}$$

where $\mathbf{U}(r, \theta, t)$ is the fluid velocity at point $(r, \theta, 0)$ and time t, in cylindrical coordinates (r, θ, z). $\Sigma(r, \theta, t)$ is the surface density of the disk, and ϕ_p and ψ_s, respectively, are the potential of the central planet and the perturbing satellite.

The system (1) is linearized around the unperturbed quantities (indexed by zero). The planet's potential is assumed to be spherically symmetric: i.e., $\phi_p = -GM_p/r$, where G is the gravitational constant and M_p the planet's mass. Finally, the satellite's orbit is taken as circular, so that

$$\psi_s(r, \theta, t) = \sum_{m=-\infty}^{+\infty} \psi_m(r) \exp[im(\theta - \omega_s t)] \quad (2)$$

where $\omega_s = (M_p G/a_s^3)^{1/2}$ is the satellite orbital frequency and

$$\psi_m(r) = -(GM_s/2a_s) b_{1/2}^{(m)}(r/a_s). \quad (3)$$

Here, M_s is the satellite's mass, a_s the satellite's orbit radius, and $b_{1/2}^{(m)}$ (m integer) is the classical Laplace coefficient (see Shu, 1984)

$$b_\gamma^{(m)}(\beta) = \frac{2}{\pi} \int_0^\pi \cos(m\theta)/(1 + \beta^2 - 2\beta\cos\theta)^\gamma d\theta. \quad (4)$$

Taking into account the oblateness of the planet would slightly displace the location of the Lindblad resonances studied below (Shu, 1984). On the other hand, considering an inclined or eccentric satellite orbit would create corotation resonances (Goldreich and Tremaine, 1982). Although sometimes important, these two effects will not be considered in this paper in order to enlighten more simply the main physics of the problem.

The linearized system (1) reads

$$\left(\frac{\partial}{\partial t} + \Omega\frac{\partial}{\partial\theta}\right) u_r - 2\Omega u_\theta = -\frac{\partial}{\partial r}(\psi_s)$$

$$\left(\frac{\partial}{\partial t} + \Omega\frac{\partial}{\partial\theta}\right) u_\theta + \frac{\Omega}{2} u_r = -\frac{1}{r}\frac{\partial}{\partial\theta}(\psi_s)$$

$$\left(\frac{\partial}{\partial t} + \Omega\frac{\partial}{\partial\theta}\right) \sigma = -\frac{\Sigma_0}{r}\left[\frac{\partial}{\partial r}(ru_r) + \frac{\partial u_\theta}{\partial\theta}\right] \quad (5)$$

where u_r, u_θ, and σ are the perturbations of the radial velocity, tangential velocity, and surface density, respectively.

In the *stationary* case, each perturbed quantity takes the same form as the forced periodic perturbation of the satellite given by Eq. (2). Then $\partial/\partial t = -im\omega_s$ and $\partial/\partial\theta = im$, and Eqs. (5) yield

$$u_{rm}(r) = -\frac{im}{rD}\left[(\Omega - \omega_s) r\frac{d}{dr} + 2\Omega\right]\psi_m(r)$$

$$u_{\theta m}(r) = \frac{1}{2rD}\left[\Omega r\frac{d}{dr} + 2m^2(\Omega - \omega_S)\right]\psi_m(r)$$

$$\sigma_m(r) = \frac{-\Sigma_0}{imr(\Omega - \omega_s)]}\left[\frac{d}{dr}(ru_{rm}) + imu_{\theta m}\right] \quad (6)$$

where $D(r) = \Omega^2 - m^2(\Omega - \omega_s)^2$, and $\Omega(r) = (M_p G/r^3)^{1/2}$ is the orbital frequency at radius r.

$D(r) = 0$ defines Lindblad resonance radii a_m at which $\Omega_m = m/(m \mp 1)\omega_s$. There, the perturbation frequency felt by a fluid element, $\Omega - \omega_s$, is an integer submultiple of the natural oscillation frequency Ω of the element. The behavior of the disk close to a Lindblad resonance is thus similar to the forced motion of a harmonic oscillator close to resonant condition.

By symmetry, only $m > 0$ needs to be considered hereafter, so that $\Omega_m = m/(m - 1)\omega_s$ (resp., $\Omega_m = m/(m + 1)\omega_s$) will define, respectively, an *inner* or *outer* Lindblad resonance. For purposes of conciseness, in all the formulas the *upper* (*lower*) sign will refer to *inner* (*outer*) Lindblad resonance.

The torque exerted by the satellite on the disk is $\Gamma = -\iint_{\text{Disk}} d^2r(\mathbf{r} \times \nabla\psi_s)\,\Sigma$. After linearization and from Parseval's theorem this yields, in terms of the Fourier components defined in Eq. (2), $\Gamma = \sum_{m=1}^{\infty} \Gamma_m$ with

$$\Gamma_m = -4\pi m \,\mathrm{Im}\left(\int_0^\infty dr r\sigma_m\psi_m\right) \quad (7)$$

(where Im denotes the imaginary part).

Close enough to a resonance (the term "close" being quantitatively specified below), the system (6) is degenerate and $u_{\theta m} \sim \pm(i/2)u_{rm}$, where

$$u_{rm} \sim -i\Omega_m A/a_m D \quad (8)$$

and $A(r) = 2m\psi_m \pm a_m\dot{\psi}_m$, the dot standing for the space derivative d/dr. Thus,

$$A(r) = -(GM_s/2a_s)\left[2mb_{1/2}^{(m)} \pm \beta\frac{db_{1/2}^{(m)}}{d\beta}\right]_{\beta=r/a_s}. \quad (9)$$

Equations (6) to (8) then yield, after integration by parts,

$$\Gamma_m = 4\pi \Sigma_0 \operatorname{Re}\left(\int dr \left[r \frac{d}{dr}\left(\frac{\psi_m}{\Omega - \omega_s}\right) \pm \frac{m}{2}\frac{\psi_m}{\Omega - \omega_s}\right] u_{rm}\right). \quad (10)$$

Assuming that u_{rm} varies sufficiently rapidly near the resonance so that the term in brackets can be taken out of the integral with $\Omega \sim \Omega_m$, Eq. (10) yields

$$\Gamma_m = \left(4\pi m \Sigma_0 A/\Omega_m\right) \operatorname{Re}\left(\int dr u_{rm}\right). \quad (11)$$

Indeed, we will see that in many astrophysical applications, the main contribution to the integral of Eq. (11) is furnished in a narrow region around the resonance.

Using for the moment the expression of u_{rm} given in Eq. (8), Eq. (11) yields

$$\Gamma_m = 4\pi m \Sigma_0 A^2/(a_m \mathscr{D}(a_m)) \operatorname{Im}\left(\int \frac{dx}{x}\right) \quad (12)$$

where $\mathscr{D} = dD/dr$ and $x = (r - a_m)/a_m$ is a dimensionless measure of the distance from a resonance.

The integral of Eq. (12) is not defined; even avoiding the singular point $x = 0$ and defining the integral as a principal value, which is not justified, it would yield $\Gamma_m = 0$. This is because all the streamlines in the disk are aligned with the satellite, so that no torque is possible for obvious reasons of symmetry.

Indeed from Eqs. (6), the polar equation $\rho(\theta)$ for a streamline close to a resonance is

$$\rho = a_m \left\{1 \pm \frac{2}{a_m\Omega_m}[\operatorname{Re}(u_{rm}) \sin m(\theta - \omega_s t) + \operatorname{Im}(u_{rm}) \cos m(\theta - \omega_s t)]\right\}. \quad (13)$$

Thus, if u_{rm} is purely imaginary (as in Eq. (8)), ρ is in phase with the perturbing potential: $2\psi_m \cos m(\theta - \omega_s t)$.

In order to eliminate the singularity in Eq. (12), let us assume that ω_s has a small imaginary part, i.e., $\omega_s^c = \omega_s(1 + i\alpha')$. As will be shown in the next sections, this can be justified by taking properly into account the actual physics of the problem. This yields in turn an imaginary part of D and thus a phase lag between the streamlines and the perturbing potential ψ_s (Eqs. (8) and (13)). Equivalently, to first order in $\alpha = (2/3)\alpha'$, x is replaced by $x_c = x + i\alpha$. Thus

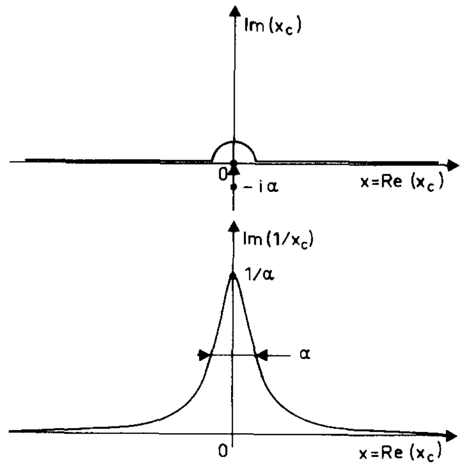

FIG. 1. (Top) A small imaginary part α in the satellite frequency displaces the resonance (i.e., the pole of the function $u_{rm}(x)$) out of the real axis, so that singularity is avoided at exact resonance ($\operatorname{Re}(x_c) = 0$). (Bottom). The torque density, proportional to $\operatorname{Im}(1/x_c)$ is well defined, with a width $\Delta x \sim \alpha$ and a height proportional to $1/\alpha$, so that the total area (the torque) is independent of α. This holds as long as the width and the height both remain small enough. See Section II for details.

$$\operatorname{Im}\left(\int \frac{dx}{x_c}\right) = -\int \frac{\alpha dx}{x^2 + \alpha^2} = -\pi \operatorname{sgn}(\alpha). \quad (14)$$

As shown in Fig. 1, the introduction of a small imaginary part allows one to avoid the singularity at resonance, and the total area under the bell-shaped curve $\operatorname{Im}(1/x_c)$ is independent of α. This yields

$$\Gamma_m = -4\pi^2 m \Sigma_0 A^2/[a_m \mathscr{D}(a_m)] \operatorname{sgn}(\alpha) \quad (15)$$

For Keplerian disks $\mathcal{D}(a_m) = \pm 3m\Omega_m\omega_s/a_m$, so that

$$\Gamma_m = \mp 4\pi^2 \Sigma_0 A^2/(3\Omega_m\omega_s) \operatorname{sgn}(\alpha). \quad (16)$$

This is identically the formula found by Goldreich and Tremaine (1979, Eq. (46)) for the torque exerted by a satellite on a spiral density wave pattern, except that it is derived more directly and that it includes the term sgn(α). We shall see that physical considerations lead us to choose sgn(α) = +1, so that the torque is always negative (positive) at inner (outer) Lindblad resonance.

(2) Conditions of Validity

Several approximations have been made up to now. We will examine them and set some limits of validity for the torque formula (16).

Linearization. The perturbations must remain small compared to the nonperturbed values. Equations (6), (8), (9) yield, near the resonance,

$$|\sigma_m| / \Sigma_0 = \frac{1}{6}\left(\frac{m}{m \mp 1}\right)^{1/3}\left[2b_{1/2}^{(m)} \pm \frac{\beta}{m}\frac{db_{1/2}^{(m)}}{d\beta}\right] \frac{1}{x^2 + \alpha^2}\frac{M_s}{M_p}. \quad (17)$$

The term in brackets is of order 1. Thus $|\sigma_m|/\Sigma_0$ peaks at value $M_s/(M_p\alpha^2)$ at exact resonance, so that the linearity condition reads

$$(M_s/M_p)^{1/2} \ll \alpha. \quad (18)$$

Note that $u_{rm}/a_m\Omega_m$ and $u_{\theta m}/a_m\Omega_m$ remain small under the less restrictive condition $M_s/M_p \ll \alpha$.

From Eq. (13), it is straightforward to show that when $\alpha \sim (M_s/M_p)^{1/2}$, the streamlines begin to intersect in a region of width $\Delta a \sim \alpha a_m$ around the resonance.

Tightly wound wave condition. For the term in brackets in Eq. (10) to vary slowly, it is necessary to avoid $\Omega = \omega_s$, i.e., to avoid crossing the satellite orbit. As shown in Fig. 1, the torque is mainly exerted over a width of a few times αa_m. Thus, to avoid $\Omega = \omega_s$ in the relevant part of the integral, it is necessary that αa_m be smaller than the distance of the resonance to the satellite orbit $\sim a_m/m$, i.e.:

$$\alpha \ll \frac{1}{m}. \quad (19)$$

It is easy to show that when this condition holds, the variation of A over the resonance width αa_m is small. This validates the expression of u_{rm} and $u_{\theta m}$ in Eqs. (10), where only D^{-1} is assumed to vary and allows us to take $D \sim \mathcal{D}x$, neglecting the higher order terms. We will see in Section IV that condition (19) is analogous to a tightly wound wave condition when spiral waves are propagating in the disk.

If $\alpha \sim 1/m$, the resonances are spread out of the disk, the inner Lindblad resonances $m:m-1$ being "contaminated" by the outer Lindblad resonances $m:m+1$. Since the torques are opposite for the two resonances, they cancel each other out so that the total resulting torque will be reduced with respect to the standard formula (16).

III. EFFECTS OF FRICTION AND NONSTATIONARITY

(1) Disks with Friction

In this section, we consider a disk with a friction force acting on the particles, and assume that it yields, in the linearized Eqs. (5), a force per unit mass $-Q\mathbf{u}$. Goldreich and Tremaine (1981) and Greenberg (1983) used a similar simple damping model, applied to the particles' orbital parameters.

In the Fourier equations giving u_{rm} and $u_{\theta m}$, adding $-Q\mathbf{u}$ is strictly equivalent to replacing ω_s by $\omega_s + iQ/m$. Thus, the coefficient α defined in Section II has the value $\alpha_Q = 2Q/(3m\omega_s)$. The torque density is proportional to Re[$u_{rm}(x)$] (Eq. (11)), where from Eq. (8)

$$\mathrm{Re}[u_{rm}(x)] = C_m \frac{\alpha_Q}{\alpha_Q^2 + x^2} \quad (20)$$

with $C_m = -\Omega_m A/(a_m^2\mathcal{D})$; $\alpha > 0$ is ensured by $Q > 0$ (the energy is dissipated by friction), so that the torque Γ_m has the correct sign. The integration of (20) shows that 80% of the total standard torque is exerted over a region of width $\Delta x \sim 3\alpha_Q$.

(2) The Nonstationary Problem: "Creating" the Satellite

The above result shows that dissipation removes the singularity at resonance and justifies the displacement of the pole performed in Section II. But what happens if we insist on studying an ideal problem without dissipation, or waves? Then, since the angular momentum cannot be transferred to the particles through dissipation, it accumulates at the resonances and the problem turns out to be nonstationary. In this case, the temporal variation of the perturbed quantities assumed in Section II is incorrect. This explains why the problem in Section II was ill defined and singular.

In this section, we study the following problem: at $t < 0$ the mass of the satellite is zero, and at $t = 0$ the satellite is "created." The satellite potential (Eq. (2)) thus becomes

$$\psi_s(r, \theta, t) = \sum_{-\infty}^{+\infty} \psi_m(r, t)\, e^{im\theta}$$

$$\psi_m(r, t) = \psi_m(r) H(t) \exp(-im\omega_s t) \quad (21)$$

where $H(t)$ is the unit-step function.

This problem is appropriate to the study of the perturbation of an initially unperturbed circular streamline after one or more encounters with the satellite. Another application is the study of transitory phenomena in numerical simulations of disks perturbed by a satellite: in that case, it is most often assumed that the satellite is "created" at the start of the simulation.

We calculate the actual temporal variation of the perturbed quantities (a priori different from $\exp(-im\omega_s t)$) by using a causality condition, i.e., assuming that they are equal to zero for $t < 0$ since the satellite does not exist. Then Laplace transform is the most appropriate tool for solving the problem:

$$\tilde{f}(p) = \int_0^\infty f(t) e^{-pt} dt$$

$$f(t) = \frac{1}{2i\pi} \int_{p_0 - i\infty}^{p_0 + i\infty} \tilde{f}(p) e^{pt} dp \quad (22)$$

with p_0 real and positive; $\tilde{f}(p)$ is defined in the complex plane for $\mathrm{Re}(p) > 0$; then it is easy to show that $f(t) = 0$ for $t < 0$.

From Eqs. (21) and (22), we have $\tilde{\psi}_m(r, p) = \psi_m(r)/(p + im\omega_s)$.

(*a*) *Case 1: There is no friction.* Taking the Laplace transform of Eqs. (5) and considering the component of order m, we obtain the same equations as in (6), except that any quantity $f(r, t)$ is replaced by $\tilde{f}(r, p)$ and that ω_s is replaced by ip/m. This yields

$$\tilde{u}_{rm}(r, p) = -i[\Omega A(r, p)/rD(r, p)]/(p + im\omega_s)$$

$$D(r, p) = \Omega^2 - m^2(\Omega - ip/m)^2$$

$$A(r, p) = 2m\psi_m \pm mr \frac{\Omega - ip/m}{\Omega} \dot{\psi}_m \cdot \quad (23)$$

From now on, the variation in r of $D(p)$, $A(p)$, and Ω will be implicit. Performing the inverse Laplace transform yields $u_{rm}(r, t)$ for $t > 0$, summing up the residues at the poles of $\tilde{u}_{rm}(r, p)$, i.e., $p_s = -im\omega_s$ and $p_+ = -im\omega_s\Omega/\Omega_m^{\mp}$ (solutions of $D(p) = 0$). Since $D(p_s) = D$ (defined in Eq. (6)), one obtains

$$u_{rm}(r, t) = -\frac{i\Omega}{r}\left[\frac{A(p_s)}{D} e^{-im\omega_s t} + \sum_{\mp} \frac{A(p^{\mp})}{(im\omega_s + p^{\mp})} \frac{e^{p^{\mp} t}}{\left(\frac{\partial D}{\partial p}\right)_{p^{\mp}}}\right]. \quad (24)$$

Let us calculate this expression close to a Lindblad resonance. We note that the denominator of the respective terms $\mp$ in Eq. (24) is $-a_m \mathcal{D} x \sim -D$ close to the resonance radius $a_m^{\mp}$, the other term being mx smaller. The condition (19) ensures that $mx \ll 1$, since we assume x smaller than a few α. Developing $p^{\pm}$ near $x \sim 0$ yields

$$u_{rm}(r, t) \sim iC_m e^{-im\omega_s t}(1 - e^{ix/\alpha_t})/x \quad (25)$$

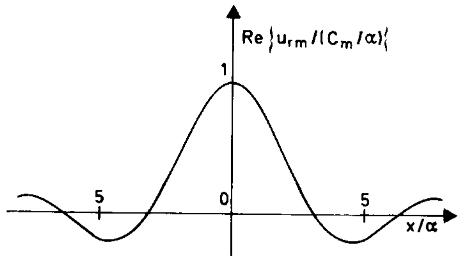

FIG. 2. The torque density, normalized to $\Gamma_m/(\pi\alpha a_m)$ (Γ_m is the standard torque, see Eq. (16)), in the case of transitory phenomena, i.e., when the satellite is "created" at $t = 0$. The analytical expression of the above function is given in Table I: $\sin(x/\alpha)/(x/\alpha)$ as well as the value of $\alpha = 2/(3m\omega_s t)$. Thus the resonance width shrinks as $1/t$ and the perturbed quantities increase as t until nonlinearity is reached.

with $\alpha_t = 2/(3m\omega_s t)$. Thus,

$$\mathrm{Re}[u_{rm}(r)] = C_m \frac{\sin(x/\alpha_t)}{x}. \qquad (26)$$

There is no more singularity at resonance (Fig. 2). So, a nonstationary evolution of the disk allows the satellite to exert a torque, exactly as in Section II. The torque is rigorously the same in the two cases, since the integral of $\sin(x)/x$ is π, and $t > 0$ implies $\alpha_t > 0$, which gives the correct sign for the torque Γ_m; this sign is clearly associated with causality.

Conditions (18) and (19) can be rewritten in terms of the satellite orbital period T_s: $T_s/(3\pi) \ll t \ll (M_p/M_s)^{1/2}T_s/(3\pi m)$.

For $t \leq T_s/3\pi$, the disk response is dominated by irrelevant transients; then for $t > T_s/3\pi$, the perturbations grow linearly with t at resonance, e.g., $\mathrm{Re}[u_{rm}(a_m)] = (3/2) m\omega_s C_m t$, as was noted by Lynden-Bell and Kalnajs (1972) for galactic disks, while the width $a_m\alpha_t$ over which the torque is exerted shrinks as $1/t$. At $t \sim T_s(M_p/M_s)^{1/2}/(3\pi m)$, the problem is no longer linear. However, in practice, the growing of the perturbations is avoided by dissipation (see below) or by carrying away the excess angular momentum through waves (Sect. IV).

(*b*) *Case 2*: *The nonstationary problem with friction. We assume a friction law in* $-Q\mathbf{u}$, as in Section 1. This replaces $\partial/\partial t$ by $(\partial/\partial t) + Q$ in the momentum equation and thus p by $p + Q$ in $A(p)$ and $D(p)$, while the term $p + im\omega_s$, which stems from $\tilde{\psi}_m(p)$, remains unchanged. The poles $p^{\mp}$, which appear in Eq. (24) are changed by $-Q$ (Fig. 3).

Thus, whenever $Qt \ll 1$, i.e., $\alpha_Q \ll \alpha_t$, the results of Section III.2.a are not modified. On the other hand, if $\alpha_Q \gg \alpha_t$, the factor e^{-Qt} stemming from $e^{p^{\mp}t}$ in Eq. (24) makes the term $\mp$ vanishingly small, and the results of Section III.1 (stationary problem with friction) are retrieved. The torque is equal to the standard torque found in Section II (Eq. (15)) for any values of α provided they satisfy inequalities (18) and (19).

The variation of $u_{rm}(r, t)$, and other perturbed quantities, is dominated by the process which has the largest α. In particular,

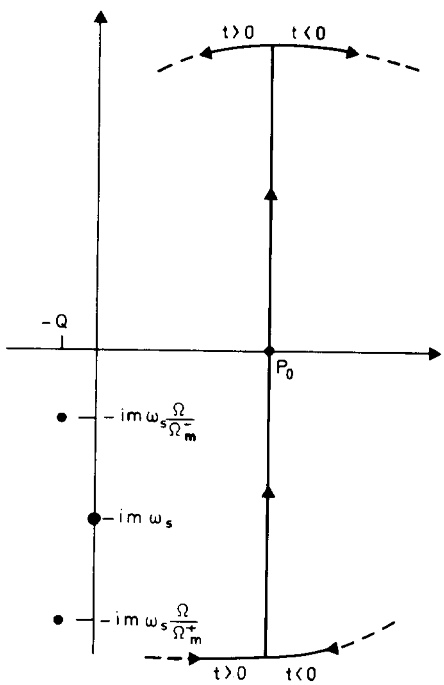

FIG. 3. Paths of integration in the Laplace transform space for disks with both transitory effects and friction (Q is the friction coefficient, see Sect. III.1). For $t < 0$, no pole is inside the path, so that all the perturbed quantities are zero. For $t > 0$, i.e., after the satellite has been created, the path must be closed on the left side where there are three poles.

if $t < 1/Q$, the nonstationarity dominates, and the perturbed quantities grow as t. When $t \geq 1/Q$, this growing stops because the disk can get rid of its excess of angular momentum through dissipation. The torque acts on a (relative) width given by the largest α, which first decreases at $1/t$, and then saturates at the value α_Q (Fig. 4).

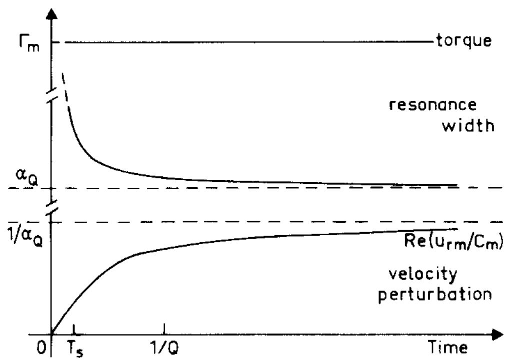

FIG. 4. Basic evolution of a disk at a Lindblad resonance under effect of transitories and friction. For $t \leq T_s$ (T_s is the satellit orbital period), the resonance width is larger than the disk, and the torque drops to zero as t tends to zero. For $t \geq T_s$, the torque remains constant with time. As t increases, the resonance width shrinks and the perturbed quantities increase, until $t \sim 1/Q$: at this time, a steady state is reached since the disk can get rid of its excess of angular momentum and energy through dissipation.

IV. WAVES IN DISKS: EFFECTS OF PRESSURE, VISCOSITY, AND SELF-GRAVITATION

(1) Basic Equations

The tensor of pressure **P** (the usual isotropic pressure being denoted P) now has to be taken into account in the momentum equation. The Poisson equation and an equation of state can close the system (1):

$$\left(\frac{\partial}{\partial t} + \mathbf{U} \cdot \boldsymbol{\nabla}\right)\mathbf{U} = -\boldsymbol{\nabla}(\phi_p + \psi_S + \psi_D) - \frac{1}{\Sigma}\boldsymbol{\nabla} \cdot \mathbf{P}$$

$$\frac{\partial}{\partial t}(\Sigma) + \boldsymbol{\nabla}\,(\mathrm{E}\mathbf{U}) = 0$$

$$\nabla^2\phi_D = 4\pi\Sigma G\delta(z)$$

$$dP/d\Sigma = c^2. \tag{27}$$

Where ψ_D stands for the disk potential, z is the coordinate perpendicular to the disk plane, δ is the Dirac function, and c is the usual sound speed, assumed uniform in the disk.

We make the following approximations:

(i) The problem is stationary.

(ii) The equations are linearized. Thus, the pressure yields a term $-c^2\boldsymbol{\nabla}\sigma/\Sigma_0$ and the viscosity yields a term $\nu\nabla^2\mathbf{u} + (\mu + \nu/3)$ $\boldsymbol{\nabla}(\boldsymbol{\nabla}\mathbf{u})$ in the momentum equation. ν and μ are the shear and bulk kinematic viscosities, respectively; their logarithmic spatial derivatives are assumed to be small with respect to those of the fluid velocity.

(iii) The spiral structures are supposed to be tightly wound; in particular, one spiral arm generated at a Lindblad resonance should not cross the orbit of the satellite when the polar angle θ varies by about one radian. Equivalently, the higher order spatial derivatives of $\mathbf{u}$, ϕ_D, and σ are assumed to be dominant as follows:

$$\frac{d^2}{dr^2} \gg \frac{m}{r}\frac{d}{dr} \gg \frac{m^2}{r^2},$$

referred to as the tightly wound wave condition.

Then, the momentum equations simplify into

$$im(\Omega - \omega_s)u_{rm} - 2\Omega u_{\theta m} = -(\dot{\psi}_{sm} + \dot{\psi}_{Dm}) - \frac{c^2}{\Sigma_0}\dot{\sigma}_m + (\mu + 4\nu/3)\ddot{u}_{rm}$$

$$im(\Omega - \omega_s)u_{\theta m} + \frac{\Omega}{2}u_{rm} = -\frac{im}{r}(\psi_{sm} + \psi_{Dm}) - \frac{im}{r}\frac{c^2}{\Sigma_0}\sigma_m + \nu\ddot{u}_{\theta m}. \tag{28}$$

The continuity equation yields

$$\sigma_m = -\frac{\Sigma_0}{im}\frac{\dot{u}_{rm}}{\Omega - \omega_s}. \tag{29}$$

We use the results of Shu (1970, 1984) to solve the Poisson equation in this approximation:

$$\dot{\psi}_{Dm} = -2\pi G i \varepsilon \sigma_m \tag{30}$$

where $\varepsilon = \pm 1$ is chosen in such a way that the disk potential $\psi_{Dm}(r + i\varepsilon|z|)$ out of the disk plane tends to zero as z goes to infinity.

(iv) We seek a solution of (27) in the vicinity of a Lindblad resonance, i.e., $x = (r - a_m)/a_m \ll 1$. Since, as previously, $u_{\theta m} \sim \pm(i/2)u_{rm}$ and $D \sim a_m\mathcal{D}x$, Eqs. (28)–(30) yield

$$-\alpha_v^3 \frac{d^2}{dx^2}(u_{rm}) + \alpha_G^2 \frac{d}{dx}(u_{rm}) - ixu_{rm} = C_m \tag{31}$$

where C_m is defined in Section III.1, and

$$\alpha_v^3 = i\alpha_P^3 + \alpha_\nu^3$$
$$\alpha_P^3 = \mp(c^2/\Omega_m)/(3ma_m^2\omega_s)$$
$$\alpha_\nu^3 = \left(\mu + \frac{7}{3}\nu\right)\Big/(3ma_m^2\omega_s)$$
$$\alpha_G^2 = \pm(2\pi\varepsilon\, \Sigma_0 G)/(3ma_m\omega_s\Omega_m). \tag{32}$$

(2) Solution

Equation (31) is solved in a standard way by Fourier transform. We assume that $u_{rm}(x)$ is square integrable and define

$$\tilde{u}_{rm}(k) = \int_{-\infty}^{+\infty} dx e^{-ikx} u_{rm}(x). \tag{33}$$

Fourier transforming Eq. (31) yields

$$\frac{d}{dk}(\tilde{u}_{rm}) + (\alpha_v^3 k^2 + i\alpha_G^2 k)\tilde{u}_{rm} = 2\pi C_m \delta(k). \tag{34}$$

This first-order equation is solved with a boundary condition allowing $\tilde{u}(k) \to 0$ for $k \to \infty$, since $\tilde{u}(k)$ is a Fourier transform. This yields the solution

$$\tilde{u}_{rm}(k) = 2\pi C_m H(k) \exp -(\alpha_v^3 k^3/3 + i\alpha_G^2 k^2/2) \tag{35}$$

($H(k)$ is the unit-step function).

Thus the solution of Eq. (31) is

$$u_{rm}(x) = C_m \int_0^\infty dk \exp[ikx - \alpha_v^3 k^3/3 - i\alpha_G^2 k^2/2]. \tag{36}$$

(3) Torque Derivation

To calculate the integral of $u_{rm}(x)$ with $\tilde{u}_{rm}(k)$ discontinuous at $k = 0$, we first transform Eq. (36) by using an integral representation of $H(k) = (i/2\pi)\int_{-\infty}^{+\infty} du(\exp(-iku)/(u + io))$, where o is an arbitrary small positive number. Then, we first integrate Eq. (36) in x, which yields $\delta(k)$, then in k, and finally in u, by residues. Thus

$$\mathrm{Re}\left(\int_{-\infty}^{+\infty} dx u_{rm}(x)\right) = \mathrm{Re}\left(iC_m \int_{-\infty}^{+\infty} \frac{du}{u + io}\right) = \pi C_m \tag{37}$$

which yields the standard torque (16) with the correct sign.

(4) Validity Conditions

(i) Stationarity: we conjecture that the response of the disk is stationary if the satellite has existed for a long enough time, such that α_t (Table 1) satisfies $\alpha_t \ll \mathrm{Max}(|\alpha_v|, |\alpha_G|)$.

(ii) Linearity: the most stringent condition is again the noncrossing of the streamlines, i.e., $|\sigma_m|/\Sigma_0 \ll 1$. However, this condition is now slightly more complicated than in Section II: although u_{rm} remains of the same order of magnitude, it oscillates more and more rapidly as the wave propagates away, so that the streamlines eventually intersect. This effect is quantitatively studied below in some particular cases. For the moment, we just note that around the resonance ($x = 0$), the main contribution to the integral in Eq. (36) is provided around $k = 0$, over a width $\Delta k \sim 1/\alpha_{\mathrm{Max}}$, where $\alpha_{\mathrm{Max}} = \mathrm{Max}(|\alpha_P|, |\alpha_\nu|, \alpha_G|)$. Thus $du_{rm}/dx \sim C_m/\alpha_{\mathrm{Max}}^2$, so that $|\sigma_m|/\Sigma_0 \ll 1$ is equivalent to $\alpha_{\mathrm{Max}} \gg (M_s/M_p)^{1/2}$, as in Section II.

(iii) The tightly wound wave condition $d/dr \gg m/r$ immediately reads $\alpha_{\mathrm{Max}} \ll 1/m$, identical to condition (19).

(iv) The condition $|x| \ll 1$ in the region relevant in calculating the torque requires $\alpha_{\mathrm{Max}} \ll 1$, which is automatically satisfied if $\alpha_{\mathrm{Max}} \ll 1/m$.

(5) Some Remarks

Equation (31) generalizes particular cases studied elsewhere:

(i) If $\alpha_v = \alpha_G = 0$ (no viscosity, no pressure, and no self-gravitation), Eq. (31) reduces to Eq. (8), with a singularity at resonance.

(ii) If $\alpha_\nu = 0$, it reduces to a case studied by Goldreich and Tremaine (1979, their Eq. (39), and 1978b, their Eq. (A9)) that they explicitly solved for $|\alpha_v| \ll |\alpha_G|$. The case $|\alpha_G| \ll |\alpha_v|$ is equivalent to a classic problem in plasma physics (see Crawford and Harker, 1972).

(iii) if $\alpha_G = 0$ (no self-gravitation), it reduces to a case studied by Lin and Papaloizou (1979) (see their Eq. (B8), with typographical errors, corrected by using their Eq. (14), with an opposite sign for α_v^3, presumably due to another typographical error).

Allowing for pressure and/or viscosity and/or self-gravitation (i.e., α_v or $\alpha_G \neq 0$) yields a nonsingular behavior of the perturbed quantities at resonance: now the ring is able to dissipate and/or carry away its excess of angular momentum deposited at resonance by the satellite. If one of the α is much larger than the other, it will determine the main properties of the solution and the scale of its spatial variation.

The main properties of the solution $u_{rm}(x)$ can be derived by simple considerations on the integral of Eq. (36).

First, we see that the boundaries of the integral (0 to $+\infty$ and not 0 to $-\infty$) stem from μ, $\nu > 0$, which sets the sign of u_{rm}. Thus, the sign of Γ_m is directly linked to the positiveness of the viscosity coefficients ν and μ. If ν and μ are rigorously zero, Eq. (31) must be solved directly. The sign of Γ_m is then given by boundary conditions: the wave must *start* at the resonance and propagate away from it. Alternatively, one could solve a nonstationary problem as in Section III.2.

Let us assume that one physical process (self-gravitation or pressure) is dominant with respect to the other. So Eq. (36) yields

$$\frac{d}{dx}(u_{rm}) = iC_m \int_0^\infty k \exp[i(kx - \alpha^n k^n/n) - \alpha_\nu^3 k^3/3]dk \quad (38)$$

with $n = 2$ or 3 corresponding to $\alpha = \alpha_G$ or α_P, respectively.

If $|x/\alpha| \geq 1$ and $\alpha_\upsilon < |\alpha|\,|x/\alpha|^{-1/(n-1)}$, the main contribution to the integral in Eq. (38) is provided when the phase of the oscillating factor is stationary, i.e., when

$$\alpha^n k^{n-1} = x. \quad (39)$$

So that

$$\left|\frac{d}{dx}(u_{rm})\right| \sim C_m \sqrt{\frac{2\pi}{n-1}} \frac{1}{|\alpha|^2} \left|\frac{x}{\alpha}\right|^{(4-n)/(2(n-1))} \quad (40)$$

and the local wavenumber is $[(n - 1)/(|\alpha| n)] \cdot |x/\alpha|^{1/(n-1)}$.

Therefore:

(i) Equation (39) shows that $\mathrm{sgn}(x) = \mathrm{sgn}(\alpha^n)$ since $k \geq 0$. Thus, if only one physical process is at work, the wave propagates on only one side of the resonance, given by $\mathrm{sgn}(\alpha^n)$. On the other side, the wave is evanescent with a damping distance $x \sim \alpha$.

(ii) Equation (40) shows that for $n = 2$ or 3, du_{rm}/dx increases like x and $x^{1/4}$, respectively. Thus, the streamlines eventually intersect, even if they do not intersect close to the resonance radius.

(iii) The damping factor $\exp(-\alpha_\nu^3 k^3/3)$ in Eq. (38) allows the avoidance of the streamline crossing at x if $\alpha_\nu \geq |\alpha|\,|x/\alpha|^{-1/(n-1)}$. If $\alpha_\nu > |\alpha|$, the crossing is avoided for any value of x; i.e., the disk is dominated by viscosity and no more waves propagate from the resonance.

(iv) In the absence of viscosity the wave oscillates more and more rapidly as the distance to the resonance increases.

(6) Some Explicit Solutions

(a) The disk is pressure dominated: $|\alpha_G|$, $\alpha_\nu \ll |\alpha_P|$; therefore,

$$u_{rm}(x) = C_m \int_0^\infty dk \exp[i(kx - \alpha_P^3 k^3/3)]. \quad (41)$$

Since $\alpha_P^3 = \mp c^2/3ma_m^2\omega_s\Omega_m$, the acoustic waves propagate *outside* the region delimited by the inner and outer resonances. The velocity u_{rm} can be expressed as a combination of Airy-type functions (Abramowitz and Stegun, 1968):

$$u_{rm}(x) = \frac{\pi C_m}{|\alpha_P|}[Ai(\pm x/|\alpha_P|) \pm iGi(\pm x/|\alpha_P|)]. \quad (42)$$

Re(u_{rm}), which reduces to the first term of Eq. (42), is shown in Fig. 5.

As noted above, $|\sigma_m/\Sigma_0|$ increases as $|x|^{1/4}$ for $\mp x > 0$ and is evanescent for $\mp x < 0$ on a distance $|x| \sim |\alpha_P|$. The wavenumber increases as $|x|^{1/2}$ for $\mp x > 0$. Numerical integration shows that 85% of the standard torque is exerted at $x = \pm 3|\alpha_P|$.

(b) The disk is viscosity dominated: $|\alpha_G|$, $|\alpha_P| \ll \alpha_\nu$: therefore,

$$u_{rm}(x) = C_m \int_0^\infty dk \exp[ikx - \alpha_\nu^3 k^3/3)] = \frac{\pi C_m}{\alpha_\nu} Hi(ix/\alpha_\nu) \quad (43)$$

with α_ν given in Table I. The result decreases exponentially on both sides of the resonance, over a scale $x \sim \alpha_\nu$. Ninety-five percent of the standard torque is provided between $-3\alpha_\nu$ and $3\alpha_\nu$ (see Fig. 6).

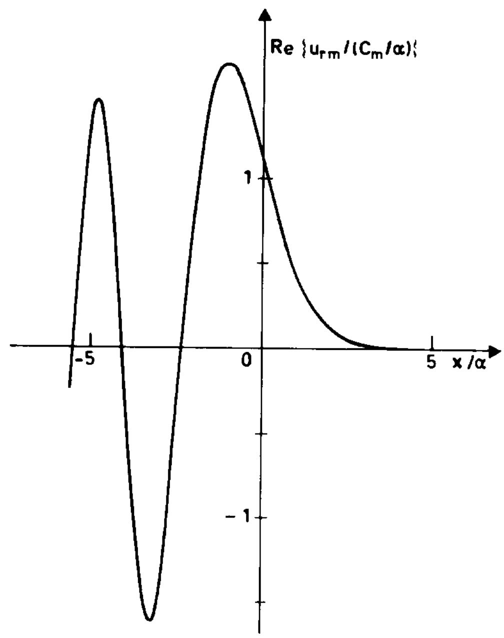

FIG. 5. The same as in Fig. 2, except that the disk is now pressure dominated. The torque density function is shown here at an inner Lindblad resonance; i.e., the satellite is far away on the right. An acoustic wave, launched at the resonance propagates *away* from the satellite. The analytical expression of the above function is $\pi Ai(x/|\alpha_P|)$ (see Table I).

TABLE I

RESONANCE WIDTH AND TORQUE DENSITY FOR VARIOUS PHYSICAL PROCESSES

Physical process	Dimensionless width of the resonance	Torque density normalized to the standard torque: $\frac{a_m}{\Gamma_m}\frac{d\Gamma}{dr}$
Transitory effects (satellite created at $t = 0$)	$\alpha_t = 2/(3m\omega_s t)$	$\sin(x/\alpha_t)/(\pi_x)$
Friction (Stokes-like law in $-Q\mathbf{u}$)	$\alpha_Q = 2Q/(3m\omega_s)$	$\alpha_Q/\pi(\alpha_Q^2 + x^2)$
Viscosity, pressure, and self-gravity	$\alpha_V^3 = i\alpha_P^3 + \alpha_\nu^3$	$\frac{1}{\pi}\int_0^\infty dk \exp[i(kx - \alpha_G^2k^2/2) - \alpha_\nu^3k^3/3]$
Viscosity	$\alpha_\nu^3 = \left(\mu + \frac{7}{3}\nu\right)\Big/(3ma_m^2\omega_s)$	$Hi(ix/\alpha_\nu)/\alpha_\nu$
Pressure	$\alpha_P^3 = \mp(c^2/\Omega_m)/(3ma_m^2\omega_s)$	$Ai(\pm x/|\alpha_P|)/|\alpha_P|$
Self-gravity	$\alpha_G^2 = \pm(2\pi\Sigma_0 G)/(3ma_m\omega_s\Omega_m)$	$g(\mp x/\sqrt{\pi}|\alpha_G|)\sqrt{\pi}|\alpha_G|$

Note. *Ai* and *Hi* are Airy-type functions and *g* is a Fresnel-type function (Abramowitz and Stegun, 1968).

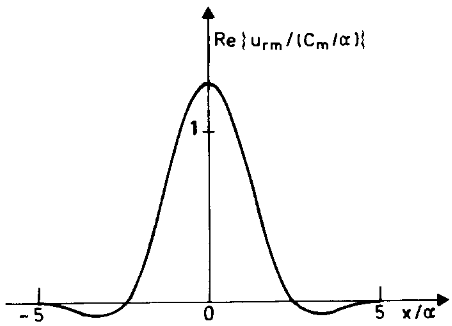

FIG. 6. The same as in Fig. 2, except that the disk is now dominated by viscosity. The analytical expression of the above function is $\pi Hi(ix/\alpha_\nu)$ (see Table I).

(c) The disk is dominated by self-gravitation, i.e., $|\alpha_P|$, $\alpha_\nu \ll |\alpha_G|$; therefore,

$$u_{rm}(x) = C_m \int_0^\infty dk \exp i(kx - \alpha_G^2 k^2/2)dk \tag{44}$$

with α_G given in Table 1.

The condition $\lim_{z\to\infty}[\phi_{Dm}(r + im|z|)] = 0$ requires $\varepsilon k > 0$, and thus $\varepsilon > 0$ since $k > 0$ (Eq. (36)). Thus, the density wave propagates *inside* the region delimited by the inner and outer resonances. The quantity $|\sigma_m/\Sigma_0|$ increases now as $|x|$ for $\pm x > 0$, as well as the wavenumber.

u_{rm} can be expressed in terms of tabulated Fresnel-type integrals (Abramowitz and Stegun, 1968):

$$u_{rm}(x) = C_m \frac{\sqrt{\pi}}{|\alpha_G|}\left[g\left(\mp\frac{x}{\sqrt{\pi}|\alpha_G|}\right) \mp if\left(\mp\frac{x}{\sqrt{\pi}|\alpha_G|}\right)\right]. \tag{45}$$

Re(u_{rm}) is displayed in Fig. 7. Again, it can be verified that most of the torque is exerted at $x = \pm 3|\alpha_G|$.

V. NUMERICAL APPLICATIONS

The results of Sections II–IV are summarized in Table I. The dominating physical process is that with the largest α. The radial extent over which the torque is exerted is a few times αa_m. Also given in Table I are the torque densities $d\Gamma_m/dr$ normalized to the total standard torque Γ_m^{st}. These functions describe the behavior of the disk close to a resonance, and their integrals are equal to unity.

In the framework of this general approach, we review some astrophysical problems linked to disk–satellite interaction, referring whenever possible to other authors for more detailed analysis.

(1) Isolated and Overlapping Resonances

The behavior of the torque density function is quite different for isolated and overlapping resonances: in the first case it ex-

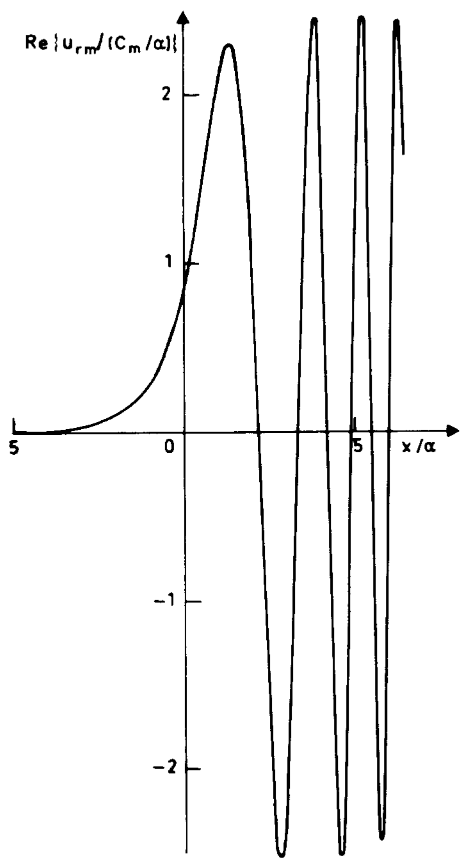

FIG. 7. The same as in Fig. 2, except that the disk is now dominated by self-gravity. A density wave, launched at the resonance propagates *toward* the satellite. The analytical expression of the above function is $\sqrt{\pi}\, g[-x/\sqrt{\pi}|\alpha_G|]$ (see Table I).

hibits discrete peaks at each of the isolated resonances, while in the second case it shows a smooth increase as one approaches the satellite orbit. The transition between the two regimes occurs when the width of one resonance, $a\alpha$, is larger than the separation between two resonances, about $2a/3m^2$. Thus overlapping occurs when

$$m^2\alpha \geq 1. \qquad (46)$$

An illustration of this effect is provided in Fig. 8.

From Eqs. (4), (9), and (16), giving the torque at an isolated resonance, it is easy to estimate the torque for the two regimes; one finds

$$|\Gamma_m| \sim fm^2\, \Sigma_0\, a_s^4\omega_s^2 \left(\frac{M_s}{M_p}\right)^2 \qquad (47a)$$

$$\left|\frac{d\Gamma}{dr}\right| \sim f'\Sigma_0\, a_s^3\omega_s^2 \left(\frac{M_S}{M_p}\right)^2 \left(\frac{a_S}{r-a_S}\right)^4 \qquad (47b)$$

for isolated and overlapping resonances, where f and f' are of order 10 and unity, respectively. For $m \gg 1$, the Laplace coefficients can be approximated in terms of Bessel functions (Goldreich and Tremaine, 1980), yielding $f = (4/3)[2K_0(2/3) + K_1(2/3)]^2 \sim 8.5, f' = (2/3)^3 f \sim 2.5$ (Goldreich and Tremaine, 1982).

We emphasize that overlapping of resonances (and thus the validity of Eq. (47b)) requires that the distance to the satellite orbit satisfy $|r - a_s|/a_s \ll \alpha^{1/2}$ (from Eq. (46) since $|a_m - a_s|/a_s \sim 1/m$, for $m \gg 1$).

(2) *Nonstationarity: Numerical Simulations and Impulse Approximation*

In any numerical simulation of colliding particles under the perturbing action of a satellite, one must remember that transitory effects produce a torque equal to the standard torque. Only when the time t is greater than the typical damping time, t_d, may this transitory effect be neglected (see Sect. III.2b).

Another consequence of Section III 2.b is the validity of the so-called "impulse approximation" sometimes used to describe

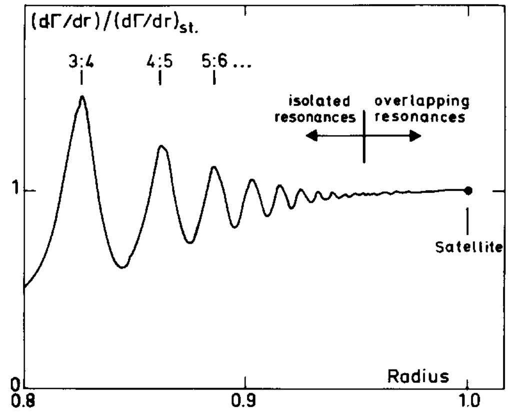

FIG. 8. The torque density $d\Gamma/dr$ normalized to the standard torque density $(d\Gamma/dr)_{st}$ obtained for overlapping resonances (Eq. (47b)). We have assumed that the disk is friction dominated, so that the torque density is given by Eq. (20). We have taken $\alpha_Q = 0.05/m$, and it can be verified that the separation between isolated and overlapping resonances occurs when $\alpha_Q m^2 \sim 1$. The origin of radius is at the planet center.

the shepherding mechanism (Lin and Papaloizou, 1979; Dermott, 1984). This approximation yields the standard torque (Eq. (47b)) (if applied to the satellite and one ring particle in circular orbits, and expressed per ring mass unit), though the numerical coefficient is about three times too small. From this approximation three major assumptions are made: (i) there is a two-body encounter between the satellite and the particle; (ii) the particle is in circular orbit before the encounter; and (iii) there are small deflections. As noted by Hénon (1984), assumption (i) is incorrect since the actual encounter duration is a few orbital periods, so the potential of the central planet cannot be ignored; this assumption is expected to be responsible for the incorrect numerical coefficient. Assumption (ii) is equivalent to "creating" the satellite as the particles get close to it. In the present formulation, this means that the system is governed by transitory effects with $\alpha = 2/(3m\omega_s t)$. Here, the time is the encounter duration, i.e., a few times $2\pi/\omega_s$. Thus, $\alpha m^2 \geq 1$, which from Eq. (46) corresponds to overlapping resonances, i.e., to the domain of application of Eq. (47b). Finally, it is easy to verify that assumption (iii) is equivalent to our linearity condition. This linearity condition, which is given by Eq. (18) for isolated resonances, is more strict for overlapping resonances since the maximum perturbed mass density is $|\sigma| = |\Sigma_m \sigma_m| \sim \alpha m^2 |\sigma_m|$. Using Eqs. (17) and (19), this condition reads $(M_s/M_p)^{1/3} \ll 1/m \sim |r - a_s|/a_s$. This is equivalent to assumption (iii) (see, for instance, Eq. (15) in Hénon (1984) where $m_1 + m_2 \sim M_s$ and $\omega^2 = M_p G/a_s^3$.

(3) Bodies in a Dissipative Medium

We now discuss the effect of gaseous friction on planetesimals in the proto-Solar System, under the influence of Jupiter.

Goldreich and Ward (1973) have studied the formation of planetesimals in the solar nebula. During a very rapid phase (about one year), bodies with size up to $R \sim 100$ m form by gravitational instabilities from the disk of dust. The typical surface density of such a disk is 8 g cm^{-2}, so that the coefficient α associated with self-gravity is $\alpha_G \sim 7 \times 10^{-3}$ (taking $a_m \sim 4.5 \times 10^{13}$ cm and $\omega_s \sim 2 \times 10^{-8}$ sec^{-1}). We shall see that the coefficient associated with gas friction is $\alpha_Q \sim 3 \times 10^{-2} > \alpha_G$ (with a gas density $\rho_G \sim 8 \times 10^{-10}$ g cm^{-3}). Thus the disk of planetesimals, under these conditions, is dominated by gaseous drag.

This drag arises for two reasons: (i) the radial pressure gradient induces a differential Keplerian motion $\Omega_g \sim \Omega - c^2/(2\Omega a^2)$, where Ω (resp., Ω_g) is the angular velocity of the planetesimals (resp., the gas) and c is the sound speed (Goldreich and Ward, 1973); (ii) close to resonances with Jupiter, the planetesimals have enhanced perturbed motions.

Let $\Delta\mathbf{V}$ be the local difference between the planetesimal and gas velocities. The acceleration due to gas drag is

$$\gamma = -Q\Delta\mathbf{V}. \tag{48}$$

Q is independent of $\Delta\mathbf{V}$ in the two following cases (Weidenschilling, 1979), depending on the particles, radius R:

(i) $R > \lambda$ (λ is the gas molecules' mean free path) and $R_e = 2R\Delta V/\nu_g < 1$ (R_e is the Reynolds number, ν_g the gas kinematic viscosity); then $Q \sim 4.5(\rho_g/\rho_b)(\nu_g/R^2)$ (ρ_b is the planetesimal density).

(ii) $R < \lambda$; then $Q \sim (\rho_g/\rho_b)(c/R)$.

If the orbits of the planetesimals do not intersect, the radial and tangential components of the drag acceleration read

$$\begin{aligned} \gamma_r &= -Qu_r \\ \gamma_\theta &= -Q(u_\theta + \delta^2 a\Omega/2) \end{aligned} \tag{49}$$

where $\delta = c/(a\Omega)$. The terms u_r and u_θ are due to the perturbation by Jupiter and the term $\delta^2 a\Omega/2$ stems from the differential keplerian motion between the planetesimals and the gas. The latter term will cause a systematic decay of the planetesimal orbits at a rate

$$\dot{a} = -Qa\delta^2, \tag{50}$$

while the terms in u_r, u_θ yield

$$\dot{a}_m = \mp \frac{\alpha_Q}{x^2 + \alpha_Q^2} m^2 a_s \omega_s \left(\frac{M_s}{M_p}\right)^2 \tag{51}$$

where x is the (dimensionless) distance to the $m:m \mp 1$ resonance and $\alpha_Q = 2Q/(3m\omega_s)$ (Sect. III.1). Thus resonance effects *counteract* (respectively, *enhance*) the orbital decay at *outer* (resp., *inner*) Lindblad resonances. This result is discussed in more detail by Greenberg (1978) and has been numerically studied by Weidenschilling and Davis (1985). According to Eqs. (50) and (51), the resonance effect becomes dominant for

$$\alpha_Q \leq (M_s/M_p)m^{1/2}/\delta. \tag{52}$$

Under this condition, planetesimals will be locked at outer resonances and gaps will be opened at inner resonances.

For $\rho_g \sim 8 \times 10^{-10}$ g cm^{-3}, the mean free path of (hydrogen) molecules $\lambda \sim 10$ cm is smaller than the size of the 1- to 100-m bodies considered here; therefore, $\alpha_Q \sim 1.6 \times 10^4/mR^2$ (cm), where we have used $\nu \sim \lambda c/6$ and $c \sim 7.6 \times 10^3 T^{1/2}$ cm s^{-1} for hydrogen. The bodies are assumed to be composed of rocky material ($\rho_b \sim 3$ g cm^{-3}). In order for the orbits of the bodies not to intersect close to resonance, one should have $\alpha_Q \geq (M_s/M_p)^{1/2}$, which can be rewritten $R \leq 720m^{-1/2}$ cm. This means that all the newly formed 100-m-sized objects entering an isolated resonance suffer collisions, possibly leading to fragmentation processes. Collisions are avoided for $R \leq 720m^{-1/2}$ cm. which may set an upper limit of particle size around a Lindblad resonance. The typical time to accumulate matter (at outer resonances) or open a gap (at inner resonances) is then $t_m \sim a_m\alpha_Q/\dot{a}_m$, i.e.:

$$t_m \sim \frac{1}{\omega_s}\left(\frac{\alpha_Q}{m}\frac{M_p}{M_s}\right)^2. \tag{53}$$

For the biggest particles, $R \sim 720m^{-1/2}$ cm, Eq. (53) yields $t_m \sim 2000/m^2$ years, and the width of the gap is $\alpha_Q \sim 3 \times 10^{-2}$. We now check if Eq. (52) holds. For a temperature of the solar nebula $T \sim 700°$K, one obtains $\delta \sim 0.15$, so $m^{1/2}(M_s/M_p)/\delta \sim 10^{-2}\sqrt{m}$. Thus accumulation or depletion of meter-sized particles around Lindblad resonances is significant and may occur in a rather short time. Note that this mechanism is more efficient as m increases. This means that the smallest particles, which can cross low-order resonances (see condition (52) where α_Q increases when R decreases) will be eventually trapped at outer resonances, or swept away at inner resonances for high enough values of m.

Obviously, many questions remain unanswered in the strict framework of this model: (i) What is the stability of the gaps when the density of the gas drops? (ii) What is the subsequent evolution of the matter locked at outer resonances? Some discussions are given in Greenberg (1978) and Weidenschilling and Davis (1985).

It should be kept in mind that resonance and friction effects may have dramatically shaped the disk of newly formed planetesimals. The swarm of asteroids may have kept some of these initial conditions. For instance, the width of the Kirkwood gap associated with 2 : 1 resonance with Jupiter is of the same order of magnitude as the width that we predicted from friction effects ($\alpha \sim 3 \times 10^{-2}$).

(4) *Planetary Rings*

Extensive efforts have been made to understand the dynamics of rings perturbed by a nearby satellite (cf. Goldreich and Tremaine, 1982; Borderies *et al.*, 1984; Dermott, 1984; Greenberg, 1983). The shepherding of rings by satellites was initially attributed to self-gravity (Goldreich and Tremaine, 1980), and then reexplained in terms of the impulse approximation or presence of dissipation (Greenberg, 1983). We shall see that the parameter α associated with self-gravity is actually the largest one in planetary rings, so that self-gravity determines the particles' behavior near a resonance.

For an optical depth of order one, the kinematic viscosity ν is a few times $R^2\Omega$, where R is the radius of the biggest particles. Measurements of the damping of density waves in Saturn's rings give $\nu \sim 150$ cm^2 sec^{-1} (Cuzzi *et al.*, 1984; Esposito *et al.*, 1983), yielding R of several meters. Numerical experiments (Brahic, 1977) and theoretical considerations (Cuzzi *et al.*, 1979; Goldreich and Tremaine, 1978a) predict a velocity dispersion c of a few times $R\Omega$, so that α_ν and α_P given in Table 1 are of the same order of magnitude. Using as a conservative upper limit $\nu \leq 200$ cm^2 sec^{-1}, one obtains $\alpha_\nu \sim \alpha_P \leq 2 \times 10^{-5}m^{-1/3}$. On the other hand, an estimated surface density 50 g cm^{-2} yields $\alpha_G \sim 10^{-4}m^{-1/2}$, so that typical regions of Saturn's rings are dominated by self-gravity, which justifies the approach of Goldreich and Tremaine (1980).

Nevertheless, the difference between α_ν (or α_P) and α_G is not so overwhelming. In particular, low-density rings ($\Sigma_0 \leq 2$ g cm^{-2}) should be dominated by viscosity and pressure.

(5) *Ring Edges*

Dissipation and conservation of angular

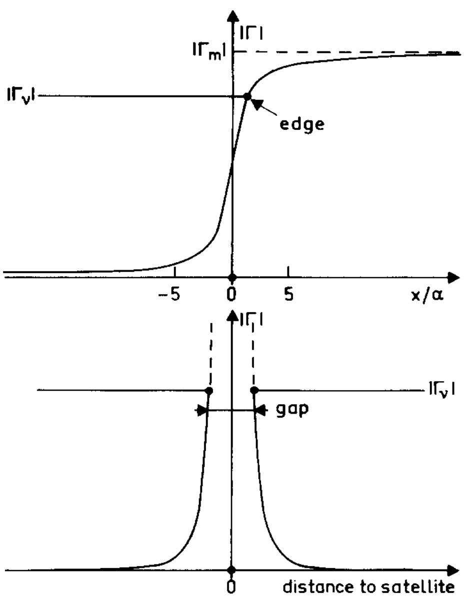

FIG. 9. (Top) Creating an edge at an isolated Lindblad resonance. The S-shaped function is the satellite torque density integrated over the disk from $x/\alpha = -\infty$ to the current value of x/α (assuming a torque density as in Eq. (20)). An edge is created when this torque balances the flux of angular momentum due to viscosity. Note that the satellite exerts a torque of only Γ_ν once an edge is created, and that no edge can appear when $\Gamma_\nu > |\Gamma_m|$ because the viscous spreading overcomes the resonance effect. (Bottom) Creating a gap with overlapping resonances around an embedded moonlet. A gap appears when the integrated torque density due to the moonlet (Eq. (47b)) balances the flux of angular mometum due to viscosity. The gap is stable if its width is larger than the moonlet diameter, which requires moonlets larger than about 1 km in the case of Saturn's rings (see Sect. V.5.b; Hénon, 1981, 1984).

momentum lead to the spreading of an unperturbed disk (Brahic, 1977). The flux of angular momentum due to viscosity in a *Keplerian* disk is

$$\Gamma_\nu = 3\pi\nu\Sigma_0 a^2\Omega \qquad (54)$$

(Lynden-Bell and Pringle, 1974). Note that this formula is valid as long as the streamlines in the disk are *not* severely distorted due to the satellite perturbations.

This disk is truncated; i.e., an edge is created when the shepherding torque Γ_m is larger than the viscous torque Γ_ν. Two different situations may arise, depending on the isolated or overlapping behavior of the resonances.

(*a*) *Isolated resonances.* For $|\Gamma_m| \geq \Gamma_\nu$, i.e., $M_s/M_p \geq (\nu/a_s^2\omega_s)^{1/2}/m$, the edge of the disk can adjust itself so that the satellite torque integrated over the ring exactly compensates for the viscous torque (Fig. 9). Using $a_s \sim 5 \times 10^9$ cm and $M_p \sim 8.7 \times 10^{28}$ g, relevant for Uranus rings, the inequality $|\Gamma_m| \geq \Gamma_\nu$ reads

$$M_s \geq \frac{1.2 \times 10^{22}}{m}\left(\frac{\nu}{100\ \mathrm{cm^2\ sec^{-1}}}\right)^{1/2} \mathrm{g}. \qquad (55)$$

Similar figures are derived for Saturn's rings. So, Mimas and 1980 S1, whose respective masses are 3.7×10^{22} and 4.4×10^{21} g, are able to truncate Saturn's rings if one assumes $\nu \leq 200\ \mathrm{cm^2\ sec^{-1}}$. For Mimas, the corresponding edge is at the 2 : 1 resonance (outer edge of the B ring), and for 1980 S1, the edge is at the 7 : 6 resonance (outer edge of the A ring).

(*b*) *Overlapping resonances.* Since for Uranus or Saturn rings, $\alpha_G \sim 10^{-4}/m^{1/2}$, resonances overlap for $m \geq 400$, i.e., when the distance d between the ring and the satellite is less than about 80 km. According to Eq. (47b), the inequality $|\Gamma| \geq \Gamma_\nu$ now reads $M_s/M_p \geq 3.36(\nu/(a_s^2\omega_s))^{1/2}(d/a_s)^{3/2}$; i.e.:

$$M_s \geq 3.7 \times 10^{15}\left(\frac{\nu}{100\ \mathrm{cm^2\ sec^{-1}}}\right)^{1/2}\left(\frac{d}{\mathrm{km}}\right)^{3/2} \mathrm{g}, \qquad (56)$$

equivalent to $R \geq (\nu/100\ \mathrm{cm^2\ sec^{-1}})^{1/6}(\rho/\mathrm{g\ cm^{-3}})^{-1/3}(d/\mathrm{km})^{1/2}$ km, where R and ρ are, respectively, the satellite's radius and density. This shows that bodies with radii greater than about one kilometer are able to open a gap wider than their own diameter in Saturn or Uranus rings. (Hénon, 1981, 1984; see Fig. 9).

(*c*) *Time scales.* If the satellite is between two rings, the torque it exerts on each of them can balance so that the whole system is stable. If not, the satellite will be pushed away from the disk at a rate $\dot{a}_s = 2\Gamma/(a_s\omega_s M_s)$, so that the satellite doubles its distance from the edge on a time scale $t_s \sim d/\dot{a}_s$. For *isolated* resonances, one obtains

$$t_s \sim M_s/(9\pi m\nu\Sigma_0) \qquad (57)$$

since only $\Gamma = -\Gamma_\nu$ is exerted by the satellite on the ring. For the system to be stable over the age of the Solar System, i.e., 4.5×10^9 years, one needs $M_s \geq 4 \times 10^{22} m(\nu/100\ \text{cm}^2\ \text{sec}^{-1})\ (\Sigma_0/100\ \text{g cm}^{-2})$ g. This shows that Mimas is massive enough to confine the outer edge of the B ring ($m = 2$) over 4.5 byr, but 1980 S1 is not ($m = 7$), *if* one assumes $\nu \sim 100\ \text{cm}^2\ \text{sec}^{-1}$. A low viscosity ($\nu \leq 3\ \text{cm}^2\ \text{sec}^{-1}$) would allow 1980 S1 to confine the A ring edge over 4.5 byr. Such a value of the kinematic viscosity is not absurdly small in view of the measurements made in Saturn's ring through the damping of density waves (Esposito *et al.*, 1983), and would imply a local thickness of the rings of a few meters.

For *overlapping* resonances, t_s reads

$$t_s \sim 0.6 \frac{1}{\Sigma_0 a_s^2 \omega_s} \frac{M_p^2}{M_s} \left(\frac{d}{a_s}\right)^4; \qquad (58)$$

thus, $t_s \geq 4.5 \times 10^9$ years requires $M_s \leq 10^4\ (100\ \text{g cm}^{-2}/\Sigma_0)(d/\text{km})^4$ g. This condition would be compatible with inequality (56) only for unrealistically low values of ν.

In conclusion, it appears that *overlapping* resonances are efficient to create little gaps inside the rings, whereas *isolated* resonances are able to truncate the whole disk. Even so, the stability of the system over the age of the Solar System requires values for the kinematic viscosity ν somewhat smaller ($\leq 3\ \text{cm}^2\ \text{sec}^{-1}$) than that inferred from the damping of density waves in Saturn's rings ($\sim 100\ \text{cm}^2\ \text{sec}^{-1}$). The exact value of ν remains the weak link as long as the detailed description of collisions in the rings is not made. Unfortunately such a description requires a precise knowledge of the size distribution, the surface properties, and the shape of the particles themselves, which are presently poorly known.

VI. CONCLUDING REMARKS

This paper generalizes and simplifies the derivation of the torque Γ_m exerted by a satellite on a disk at a Lindblad resonance. Our main conclusions may be summarized as follows:

Provided that (i) the problem remains linear, i.e., the surface density perturbations of the disk are small compared to the unperturbed surface density (there is no streamline crossing), and (ii) the width of the resonance is small compared to the distance of the resonance to the satellite, then:

(1) The torque exerted by the satellite at a Lindblad resonance $m:m \mp 1$ appears as soon as some interactions between the particles allow the disk to avoid the singularity at resonance. Moreover, the value of this torque per unit surface density is *independent* on the parameters describing these interactions (viscosity, pressure, surface density, friction coefficient, etc.) and only depends on the planet and satellite masses and on the order m of the resonance (Eq. (47a)). In the absence of any physical process such as dissipation or waves, the problem is nonstationary, which provides the same Γ_m, as long as linearity holds.

(2) Nevertheless, the torque density $d\Gamma/dr$ (torque per unit radius) at an isolated resonance does depend on the physical process at work in the disk. $d\Gamma/dr$ is proportional to the velocity perturbation $u_{rm}(r)$, and the surface density perturbation σ_m is proportional to the derivative of that quantity: du_{rm}/dr. The dimensionless velocity perturbation $u_{rm}(r)/a_m\Omega_m$ is characterized by a coefficient α such that αa_m is the typical width of the resonance and $1/\alpha$ is the typical maximum value of $u_{rm}(r)/a_m\Omega_m$ reached at resonance. Some examples of functions $u_{rm}(r)$ are given in Figs. 1, 2, 5, 6, and 7. The α coefficients, which depend on the physical properties of the disk, together with analytical expressions of $u_{rm}(r)$ are indicated in Table I. The integral of $d\Gamma/dr$ at an isolated resonance is independent on α, see point (1).

(3) When one coefficient α is significantly larger than the others, the corresponding physical process commands the behavior of the disk around the resonance. Thus Table I provides a convenient way to determine the dominant process in many astrophysical objects. When several coefficients α are of the same order of magnitude, the torque density function is more complicated, but the total torque remains unchanged.

(4) The validity conditions (i) and (ii) quoted above simply read $(M_s/M_p)^{1/2} \ll \alpha \ll 1/m$. The first inequality warrants linearity and the second one prevents the spreading of the resonance over the disk.

(5) The sign of α is directly linked to the sign of the torque. It is positive in all the examples that we have considered. Thus, the satellite tends to push the disk *away* from its own orbit. The positiveness of α is linked either to dissipation of energy or to causality effects when no dissipation is present (e.g., the satellite is "switched on" at a given time for transitory effects to appear the waves must start at the resonance).

(6) Thus, dissipation is *not* necessary for the torque to appear. Dissipation or waves allow the disk to get rid of its excess, or lack, of angular momentum and thus to reach a stationary state. However, only dissipation is able to transfer the angular momentum deposited by the satellite at the resonance to the particles themselves and thus to confine them.

(7) Several astrophysical disks may be described in the framework of the linear approach studied in this paper. Among others, we have considered planetary rings, planetesimals perturbed by Jupiter, and colliding particles in numerical simulation. Other objects of interest are the accretion disks in binary systems and galactic disks. In spite of the very different physics at work in each of these systems, the angular momentum exchange between the disk and the perturber bears essentially the same form.

One of the main results of this paper is the determination of the dominant physical process in a disk, the typical width of the resonances, and the validity conditions of the equations through only one dimensionless coefficient α which depends simply on the physical parameters of the disk.

Extension of this work may be provided by studying corotation resonances and other Lindblad resonances which appear when the satellite orbit is eccentric and/or inclined.

Another point of interest is of course the estimation of the value of the torque once the validity conditions $(M_s/M_p)^{1/2} \ll \alpha \ll 1/m$ break down. This is far from being an academic problem since Saturn's rings, for instance, clearly exhibit nonlinear behavior at some resonances. Also, the system can secularly evolve, so that these validity conditions may be verified at one time and not at the other (e.g., the planetesimal disk in the solar nebula as the gas density drops). Important insights to nonlinear wave behavior in planetary rings can be found in recent papers by Shu *et al.* (1985) and Borderies *et al.* (1986). Although the linear approach can serve as a reference point for a wide variety of physical processes and astrophysical objects, only in a more general study is one able to describe the observed phenomena in a satisfactory way.

ACKNOWLEDGMENTS

We thank S. Tremaine and an anonymous referee for a careful reading of the first version of this paper and for useful comments.

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