RADIATIVE PROCESSE S IN ASTROPHYSICS
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5

BREMSSTRAHLUNG

Radiation due to the acceleration of a charge in the Coulomb field of another charge is called *bremsstrahlung* or *free-free emission*. A full understanding of this process requires a quantum treatment, since photons of energies comparable to that of the emitting particle can be produced. However, a classical treatment is justified in some regimes, and the formulas so obtained have the correct functional dependence for most of the physical parameters. Therefore, we first give a classical treatment and then state the quantum results as corrections (Gaunt factors) to the classical formulas.

First of all we shall treat nonrelativistic bremsstrahlung. Relativistic corrections are treated in §5.4. We note that bremsstrahlung due to the collision of like particles (electron-electron, proton-proton) is zero in the dipole approximation, because the dipole moment $\Sigma e_i \mathbf{r}_i$ is simply proportional to the center of mass $\Sigma m_i \mathbf{r}_i$, a constant of the motion. We therefore must consider two different particles. In electron-ion bremsstrahlung the electrons are the primary radiators, since the relative accelerations are inversely proportional to the masses, and the charges are roughly equal. Since the ion is comparatively massive, it is permissible to treat the electron as moving in a fixed Coulomb field of the ion.

5.1 EMISSION FROM SINGLE-SPEED ELECTRONS

Let us assume that the electron moves rapidly enough so that the deviation of its path from a straight line is negligible. This is the *small-angle scattering* regime. This approximation is not necessary, but it does simplify the analysis and leads to equations of the correct form. Consider an electron of charge $-e$ moving past an ion of charge Ze with impact parameter b (see Fig. 5.1). The dipole moment is $\mathbf{d}=-e\mathbf{R}$, and its second derivative is

$$\ddot{\mathbf{d}}=-e\dot{\mathbf{v}}, \tag{5.1}$$

where $\mathbf{v}$ is the velocity of the electron. Taking the Fourier transform of this equation, noting that the Fourier transform of $\ddot{\mathbf{d}}$ is $-\omega^2\hat{\mathbf{d}}(\omega)$, [cf. Eq. (3.25a)], we have the result:

$$-\omega^2\hat{\mathbf{d}}(\omega)=-\frac{e}{2\pi}\int_{-\infty}^{\infty}\dot{\mathbf{v}}e^{i\omega t}dt. \tag{5.2}$$

It is easy to derive expressions for $\hat{\mathbf{d}}(\omega)$ in the asymptotic limits of large and small frequencies. First we note that the electron is in close interaction with the ion over a time interval, called the *collision time*, which is of order

$$\tau=\frac{b}{v}. \tag{5.3}$$

For $\omega\tau\gg 1$ the exponential in the integral oscillates rapidly, and the integral is small. For $\omega\tau\ll 1$ the exponential is essentially unity, so we may write

$$\hat{\mathbf{d}}(\omega)\sim\begin{cases}\dfrac{e}{2\pi\omega^2}\Delta\mathbf{v}, & \omega\tau\ll 1\\ 0, & \omega\tau\gg 1,\end{cases} \tag{5.4}$$

where $\Delta\mathbf{v}$ is the change of velocity during the collision. Referring to Eq.

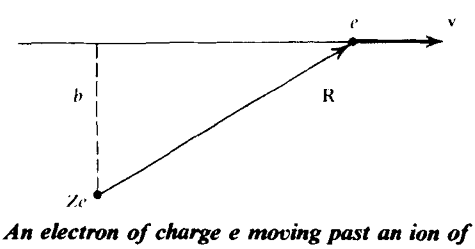

***Figure 5.1** An electron of charge e moving past an ion of charge Ze.*

(3.26b) and using Eq. (5.4), we have

$$\frac{dW}{d\omega}=\begin{cases}\dfrac{2e^2}{3\pi c^3}|\Delta\mathbf{v}|^2, & \omega\tau\ll 1\\ 0, & \omega\tau\gg 1.\end{cases} \tag{5.5}$$

Let us now estimate $\Delta\mathbf{v}$. Since the path is almost linear, the change in velocity is predominantly normal to the path. Thus we merely integrate that component of the acceleration normal to the path:

$$\Delta v=\frac{Ze^2}{m}\int_{-\infty}^{\infty}\frac{b\,dt}{(b^2+v^2t^2)^{3/2}}=\frac{2Ze^2}{mbv},$$

the integral being elementary. Thus for small angle scatterings, the emission from a single collision is

$$\frac{dW(b)}{d\omega}=\begin{cases}\dfrac{8Z^2e^6}{3\pi c^3m^2v^2b^2}, & b\ll v/\omega\\ 0, & b\gg v/\omega.\end{cases} \tag{5.6}$$

We now wish to determine the total spectrum for a medium with ion density n_i, electron density n_e and for a fixed electron speed v. Note that the flux of electrons (electrons per unit area per unit time) incident on one ion is simply $n_e v$. The element of area is $2\pi b\,db$ about a single ion. The total emission per unit time per unit volume per unit frequency range is then

$$\frac{dW}{d\omega\,dV\,dt}=n_en_i2\pi v\int_{b_{\min}}^{\infty}\frac{dW(b)}{d\omega}b\,db, \tag{5.7}$$

where $b_{\min}$ is some minimum value of impact parameter; its choice is discussed below.

It would seem that the asymptotic limits (5.6) are insufficient to evaluate the integral in Eq. (5.7), which requires values of $dW(b)/d\omega$ for a full range of impact parameters. However, it turns out that a very good approximation can be achieved using only its low frequency asymptotic form. To see this, substitute the $b\ll v/\omega$ result of Eq. (5.6) into Eq. (5.7). This gives

$$\frac{dW}{d\omega\,dV\,dt}=\frac{16e^6}{3c^3m^2v}n_en_iZ^2\int_{b_{\min}}^{b_{\max}}\frac{db}{b}=\frac{16e^6}{3c^3m^2v}n_en_iZ^2\ln\left(\frac{b_{\max}}{b_{\min}}\right), \tag{5.8}$$

where $b_{\max}$ is some value of b beyond which the $b \ll v/\omega$ asymptotic result is inapplicable and the contribution to the integral becomes negligible. The value of $b_{\max}$ is uncertain, but it is of order v/ω. Since $b_{\max}$ occurs inside the logarithm, its precise value is not very important, so we simply take

$$b_{\max} \equiv \frac{v}{\omega}, \tag{5.9}$$

and make a small error. It can now be seen that the use of the asymptotic forms (5.6) is justified, because equal intervals in the logarithm of b contribute equally to the emission, and over most of these intervals the emission is determined by its low frequency asymptotic limit.

The value of $b_{\min}$ can be estimated in two ways. First we can take the value at which the straight-line approximation ceases to be valid. Since this occurs when $\Delta v \sim v$, we take

$$b_{\min}^{(1)} = \frac{4Ze^2}{\pi m v^2}. \tag{5.10a}$$

A second value for $b_{\min}$ is quantum in nature and concerns the possibility of treating the collision process in terms of classical orbits, as we have done. By the uncertainty principle $\Delta x \Delta p \gtrsim \hbar$; and taking $\Delta x \sim b$ and $\Delta p \sim mv$ we have

$$b_{\min}^{(2)} = \frac{h}{mv}. \tag{5.10b}$$

When $b_{\min}^{(1)} \gg b_{\min}^{(2)}$ a classical description of the scattering process is valid, and we use $b_{\min} = b_{\min}^{(1)}$. This occurs when $\frac{1}{2}mv^2 \ll Z^2 Ry$, where $Ry = me^4/(2\hbar^2)$ is the Rydberg energy for the hydrogen atom. When $b_{\min}^{(1)} \ll b_{\min}^{(2)}$, or, equivalently, $\frac{1}{2}mv^2 \gg Z^2 Ry$, the uncertainty principle plays an important role, and the classical calculation cannot strictly be used. Nonetheless, results of the correct order of magnitude are obtained by simply setting $b_{\min} = b_{\min}^{(2)}$.

For any regime the exact results are conveniently stated in terms of a correction factor or *Gaunt factor* $g_{ff}(v,\omega)$ such that

$$\frac{dW}{d\omega\, dV\, dt} = \frac{16\pi e^6}{3\sqrt{3}\, c^3 m^2 v} n_e n_i Z^2 g_{ff}(v,\omega). \tag{5.11}$$

Comparison of Eqs. (5.8) and (5.11) gives g_{ff} in terms of an effective logarithm

$$g_{ff}(v,\omega)=\frac{\sqrt{3}}{\pi}\ln\left(\frac{b_{\max}}{b_{\min}}\right). \tag{5.12}$$

The Gaunt factor is a certain function of the energy of the electron and of the frequency of the emission. Extensive tables and graphs of it exist in the literature. See, for instance, the review article by Bressaard and van de Hulst, (1962) and the article by Karzas and Latter (1961).

5.2 THERMAL BREMSSTRAHLUNG EMISSION

The most interesting use of these formulas is their application to *thermal bremsstrahlung*; that is, we average the above single-speed expression over a thermal distribution of speeds. The probability dP that a particle has velocity in the velocity range $d^3\mathbf{v}$ is

$$dP \propto e^{-E/kT}d^3\mathbf{v}=\exp\left(-\frac{mv^2}{2kT}\right)d^3\mathbf{v}.$$

Since $d^3\mathbf{v}=4\pi v^2dv$ for an isotropic distribution of velocities, the probability that a particle has a speed in the speed range dv is

$$dP \propto v^2\exp\left(-\frac{mv^2}{2kT}\right)dv. \tag{5.13}$$

Now we want to integrate Eq. (5.11) over this function. What are the limits of integration? At first guess, one would choose $0\leqslant v<\infty$. But at frequency ν, the incident velocity must be at least such that

$$h\nu \leqslant \tfrac{1}{2}mv^2$$

because otherwise a photon of energy $h\nu$ could not be created. This cutoff in the lower limit of the integration over electron velocities is called a *photon discreteness effect*. Performing the integral

$$\frac{dW(T,\omega)}{dV\,dt\,d\omega}=\frac{\displaystyle\int_{v_{\min}}^{\infty}\frac{dW(v,\omega)}{d\omega\,dV\,dt}v^2\exp(-mv^2/2kT)\,dv}{\displaystyle\int_0^{\infty}v^2\exp(-mv^2/2kT)\,dv},$$

where $v_{\min} \equiv (2h\nu/m)^{1/2}$, and using $d\omega = 2\pi d\nu$, we obtain

$$\frac{dW}{dV\,dt\,d\nu} = \frac{2^5\pi e^6}{3mc^3}\left(\frac{2\pi}{3km}\right)^{1/2} T^{-1/2} Z^2 n_e n_i e^{-h\nu/kT} \bar{g}_{ff}. \qquad (5.14a)$$

Evaluating eq. (5.14) in CGS units, we have for the emission (erg s^{-1} cm^{-3} Hz^{-1})

$$\epsilon_\nu^{ff} \equiv \frac{dW}{dV\,dt\,d\nu} = 6.8 \times 10^{-38} Z^2 n_e n_i T^{-1/2} e^{-h\nu/kT} \bar{g}_{ff}. \qquad (5.14b)$$

Here $\bar{g}_{ff}(T,\nu)$ is a *velocity averaged Gaunt factor*. The factor $T^{-1/2}$ in Eq. (5.14) comes from the fact that $dW/dV\,dt\,d\omega \propto v^{-1}$ [cf. Eq. (5.11) and $\langle v \rangle \propto T^{1/2}$. The factor $e^{-h\nu/kT}$ comes from the lower-limit cutoff in the velocity integration due to photon discreteness and the Maxwellian shape for the velocity distribution.

Approximate analytic formulas for $\bar{g}_{ff}$ in the various regimes in which large-angle scatterings and small-angle scatterings are dominant, in which

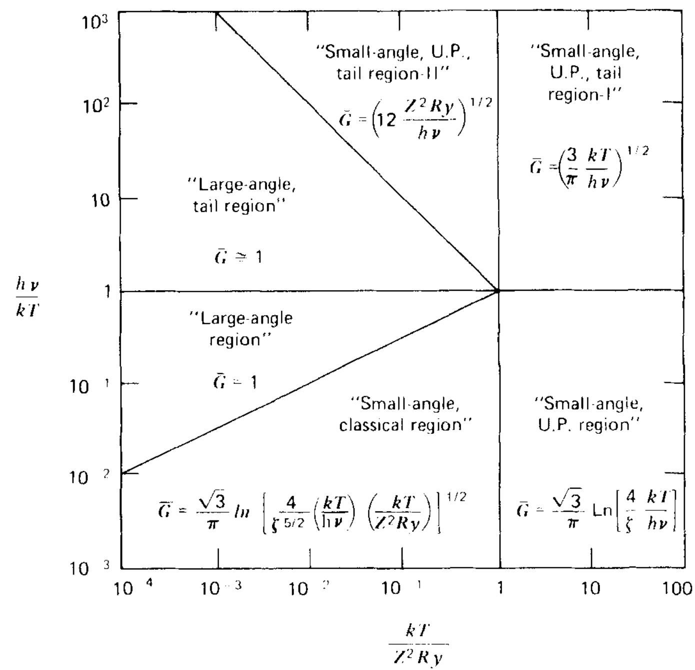

***Figure 5.2** Approximate analytic formulae for the gaunt factor $\bar{g}_{ff}(\nu, T)$ for thermal bremsstrahlung. Here $\bar{g}_{ff}$ is denoted by $\bar{G}$ and the energy unit $Ry = 13.6$ eV. (Taken from Novikov, I. D. and Thorne, K. S. 1973 in Black Holes, Les Houches, Eds. C. DeWitt and B. DeWitt, Gordon and Breach, New York.)*

the uncertainty principle (U. P.) is important in the minimum impact parameter, and so on are indicated in Fig. 5.2. Figure 5.3 gives numerical graphs of $\bar{g}_{ff}$. The values of $\bar{g}_{ff}$ for $u \equiv h\nu/kT \gg 1$ are not important, since the spectrum cuts off for these values. Thus $\bar{g}_{ff}$ is of order unity for $u \sim 1$ and is in the range 1 to 5 for $10^{-4} < u < 1$. We see that good order of magnitude estimates can be made by setting $\bar{g}_{ff}$ to unity.

We also see that bremsstrahlung has a rather "flat spectrum" in a log-log plot up to its cutoff at about $h\nu \sim kT$. (This is true only for optically thin sources. We have not yet considered *absorption* of photons by free electrons.)

To obtain the formulas for *nonthermal bremsstrahlung*, one needs to know the actual distributions of velocities, and the formula for emission from a single-speed electron must be averaged over that distribution. To do this one also must have the appropriate Gaunt factors.

Let us now give formulas for the total power per unit volume emitted by thermal bremsstrahlung. This is obtained from the spectral results by integrating Eq. (5.14) over frequency. The result may be stated as

$$\frac{dW}{dt\,dV} = \left(\frac{2\pi kT}{3m}\right)^{1/2} \frac{2^5\pi e^6}{3hmc^3} Z^2 n_e n_i \bar{g}_B, \tag{5.15a}$$

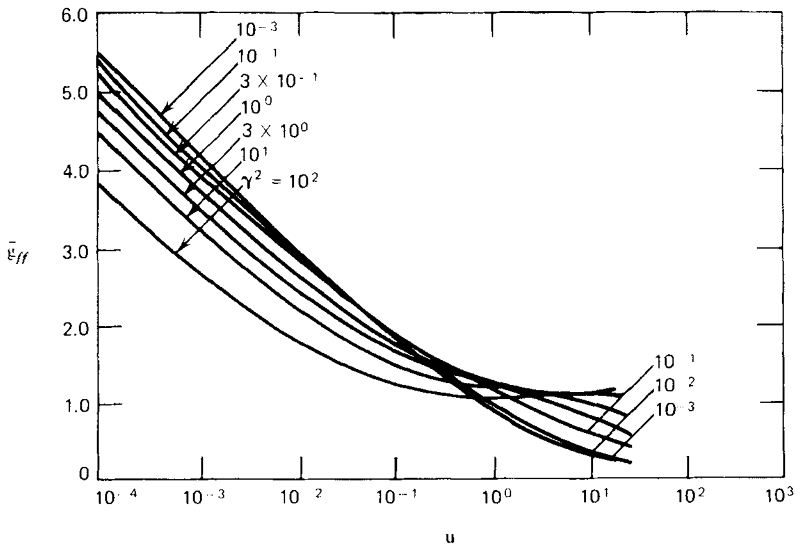

***Figure 5.3** Numerical values of the gaunt factor $\bar{g}_{ff}(\nu, T)$. Here the frequency variable is $u = 4.8 \times 10^{11}\nu/T$ and the temperature variable is $\gamma^2 = 1.58 \times 10^5 Z^2/T$. (Taken from Karzas, W. and Latter, R. 1961, Astrophys. J. Suppl., 6, 167.)*

or numerically, again in CGS units, the emission (erg s^{-1} cm^{-3}) is

$$\epsilon^{ff} \equiv \frac{dW}{dt\,dV} = 1.4\times 10^{-27} T^{1/2} n_e n_i Z^2 \bar{g}_B. \qquad (5.15b)$$

Here $\bar{g}_B(T)$ is a frequency average of the velocity averaged Gaunt factor, which is in the range 1.1 to 1.5. Choosing a value of 1.2 will give an accuracy to within about 20%.

5.3 THERMAL BREMSSTRAHLUNG (FREE-FREE) ABSORPTION

It is possible to relate the absorption of radiation by an electron moving in the field of an ion to the preceding bremsstrahlung emission process. The most interesting case is thermal free-free absorption. In that case we have Kirchhoff's law [cf. Eq. (1.37)]

$$j_\nu^{ff} = \alpha_\nu^{ff} B_\nu(T). \qquad (5.16)$$

Here α_ν^{ff} is the free-free absorption coefficient, and j_ν^{ff} is related to the preceding emission formula by

$$\frac{dW}{dt\,dV\,d\nu} = 4\pi j_\nu^{ff}. \qquad (5.17)$$

With the form for the Planck function [Eq. (1.51)], we have then

$$\alpha_\nu^{ff} = \frac{4e^6}{3mhc}\left(\frac{2\pi}{3km}\right)^{1/2} T^{-1/2} Z^2 n_e n_i \nu^{-3}(1-e^{-h\nu/kT})\bar{g}_{ff}. \qquad (5.18a)$$

Evaluating Eq. (5.18a) in CGS units, we have for $\alpha_\nu^{ff}(\mathrm{cm}^{-1})$:

$$\alpha_\nu^{ff} = 3.7\times 10^{8} T^{-1/2} Z^2 n_e n_i \nu^{-3}(1-e^{-h\nu/kT})\bar{g}_{ff}. \qquad (5.18b)$$

For $h\nu \gg kT$ the exponential is negligible, and α_ν is proportional to ν^{-3}. For $h\nu \ll kT$, we are in the Rayleigh–Jeans regime, and Eq. (5.18a) becomes

$$\alpha_\nu^{ff} = \frac{4e^6}{3mkc}\left(\frac{2\pi}{3km}\right)^{1/2} T^{-3/2} Z^2 n_e n_i \nu^{-2}\bar{g}_{ff}, \qquad (5.19a)$$

or, numerically,

$$\alpha_\nu^{ff} = 0.018\, T^{-3/2} Z^2 n_e n_i \nu^{-2}\bar{g}_{ff}. \qquad (5.19b)$$

The Rosseland mean of the free-free absorption coefficient [Eq. (1.109)] is, in CGS units,

$$\alpha_R^{ff} = 1.7\times 10^{-25} T^{-7/2} Z^2 n_e n_i \bar{g}_R, \qquad (5.20)$$

where $\bar{g}_R$ is an appropriately weighted frequency average of $\bar{g}_{ff}$, and is of order unity.

5.4 RELATIVISTIC BREMSSTRAHLUNG

Our previous discussion of bremsstrahlung was for nonrelativistic particles. We now show how the relativistic case can be treated by an interesting and physically picturesque method called the *method of virtual quanta.* A classical treatment provides useful insight, even though a full understanding would require quantum electrodynamics.

We consider the collision between an electron and a heavy ion of charge Ze. Normally, the ions move rather slowly in comparison to the electrons (in the rest frame of the medium as a whole), but it is possible to view the process in a frame of reference in which the electron is initially at rest. In that case the ion appears to move rapidly toward the electron. With no loss of generality we can assume that the ion moves along the x axis with velocity v while the electron is initially at rest on the y axis, a distance b from the origin. From the discussion of §4.6 we recall that the electrostatic field of the ion is transformed into an essentially transverse pulse with $|\mathbf{E}|\sim|\mathbf{B}|$, which appears to the electron to be a pulse of electromagnetic radiation (see Fig. 5.4). This radiation then Compton scatters off the electron to produce emitted radiation. Transforming back to the rest frame

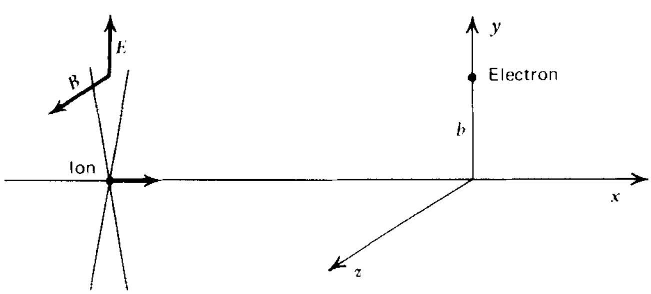

***Figure 5.4** Electric and magnetic fields of an ion moving rapidly past an electron.*

of the ion (or lab frame) we obtain the bremsstrahlung emission of the electron. Thus relativistic bremsstrahlung can be regarded as the Compton scattering of the *virtual quanta* of the ion's electrostatic field as seen in the electron's frame.

In the (primed) electron rest frame, the spectrum of the pulse of virtual quanta has the form, [cf. Eq. (4.72b)]

$$\frac{dW'}{dA'd\omega'}(\text{erg cm}^{-2}\text{ Hz}^{-1}) = \frac{(Ze)^2}{\pi^2 b'^2 c}\left(\frac{b'\omega'}{\gamma c}\right)^2 K_1^2\left(\frac{b'\omega'}{\gamma c}\right), \tag{5.21}$$

where we have set $v=c$ in the ultrarelativistic limit. Now, in this frame the virtual quanta are scattered by the electron according to the Thomson cross section for $\hbar\omega' \lesssim mc^2$, and according to the Klein-Nishina cross section for $\hbar\omega' > mc^2$ [see Chapter 7]. In the low-frequency limit, the scattered radiation is

$$\frac{dW'}{d\omega'} = \sigma_T \frac{dW'}{dA'd\omega'}, \tag{5.22}$$

where σ_T is the Thomson cross section. Now, since energy and frequency transform identically under Lorentz transformations, we have for the energy emitted per frequency in the lab frame, $dW/d\omega = dW'/d\omega'$. To write $dW/d\omega$ as a function of b and ω, rather than b' and ω', we note that transverse lengths are unchanged, $b=b'$, and that $\omega = \gamma\omega'(1+\beta\cos\theta')$, [cf. Eq. (4.12b), where θ' is the scattering angle in the electron rest frame]. Because such scattering is forward-backward symmetric, we have the averaged relation $\omega = \gamma\omega'$. Thus the emission in the lab frame is

$$\frac{dW}{d\omega} = \frac{8Z^2e^6}{3\pi b^2c^5m^2}\left(\frac{b\omega}{\gamma^2 c}\right)^2 K_1\left(\frac{b\omega}{\gamma^2 c}\right). \tag{5.23}$$

Equation (5.23) is the energy per unit frequency emitted by the collision of an ion and a relativistic electron at impact parameter b. For a plasma with electron and ion densities n_e and n_i, respectively, we can repeat the arguments leading to Eq. (5.7), where v is replaced by c and where $b_{\min} \sim h/mc$ according to the uncertainty principle. The integral in Eqs. (5.7) and (5.23) is identical to that in Eq. (4.74a), except for an additional factor of γ in the argument. Thus we have the low-frequency limit, $\hbar\omega \ll \gamma mc^2$,

$$\frac{dW}{dt\,dV\,d\omega} \sim \frac{16Z^2e^6n_en_i}{3c^4m^2}\ln\left(\frac{0.68\gamma^2 c}{\omega b_{\min}}\right). \tag{5.24}$$

At higher frequencies Klein–Nishina corrections must be used.

For a thermal distribution of electrons, a useful approximate expression for the frequency integrated power (erg s^{-1} cm^{-3}) in CGS units is [see Novikov and Thorne 1973]

$$\frac{dW}{dV\,dt} = 1.4\times 10^{-27} T^{1/2} Z^2 n_e n_i \bar{g}_B (1 + 4.4\times 10^{-10} T). \tag{5.25}$$

The second term in brackets is a relativistic correction to Eq. (5.15b).

PROBLEMS

5.1—Consider a sphere of ionized hydrogen plasma that is undergoing spherical gravitational collapse. The sphere is held at constant isothermal temperature T_0, uniform density and constant mass M_0 during the collapse, and has decreasing radius $R(t)$. The sphere cools by emission of bremsstrahlung radiation in its interior. At $t = t_0$ the sphere is optically thin.

a. What is the total luminosity of the sphere as a function of M_0, $R(t)$ and T_0 while the sphere is optically thin?

b. What is the luminosity of the sphere as a function of time after it becomes optically thick?

c. Give an implicit relation, in terms of $R(t)$, for the time t_1 when the sphere becomes optically thick.

d. Draw a qualitative curve of the luminosity as a function of time.

5.2—Suppose X-rays are received from a source of known distance L with a flux F (erg cm^{-2} s^{-1}). The X-ray spectrum has the form of Fig. 5.5 It is conjectured that these X-rays are due to bremsstrahlung from an optically thin, hot, plasma cloud, which is in hydrostatic equilibrium around a central mass M. Assume that the cloud thickness ΔR is roughly its radius R, $\Delta R \sim R$. Find R and the density of the cloud, ρ, in terms of the known observations and conjectured mass M. If $F = 10^{-8}$ erg cm^{-2} s^{-1}, $L = 10$ kpc, what are the constraints on M such that the source would indeed be effectively thin (for self-consistency)? Does electron scattering play any role? Here 1 kpc$\equiv$one kiloparsec, a unit of distance$\approx 3.1\times 10^{21}$ cm.

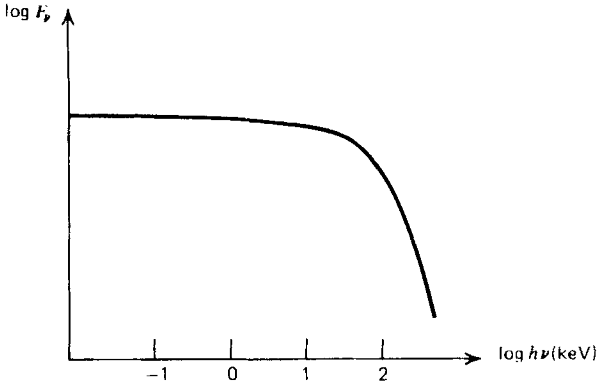

***Figure 5.5** Detected spectrum from an X-ray source.*

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