Pollack et al. (1979) and Pollack and Fanale (1982). Numerical integrations demonstrate that, for capture to happen while traversing the nebula, the planetoid's speed must be slowed by at least tens of percent. This in turn means that the planetoid, during its flight across the nebula, must interact with gas that has mass equal to tens of percent, defined as the fraction β, of its own mass. Since β is inversely proportional to the planetoid's radius, small objects are preferentially acquired. Nevertheless, for expected nebula models (Pollack et al. 1977), objects up to 10 to 10^2 km can be slowed enough to be captured near the edges of the nebula; deeper penetrations of the nebula are necessary to ensnare a satellite like Triton. The abrupt deceleration that transpires in the capture event will likely cause fragmentation. The relative speed between members of either Jovian cluster are of the same order as escape velocities off the largest member; they hint that each cluster was born when a primordial object barely split apart.

Once capture by gas drag is accomplished, orbital evolution proceeds swiftly because the acquired planetoid continues to sweep through the circumplanetary envelope. The characteristic time for evolution is $\sim P/\beta$ (where P is the satellite's orbital period) or ~ 10 yr. This rapid evolution has an important implication: post-capture orbital evolution can quickly turn irregular orbits into fairly circular, uninclined ones that ultimately crash into the planet. Indeed some process must halt the evolution (i.e., remove the nebula) if any satellites are to be found in orbit. Fortunately models of nebular evolution do undergo such a collapse phase. If this scenario is correct, the current outer satellites are only the last of many captured objects, their predecessors having fallen into the planet proper.

IV. CONCLUSION

Even though the solar system itself has not changed much in the decade since the last review of satellite evolution (Burns 1977*b*), our understanding of it has. I foresee a similar evolution over the next decade, with resonances and ring-satellite interactions better comprehended. The advent of supercomputers should help sort out the complex interactions of many-body systems. Just as surely, new and equally interesting puzzles will arise to be solved.

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5. ORBITAL RESONANCES, UNUSUAL CONFIGURATIONS AND EXOTIC ROTATION STATES AMONG PLANETARY SATELLITES

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Several examples of satellite dynamics are presented where significant progress has been made in understanding a complex problem, where a long-standing problem has finally been solved, where newly discovered configurations have motivated novel descriptions or where an entirely new phenomenon has been revealed. The origin of orbital resonances is shown in the demonstration of the evolution of a pair of planetary satellites through a commensurability of the mean motions by a sequence of diagrams of constant energy curves in a two-dimensional phase space, where the closed curve corresponding to the motion in each successive diagram is identified by its adiabatically conserved area. All of the major features of orbital resonance capture and evolution can be thus understood with a few simple ideas. Qualifications on the application of the theory to real resonances in the solar system are presented. The two-body resonances form a basis for the solution of the problem of origin and evolution of the three-body Laplace resonance among the Galilean satellites of Jupiter. Dissipation in Io is crucial to the damping of the amplitude of the Laplace libration to its observed small value. The balance of the effects of tidal dissipation in Io to that in Jupiter leads to rather tight bounds on the rate of dissipation of tidal energy in Jupiter. Motion in the relative horseshoe orbits of Saturn's coorbital satellites is described very well by a simple expansion about circular reference orbits. The coorbitals are currently very stable, and their relative motions can be used for the determination of the masses of both satellites. Pluto and its relatively large satellite Charon form an unusual system where the relative size and proximity of Charon lead to a most probable state where both Pluto and Charon are rotating synchronously with their orbital motion. The normal tidal evolution of a satellite spin toward synchronous rotation is frustrated in the case of Saturn's satellite Hyperion where gravitational torques on the large permanent

asymmetry cause it to tumble chaotically. Observations of Hyperion's lightcurve are consistent with the chaotic rotation but do not verify it with certainty.

I. INTRODUCTION

The planetary satellite systems have provided a long list of puzzles involving origins and evolutions of various configurations that have slowly yielded to solution over the years. Much more detailed information about these systems from recent spacecraft observations has motivated a flurry of dynamical analysis—some of which does not even depend much on the new observations. We describe several examples of this recent activity, where the emphasis is on resonances and the consequences of the dissipation of tidal energy. By resonance we mean a situation in which the mean angular velocities of two satellites have a ratio near that of two small integers.

The dynamical evolutions of various satellite configurations in the solar system include the effects of energy dissipation on both the orbits and the spins. We discuss first (Sec. II) the origin and evolution of the two-body orbital resonances among the satellites, where significant progress has been made in simplifying the rather complex and diverse mathematical descriptions. It is now possible to understand the origin and evolution of orbital resonances in terms of a few simple principles. We begin here with a heuristic discussion of a two-body orbital resonance as an introduction to the description of the mathematical development. This development starts with first principles, and will require the reader to be familiar with graduate level classical mechanics but only the more commonly known jargon of celestial mechanics. The reward for the persevering reader will be a knowledge of how the relatively complex system is reduced to a single degree of freedom to yield the simplified Hamiltonian used in the analysis and of the approximations used in getting it. We show how the evolution of a two-body orbital resonance due to differential tidal expansion of the respective orbits can be followed through a sequence of diagrams showing curves of constant Hamiltonian in the two-dimensional phase space. The trajectory of the system in this phase space is essentially one of the curves of constant Hamiltonian in each diagram. Since the Hamiltonian is not conserved as the system evolves, the trajectory corresponding to the motion in successive diagrams is identified by the adiabatically conserved action. The conditions for capture into an orbital resonance as tides push the system toward a commensurability of the mean motions and the subsequent evolution within the resonance are easily seen in the sequence of diagrams. The model also allows analytic determinations of the probability of capture into the resonance when the system reaches a commensurability of mean motions.

This analysis has greatly eased the understanding of the establishment and current configurations of orbital resonances among the satellites. We indicate how it can give reasonably accurate quantitative descriptions of real resonances in some cases, but many existing resonances require a more accurate analysis—often with much more elaborate models including dissipation within the satellites and, in the case of Saturn's satellites, strong interactions with the rings.

In Sec. III we show how the analysis of the two-body orbital resonances described in Sec. II suffices for the description of the two-body resonances among the Galilean satellites of Jupiter. This description works in spite of the existence of the three-body Laplace resonance and the simultaneous libration of more than one resonance variable at the same commensurability of the mean motions. The two-body resonances form the basis for understanding the evolution to, and capture within, the more complex Laplace orbital resonance involving the inner three Galilean satellites. The subsequent analysis describes the evolution of the Galilean satellite system to its current configuration of a very small libration amplitude for the three-body Laplace resonance, where a high rate of dissipation of tidal energy in the satellite Io is crucial to the damping of the libration. This theory for the origin of the Laplace resonance, especially the understanding of the almost zero amplitude of libration of the Laplace angle, must rank as one of the major accomplishments in the study of dynamical evolution in the last decade. A perhaps unexpected bonus from this work is the establishment of rather tight constraints on the rate of dissipation of tidal energy in Jupiter. As in Sec. II, sufficient mathematical detail is presented here to enable one to follow the development without mystery concerning the source of the conclusions and the approximations involved. In Sec. IV we describe the coorbital satellites of Saturn, where the evolutionary aspects of this system apply more to the future than to the past. The interest in this system lies in its uniqueness and in the very simple modification of the analysis of the restricted three-body problem which suffices for its description. The relative horseshoe orbits for the satellites are shown to be currently very stable. In principle, the masses and densities of both coorbitals can be found by inserting observational parameters into the analysis, but accurate values will require more precise observations from a spacecraft orbiting Saturn. From an emphasis on orbital configuration and evolution, we turn (Sec. V) to a description of the Pluto-Charon system, where both planet and satellite appear to rotate synchronously with the orbital motion. This isolated system has thus reached the ultimate endpoint of tidal evolution, and its recent discovery, its uniqueness and the interesting dynamics of the dual-synchronous rotation state warrants its inclusion here. Two possible states of dual-synchronous rotation for a given total angular momentum are shown to exist—an unstable state at close separation and a stable state at a more distant separation. The Pluto-Charon system occupies the latter. The tidal evolution of the spins of the satellites discussed to this point have been a relentless retardation by tides toward the currently observed synchronous rotation. Except for the Pluto-Charon example, this evolution has not been elaborated, since it has been long understood. Hyperion, on the other hand, is distinguished from other satellites of its size by being very nonspherical. In addition, it occupies an orbital resonance with Titan which maintains its very large orbital eccentricity of about 0.1. We show in Sec. VI that this combination of circumstances forces Hyperion to evolve not toward an orderly synchronous or other commensurate spin-orbit state, but to a region in phase space where it will most likely tumble chaotically, essentially forever.

II. ORBITAL RESONANCES

If two satellites orbiting the same primary have mean orbital angular velocities (mean motions) which have a ratio very near that of two small integers, the satellite motions are said to be commensurate and to define an orbital resonance. Orbital resonances among the satellites of the major planets are of interest because many more such resonances exist than can be accounted for by a random distribution of orbits (Roy and Ovenden 1955). The inner three Galilean satellites of Jupiter, Io, Europa and Ganymede, are locked in the famous Laplace relation where $n_1 - 3n_2 + 2n_3 = 0$ with $\lambda_1 + 3\lambda_2 + 2\lambda_3$ librating about 180° with very small amplitude. The subscripts on the mean motions and mean longitudes (n_i, λ_i) number the satellites sequentially from the closest to the farthest from Jupiter. In addition, $\lambda_1 - 2\lambda_2 + \tilde{\omega}_1$ and $\lambda_2 - 2\lambda_3 + \tilde{\omega}_2$ librate about 0° and $\lambda_1 - 2\lambda_2 + \tilde{\omega}_2$ librates about π, where the $\tilde{\omega}_i$ are the longitudes of periapses. At Saturn, Enceladus-Dione, Mimas-Tethys and Titan-Hyperion are locked in orbital resonances with $n_i/n_j = 1/2$, $1/2$ and $3/4$, respectively. For Mimas-Tethys, $2\lambda_1 - 4\lambda_3 + \Omega_1 + \Omega_3$ librates about 0° (Ω_i are the longitudes of ascending nodes), for Enceladus-Dione, $\lambda_2 - 2\lambda_4 + \tilde{\omega}_2$ librates about 0° and for Titan-Hyperion, $3\lambda_6 - 4\lambda_7 + \tilde{\omega}_7$ librates about π. Voyagers 1 and 2, as well as contemporaneous groundbased observations, revealed several examples of stable commensurabilities in the Saturn system where two satellites' periods were nearly identical. Some were librating about the Trojan points 60° in front of or behind a large satellite whereas one pair, the so-called coorbitals, have horseshoe-shaped orbits relative to each other. We shall address the analysis of this last 1:1 resonance in Sec. IV and consider here only the resonances where the mean orbital motions are near the ratio $j:j+k$ where j and k are nonzero integers.

The hypothesis of the formation of orbital resonances by the differential tidal expansion of initially randomly distributed orbits was suggested by T. Gold (personal communication, 1960). Goldreich (1965) showed the resonances to be stable against continued tidal expansion of the orbits and Allan (1969) demonstrated how the age of a resonance follows from the amplitude of libration if the rate of tidal expansion of the orbits is assumed. Greenberg et al. (1972) and Greenberg (1973*a*) demonstrated the automatic capture of a Titan-Hyperion type resonance and the subsequent evolution within the resonance, and Sinclair (1972) calculated capture probabilities for the resonances involving Mimas-Tethys and Enceladus-Dione, respectively, as the resonance

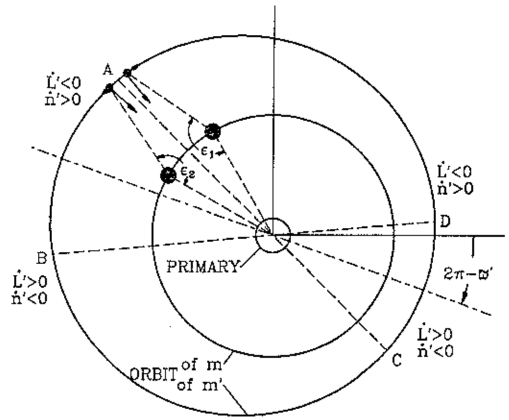

Fig. 1. Schematic diagram of resonant orbits demonstrating stability. Arbitrary positions of repetitive conjunctions are at points A, B, C and D. L and n are angular momentum and mean motion, respectively.

variable of each system passed from circulation to libration. Yoder (1973, 1979*a*) developed the first analytic theory which described the complete origin and evolution of an arbitrary two-body resonance including analytically determined capture probabilities. The field was reviewed by Peale (1976*b*) and by Greenberg (1977). Heuristic descriptions of several resonance properties and processes are given in both reviews. One of these descriptions of stability and some evolutionary aspects are given here to illuminate the physical basis for the mathematical development to follow.

The stability of two-body orbital resonances was understood at the time of Laplace and follows from the ensuing argument. Consider two satellites of masses $m >> m'$ in coplanar orbits about a primary. The inner satellite m is assumed to be in a circular orbit and is so much more massive than m' that perturbations by the latter can be ignored. The mean motions are assumed to be nearly commensurate, and m' to be in an eccentric orbit. The orbits are shown schematically in Fig. 1, where $\tilde{\omega}'$ is the longitude of the outer orbit's pericenter. Four arbitrary positions of repetitive conjunctions are indicated by dashed lines, and the relative positions of each satellite just before and just after a conjunction are also shown along with the radial and tangential components of the perturbing force by m on m'. A dot over a symbol indicates time differentiation.

During the period from opposition to conjunction, m removes angular momentum from m' via the tangential component of the perturbing force and, from conjunction to the next opposition, it adds angular momentum. If the conjunction occurs exactly at pericenter or exactly at apocenter, the effects of the tangential forces integrate to zero, and there is no net transfer. Repetitive conjunctions at any other point destroy this symmetry. If we assume that the line of apsides is fixed and also assume precise commensurability of the mean motions, a conjunction at point A in Fig. 1, for example, would be followed by successive conjunctions at the same point for noninteracting satellites. However, the tangential component of the perturbing force is larger prior to conjunction than after (for $\varepsilon_1 = \varepsilon_2$ in Fig. 1) because the orbits are diverging. In addition, the angular velocity of m' is closer to that of m prior to conjunction, as m' is slowing down as it approaches apocenter. This means m catches up with m' more slowly than it recedes after conjunction, so the larger tangential force opposing the motion of m is also applied for a longer time than the smaller tangential force in the opposite sense after conjunction. Hence, a conjunction at A leads to a net loss of angular momentum by m' over an entire synodic period. The resulting increase in the mean orbital angular velocity n' means that the next conjunction is closer to apocenter.

Similarly, a conjunction after apocenter (point B in Fig. 1) results in a net gain of angular momentum by m', and a tendency for the next conjunction to be again closer to apocenter. The conjunctions thus librate stably about the apocenter of m', preserving the commensurability. Allowing a secular variation of $\tilde{\omega}'$ does not change this conclusion as the ratio n/n' is adjusted such that conjunctions still librate about the apocenter.

The same arguments applied to a conjunction at points C or D near pericenter show that conjunctions are again driven toward apocenter. The pericenter conjunctions thus correspond to an unstable equilibrium configuration like that of a pendulum near the top of its support. The stable point of the analogous pendulum corresponds to the apocenter conjunction.

Now suppose conjunctions occur repetitively at apocenter with no libration and that the inner orbit is being expanded by tidal interactions with the primary. The orbital period of m will increase, and successive conjunctions will occur slightly after apocenter on the average. Angular momentum will thus be secularly transferred in just the right amount to preserve the commensurability against the tendency of the tide to disrupt it (Goldreich 1965).

Two other characteristics of a stable commensurability can now be understood. First, if conjunctions always occur at apocenter of the outer satellite in this example, the radial force of m on m' accelerates m' toward the primary, and m' follows a trajectory slightly inside the trajectory it would have followed if m were not there. This means m' will reach its closest point to the primary slightly sooner than normal and the line of apsides will have rotated in a retrograde sense. If m is sufficiently massive, this regression of the line of apsides due to the resonant perturbation (conjunction always at apocenter) can dominate the normal prograde motion due to the oblateness of the primary and the secular perturbations from other satellites. This is actually realized in the Titan-Hyperion resonance where the line of apsides of Hyperion's orbit regresses about $19°\ \mathrm{yr}^{-1}$.

The second characteristic is the secular increase of the eccentricity in

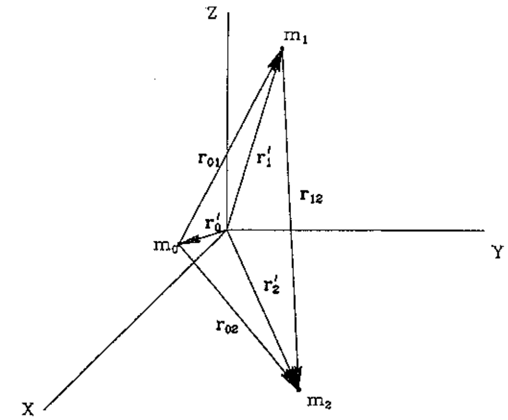

Fig. 2. Definition of variables used in orbital resonance analysis.

this type of orbital resonance. Recall that a tidal expansion of the inner orbit causes the conjunctions of the stable resonance to occur slightly after apocenter. A radial impulse force anywhere between apocenter and pericenter causes the orbiting body to fall closer to the primary, thereby increasing the eccentricity e' of its orbit. With conjunctions now occurring slightly after apocenter, the maximum of the radial perturbative force tends to increase e' secularly.

The stability of a resonance to tidal expansion of the orbits and the growth of the eccentricity of that satellite orbit whose longitude of apoapse or periapse is always near the conjunctions is thus understood from simple ideas. These properties of orbital resonances emerge naturally in the description below, and they play fundamental roles in the evolutionary process. Additional heuristic descriptions are found in Peale (1976*b*) and in Greenberg (1977). See also Greenberg (1973*b*) for a lucid description of the Mimas-Tethys resonance, which involves the inclinations of *both* orbits to Saturn's equatorial plane with libration of conjunctions about the mean of ascending node longitudes.

The relatively complex descriptions of origin and evolution of orbital resonances by differential tidal expansion of the orbits have recently been simplified enormously (Henrard 1982*a*; Henrard and Lemaître 1983). An elegant description of the capture process (Henrard 1982*a*) has led to a second, much simplified analytic evaluation of capture probability (Borderies and Goldreich 1984) applicable to the most common orbital resonances. However, a modification of these capture probabilities may be necessary when the chaotic nature of the separatrix (the curve in phase space separating circulation from libration) is taken into account (Wisdom et al. 1984) (see Sec. VI).

We shall develop here the Hamiltonian appropriate to two-body resonances from first principles and justify various approximations. Henrard and Lemaître (1983) use this Hamiltonian together with the adiabatic invariance of the system action in a description in which most of the resonance phenomena are easily followed and understood as the system evolves into and within libration due to tidal expansion of the orbits. The success of this simplified theory in explaining the origin and current states of some orbital resonances, as well as cases in which better approximations are necessary, will be demonstrated by examples.

We describe the motion of two interacting satellites orbiting a primary as a system of three point masses $m_0 >> m_1$, m_2 with position vectors $\mathbf{r}_i'$ from a fixed origin at the center of mass (Fig. 2). Relative positions are indicated by $\mathbf{r}_{ij} = \mathbf{r}_j' - \mathbf{r}_i'$. The equations of motion are

$$m_i \ddot{\mathbf{r}}_i' = \sum_{i \neq j} \frac{G\, m_i m_j\, \mathbf{r}_{ij}}{r_{ij}^3} = -\nabla_i V \tag{1}$$

where

$$V = -\sum_{i \neq j} \frac{G\, m_i m_j}{r_{ij}} \tag{2}$$

with G being the gravitational constant and ∇_i indicating differentiation with respect to the coordinates of the ith mass. By referring the position of m_1 to m_0 and that of m_2 to the center of mass of m_1 and m_0, the potential can be separated into a central term for the elliptic motion of the m_i about their respective centers (i.e., m_0 and the center of mass of m_0 and m_1) and a common disturbing potential which appears in the equation of motion of each m_i ($i > 0$). This system of coordinates was found by Jacobi (see, e.g., Brouwer and Clemence 1961, p. 588). There results

$$\frac{d^2\mathbf{r}_i}{dt^2} + \frac{G\, m_0 m_i \mathbf{r}_i}{r_i^3 m_i'} = -\nabla_i \frac{\Phi}{m_i'} \tag{3}$$

where

$$\mathbf{r}_i = \mathbf{r}_i' - \sum_{k=0}^{i-1} m_k \mathbf{r}_k' \Big/ \sum_{k=0}^{i-1} m_k \tag{4}$$

locates m_i relative to their respective centers,

$$m_i' = \frac{m_i \sum_{k=0}^{i-1} m_k}{\sum_{k=0}^{i} m_k} \tag{5}$$

and

$$\Phi = -G\left[\frac{m_1 m_2}{r_{12}} + m_0 m_2\left(\frac{1}{r_{02}} - \frac{1}{r_2}\right)\right] \tag{6}$$

is the disturbing potential for the two masses m_1 and m_2, which is just V less the part that produces the second term in Eq. (3). From Eq. (6) one sees how Φ and hence Eq. (3) can be generalized to any number of masses (see Eq. 54 in Sec. III below).

The left-hand side of Eq. (3) is just the expression for two-body motion with $G(m_0 + m_i)$ replaced by $\mu_i = G m_0 m_i / m_i'$. The Cartesian coordinates and velocities can thus be transformed into a canonical set, (see, e.g., Plummer 1918, pp. 142–153) and subsequent canonical transformations used to yield

$$\begin{aligned} L_i &= m_i' \sqrt{\mu_i a_i} \quad &, \quad \lambda_i &= \varepsilon_i + \textstyle\int^t n_i dt \\ W_i &= L_i (1 - \sqrt{1 - e_i^2}) \quad &, \quad w_i &= -\tilde{\omega}_i \\ Z_i &= L_i \sqrt{1 - e_i^2}\,(1 - \cos I_i) \quad &, \quad z_i &= -\Omega_i \end{aligned} \tag{7}$$

$$\begin{aligned} \frac{dL_i}{dt} &= -\frac{\partial H}{\partial \lambda_i}, & \frac{d\lambda_i}{dt} &= \frac{\partial H}{\partial L_i} \\ \frac{dW_i}{dt} &= -\frac{\partial H}{\partial w_i}, & \frac{dw_i}{dt} &= \frac{\partial H}{\partial W_i} \\ \frac{dZ_i}{dt} &= -\frac{\partial H}{\partial z_i}, & \frac{dz_i}{dt} &= \frac{\partial H}{\partial Z_i} \end{aligned} \tag{8}$$

where a_i, e_i, I_i, λ_i, $\tilde{\omega}_i$, Ω_i, ε_i are, respectively, the semimajor axes, eccentricities, inclinations of the orbit planes to some reference plane (here the equator of m_0), mean longitudes, longitudes of the periapses, longitudes of the ascending nodes of the orbit planes on the reference plane and mean longitudes at epoch, and $n_i \equiv \sqrt{\mu_i / a_i^3}$ are the orbital mean motions. H is the Hamiltonian given by

$$H = -\sum_{i=1}^{2} \frac{m_i' \mu_i^2}{2L_i^2} + \Phi. \tag{9}$$

The two terms in Φ are called the direct and indirect terms, respectively. From Eq. (4)

$$\mathbf{r}_{02} = \mathbf{r}_2' - \mathbf{r}_0' = \mathbf{r}_2 + K_1\mathbf{r}_1 \tag{10}$$

where $K_1 = m_1/(m_0 + m_1)$ and we see that the indirect part of Φ is expandable in a rapidly converging series in $K_1(r_1/r_2)$:

$$\frac{1}{r_{02}} - \frac{1}{r_2} = \frac{-m_1}{m_0 + m_1}\frac{r_1}{r_2^2}\cos S_{12} + 0\left(\frac{m_1}{m_0}\right)^2. \tag{11}$$

We neglect terms of order $(m_1/m_0)^2$ and higher, and also let $m_0 + m_1 \approx m_0$ which means $m_i' \approx m_i$. With these approximations, we can expand Φ to the form (Allan 1969)

$$\Phi = -\sum \frac{Gm_1m_2}{a_2} CI_1{}^{|\ell - m - 2p_1|}I_2{}^{|\ell - m - 2p_2|}e_1{}^{|q_1|}e_2{}^{|q_2|}\cos\phi_{\ell m p_1 p_2 q_1 q_2} \tag{12}$$

where

$$\begin{aligned}\phi_{\ell m p_1 p_2 q_1 q_2} = {} & (\ell - 2p_1 + q_1)\lambda_1 - (\ell - 2p_2 + q_2)\lambda_2 - (\ell - m - 2p_1)\Omega_1 \\ & + (\ell - m - 2p_2)\Omega_2 - q_1\tilde{\omega}_1 + q_2\tilde{\omega}_2\end{aligned} \tag{13}$$

and where we assume $a_2 > a_1$. In Eq. (12) C is a series in a_1/a_2, e_1^2, e_2^2, I_1^2, I_2^2 whose lowest order term is of order $(a_1/a_2)^\ell$; that is, the lowest order term in C contains neither e nor I. The summation indices have the range $2 \le \ell < \infty$, $0 \le p < \ell$, $0 \le m < \ell$, $-\infty < q < \infty$. An important property of the series is the equality of the coefficient of $\tilde{\omega}$ with the lowest power of e in the coefficient of the cosine and the equality of the coefficient of Ω with the lowest power of I. When Φ is expressed in terms of the canonical variables, L, W, Z are confined to the coefficients of the cosines and λ, w, z are confined to the arguments. Rotational invariance requires that the sum of the coefficients in each argument vanish.

Near a low order commensurability of the mean motions of two satellites, the frequency of some of the terms becomes very small. Often the combination of a large coefficient and a small frequency makes the perturbations due to a single term completely dominant. Keeping only this term in the expansion of Φ is equivalent to averaging over high-frequency terms and ignoring all the remaining terms with small coefficients. This is a much more severe approximation than neglecting the terms which are higher order in the masses, because terms which have small frequencies but which are not the lowest order in e or I may have other factors in the coefficient of the cosine that make them comparable in magnitude to the lowest order term. Still, keeping only

the single term in Φ is usually a good approximation if e (or I) is not too large. It is this model we shall investigate.

For the purpose of illustration, we shall choose a simple eccentricity type resonance where a single term in Φ dominates. The disturbing potential becomes

$$\Phi = \frac{Gm_1m_2}{a_2} C_1 e_1{}^{|k|}\cos(j\lambda_1 - (j + k)\lambda_2 + k\tilde{\omega}_1). \tag{14}$$

If $j = k = 1$, Eq. (14) would be appropriate for the Enceladus-Dione resonance.

The secular motions of the nodes and periapses due to the oblateness of m_0 and the presence of the other satellites are important in separating the frequencies of several variables having the same ratio of the coefficients of λ_1 and λ_2 (e.g., $\lambda_1 - 2\lambda_2 + \tilde{\omega}_1$, $\lambda_1 - 2\lambda_2 + \tilde{\omega}_2$). Only because the secular variations of $\tilde{\omega}_i$ and Ω_i are sufficiently different for the two satellites, can we consider all but one resonance variable to be high frequency and hence ignorable. (We shall see later that two resonances with nearly identical frequencies can still be treated independently provided the respective resonance variables contain no common longitudes of periapses or common longitudes of ascending nodes.) We can include these secular motions by adding to Eq. (14) the terms from Eq. (12) with $\phi = 0$, the zero frequency terms from the disturbing functions involving the other satellites and the Sun as well as the secular terms from expansion of m_0's nonspherical field. Rather than carry this exercise out in detail, we note that

$$\left.\frac{\mathrm{d}w}{\mathrm{d}t}\right|_s = -\dot{\tilde{\omega}}_s = \frac{\partial H}{\partial W} \tag{15}$$

which we can obtain by adding the term $-W\dot{\tilde{\omega}}_s$ to H. Similarly $-Z\dot{\Omega}_s$ and $L\dot{\lambda}_s$ account for the secular motion of the node and the addition to the mean motion. The subscript s denotes secular. Then

$$\begin{aligned}H = {} & -\frac{G^2m_0^2m_1^3}{2L_1^2} - \frac{G^2m_0^2m_2^3}{2L_2^2} - \frac{G^2m_0m_1m_2^3}{L_2^2} C\left(\frac{2W_1}{L_1}\right)^{|k|/2} \\ & \times \cos(j\lambda_1 - (j + k)\lambda_2 - kw_1) - W_1\dot{\tilde{\omega}}_{s1} - Z_1\dot{\Omega}_{s1} \\ & + L_1\dot{\lambda}_{s1} - W_2\dot{\tilde{\omega}}_{s2} - Z_2\dot{\Omega}_{s2} + L_2\dot{\lambda}_{s2}\end{aligned} \tag{16}$$

is the Hamiltonian for the two-body, simple eccentricity type resonance, where from Eq. (7) $W_1 \approx L_1e_1^2/2$ has been used.

There are two sets of variables L_i, W_i, Z_i, λ_i, w_i, z_i corresponding to 6 degrees of freedom, but the above approximate form of H allows us to reduce this to 4 degrees of freedom. Since $z_i = -\Omega_1$ does not appear in Φ (i.e., z_i is cyclic), the Z_i are constants of the motion and $\partial H/\partial Z_i = -\dot{\Omega}_{si}$ are independent of the other variables. Next, the angle variables appear only in the combination $j\lambda_1 - (j+k)\lambda_2 + k\tilde{\omega}_1$ which suggests we make the further change of variables (recall $w_i = -\tilde{\omega}_i$)

$$\begin{aligned}\theta_1 &= j\lambda_1 - (j+k)\lambda_2 - kw_1\\ \theta_2 &= j\lambda_1 - (j+k)\lambda_2 - kw_2\\ \theta_3 &= \lambda_1\\ \theta_4 &= \lambda_2.\end{aligned} \tag{17}$$

The momenta Θ_i conjugate to θ_i are obtained from the differential relation (Brouwer and Clemence 1961, p. 539)

$$\sum_{i=1}^{4} \Theta_i \mathrm{d}\theta_i = \sum_{j=1}^{2} (L_j \mathrm{d}\lambda_j + w_j \mathrm{d}w_j) \tag{18}$$

which yields

$$\begin{aligned} j(\Theta_1 + \Theta_2) + \Theta_3 &= L_1\\ -(j+k)(\Theta_1+\Theta_2) + \Theta_4 &= L_2\\ -k\Theta_1 &= W_1\\ -k\Theta_2 &= W_2.\end{aligned} \tag{19}$$

Since θ_2, θ_3 and θ_4 do not appear in H, Θ_2, Θ_3 and Θ_4 are constants of the motion, and the problem is reduced to one degree of freedom with only θ_1 and Θ_1 as variables.

Now $\Theta_2 = -W_2/k = -L_2(1 - \sqrt{1-e_2^2}) \approx -L_2 e_2^2/2$ and since it is constant, we can set it equal to zero or include its value in Θ_3 and Θ_4 in Eq. (19) with negligible change in the resonance behavior. Then in the new variables

$$\begin{aligned} H = &-\frac{G^2 m_0^2 m_1^3}{2(\Theta_3 + j\Theta_1)^2} - \frac{G^2 m_0^2 m_2^3}{2[\Theta_4 - (j+k)\Theta_1]^2}\\ &- \frac{G^2 m_1 m_2^3 m_0 C(-2k\Theta_1)^{k/2}\cos\theta_1}{[\Theta_4 - (j+k)\Theta_1]^2[\Theta_3 + j\Theta_1]^{k/2}}\\ &+ \frac{\Theta_1}{k}\dot{\tilde{\omega}}_s + \frac{\Theta_2}{k}\dot{\tilde{\omega}}_{s2} + (j\Theta_1 + \Theta_3)\dot{\lambda}_{s1}\\ &+ [-(j+k)\Theta_1 + \Theta_4]\dot{\lambda}_{s2}.\end{aligned} \tag{20}$$

From Eqs. (19) and (7) we see that $|\Theta_1| \ll \Theta_3$, and $|\Theta_1| \ll \Theta_4$; hence, we can expand the first two terms in H to second order in Θ_1/Θ_3 and Θ_1/Θ_4

consistent with the magnitude of the terms kept in Φ. Since the coefficient of the cosine is a factor m_1/m_0 smaller than the first two terms, we need not expand the denominator of this term in Θ_1. We carry out the expansions, absorb all the constant terms into H, use W_1 as a variable instead of Θ_1, replace the angle variable θ_1 by $\theta_1' = \theta_1/k$ and drop the prime. Then with $H' = -H$ and $\mathrm{d}W_1/\mathrm{d}t = -\partial H'/\mathrm{d}\theta$, $\mathrm{d}\theta_1/\mathrm{d}t = \partial H'/\mathrm{d}W_1$, we arrive at the Hamiltonian used by Henrard and Lemaître (1983).

$$H' = \alpha W_1 + \beta W_1^2 + \varepsilon(2W_1)^{k/2}\cos k\theta_1 \tag{21}$$

where

$$\begin{aligned}\alpha &= (jn_1^* - (j+k)n_2^* + k\dot{\tilde{\omega}}_{s1})/k\\ \beta &= \frac{3}{2k^2}\left[\frac{j^2}{m_1 a_1^2} + \frac{(j+k)^2}{m_2 a_2^2}\right]\\ \varepsilon &= C(n_1^2)^{1-\frac{k}{4}} \frac{a_1}{a_2}^{3-k} \frac{m_2}{m_0} m_1{}^{1-\frac{k}{2}}\end{aligned} \tag{22}$$

where $n_i^* = n_i + \dot{\lambda}_{si}$ with $\Theta_3 \approx L_1$ and $\Theta_4 \approx L_2$ being used. This Hamiltonian is the same form used by Yoder (1973,1979*b*) in his original analytic solution.

The reason for transforming the angle variable to θ_1/k from the old θ_1 is to have $k\theta_1$ as the argument of the cosine in Eq. (21). We shall use a canonical transformation below similar to $x = (2W_1)^{1/2}\cos\theta_1$ and $y = (2W)^{1/2}\sin\theta_1$ where the Hamiltonian in the new variables x and y is analytic at the origin for $k > 1$ only with $k\theta_1$ as the cosine argument. This property of the disturbing potential is called the d'Alembert characteristic. Although we shall explicitly treat only the $k = 1$ case in detail, this proper choice of canonical variables is an important consideration in the analysis of higher-order resonances.

We have assumed C to be constant without expressing it in terms of the Θ_i. This is justified as follows. The lowest-order term in C involves only a_1/a_2. Then $\delta C/C \sim \delta a/a \sim \delta L/L$. Since $\Theta_3 = L_1 + jW_1/k$ is a constant of the motion, $\delta L_1 = -j\,\delta W_1/k$. But $\delta e_1/e_1 \sim \delta W_1/W_1 = 0(\delta L_1/L_1)/e_1^2$ or $0(\delta C/C)/e_1^2$. So the variation in the coefficient of the cosine during the conservative oscillations is determined almost completely by the variation in e_1, and keeping C constant is a good approximation.

All three coefficients and the "constants" absorbed in H' will be slowly varying as tides raised on m_0 cause n_1^* and n_2^* to decrease differentially. However, α is small near the resonance and the fractional change in α due to the changes in n_i^* will generally be much larger than the fractional change in β or ε. The exception to this condition occurs when the mass m_2 is sufficiently larger than m_1 that $\langle|j\mathrm{d}n_1^*/\mathrm{d}t - (j+k)\mathrm{d}n_2^*/\mathrm{d}t|\rangle$ is very small, where $\langle\rangle$ indicates averaged value. In this case substantial changes in the a_i would occur for significant change in α. Since $\delta\beta/\beta \sim \delta a/a$ and $\delta\varepsilon/\varepsilon \sim \delta a/a$, the fractional change in β and ε would not be small compared to $\delta\alpha/\alpha$. However, for the existing resonances among the satellites we can consider β and ε approximately constant compared to α. The approximation is least applicable to the Mimas-Tethys resonance where $m_2/m_1 \approx 17$, but the physical processes involved remain the same and are adequately described by the model.

The approximate constancy of β and ε compared to α is a necessary condition for the applicability of the Henrard-Lemaître model in which the Hamiltonian is reduced to a dimensionless form with a single variable parameter. We consider only the case $k = 1$, which applies to all the two-body resonances among the satellites except the Mimas-Tethys resonance and even this can be approximated as a $k = 1$ resonance because m_2/m_1 is so large. Henrard and Lemaître (1983) simplify the Hamiltonian by the following change in variables:

$$\begin{aligned} R &= \left(\frac{2\beta}{\varepsilon}\right)^{2/3} W_1 \\ \phi &= \theta_1 + \pi \qquad \text{if } \varepsilon\beta > 0 \\ \phi &= \theta_1 \qquad \text{if } \varepsilon\beta < 0 \\ \tau &= \left(\frac{\beta\varepsilon^2}{4}\right)^{1/3} t. \end{aligned} \tag{23}$$

The Hamiltonian becomes

$$K = -3(\delta + 1)R + R^2 - 2\sqrt{2R}\cos\phi \tag{24}$$

where

$$\delta = -\left(\frac{4\alpha^3}{27\beta\varepsilon^2}\right)^{1/3} - 1 \tag{25}$$

and

$$\frac{dR}{d\tau} = -\frac{\partial K}{\partial \phi}, \quad \frac{d\phi}{d\tau} = \frac{\partial K}{\partial R}. \tag{26}$$

In Eq. (24) the variations due to tides are confined to the single parameter δ.

Since K is a constant of the motion in the absence of tides, the nature of the motion can be ascertained by plotting R vs. ϕ for various values of K. These curves are called the level curves of the Hamiltonian or constant energy curves. It is more convenient to do this in Poincare variables, with

$$\begin{aligned} x &= \sqrt{2R}\cos\phi \\ y &= \sqrt{2R}\sin\phi\ . \end{aligned} \tag{27}$$

With $\sqrt{2R}$ used instead of R in the definitions of x and y, the transformation is canonical with

$$\frac{dx}{dt} = -\frac{\partial K}{\partial y}, \quad \frac{dy}{dt} = \frac{\partial K}{\partial x} \tag{28}$$

and

$$K = -\Delta\frac{(x^2+y^2)}{2} + \frac{(x^2+y^2)^2}{4} - 2x \tag{29}$$

where $\Delta = 3(1 + \delta)$. Also $\sqrt{2R} \propto e_1$, and the radius from the origin to the trajectory measures the instantaneous eccentricity.

The stationary points follow from $\partial K/\partial x = \partial K/\partial y = 0$ and are the roots of

$$\begin{aligned} x^3 - \Delta x - 2 &= 0, \\ y &= 0. \end{aligned} \tag{30}$$

The parameter δ has been constructed such that for $\delta < 0$ ($\Delta < 3$) there are no negative real roots of Eq. (30) and one positive real root, whereas for $\delta > 0$ ($\Delta > 3$), there are two negative real roots and one positive real root.

A typical set of level curves is shown in Fig. 3, where $\Delta \approx 9$. Curves for positive and negative circulation and for libration are indicated. We define libration here as any state where ϕ is bounded. The curve marked separatrix separates regions of circulation from those of libration in Fig. 3, although it is possible to have a librating system, according to our definition, without the trajectory being inside the separatrix. There are three stationary points apparent in Fig. 3 (two stable and one unstable corresponding to the roots of Eq. 30), and a trajectory near the stable point on the negative x axis is not within the separatrix but would correspond to libration about $\phi = 180°$.

Henrard (personal communication, 1985) is careful to point out that libration of a resonance variable ϕ does not necessarily imply much dynamical significance. In our example, ϕ could be librating even though n_1 and n_2 are far from commensurability when e is very small and $\tilde{\omega}$ varies rapidly. In this case, the libration has little or no effect on the system evolution unless it is maintained until n_1 and n_2 approach the commensurability. There is thus some argument for reserving the term "resonance" for those systems described by a level curve contained within the separatrix (Henrard and Lemaître 1983). However, we shall retain our definition of libration, but keep

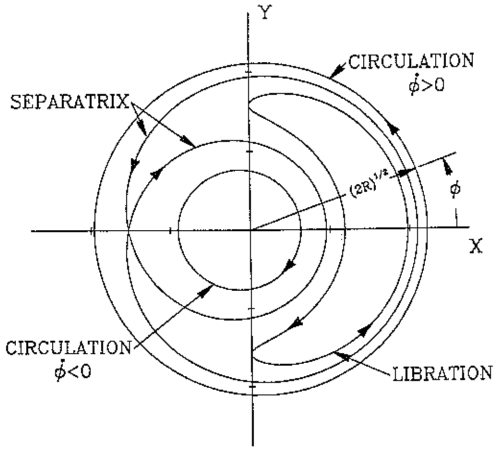

Fig. 3. A typical set of level curves of the Hamiltonian appropriate to a two-body orbital resonance. Positive and negative circulation and libration curves are indicated as well as the curve (separatrix) separating circulation from libration.

in mind that the "resonance" is significant only when n_1/n_2 is relatively close to commensurability. (See the qualifications below for more discussion on this point.)

The character of the set of level curves depends on the value of Δ (or δ). For example, there is only one stationary point and no separatrix for $\Delta < 3$, and both inner and outer curves of the separatrix expand as Δ increases above 3. As Δ is only slowly varying, the actual trajectories of the system in the xy plane are very close to the trajectories for constant Δ and the system can be thought to evolve slowly through a set of level curve diagrams as Δ varies. To follow the behavior of the system as the set of possible level curve trajectories changes with Δ, we need one more important principle.

As long as Δ does not change very much during a period of the motion, the action

$$J = \oint R d\phi = \oint x dy \tag{31}$$

is an adiabatic invariant. This has been proven for periodic systems with a slowly varying parameter by Gardner (1959), Lenard (1959), and explicitly for nonlinear oscillations appropriate to orbital resonance by Henrard (1982*a*; see also Landau and Lifshitz 1960; Yoder 1979*b*). The exception to the adiabatic conservation of J occurs when a trajectory crosses the separatrix since the period on the separatrix is infinite, due to asymptotic approach to the unstable equilibrium point. Hence, we expect the action to be conserved until a

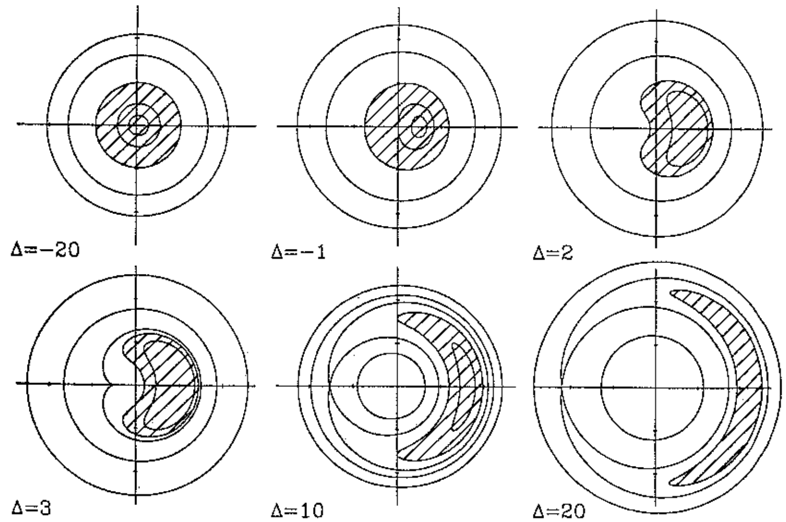

Fig. 4. Tidal evolution of two bodies toward orbital resonance as a series of level curves of the Hamiltonian. The particular curve appropriate to the motion encloses the shaded area which is the conserved system action determined by initial conditions far from resonance. The sense of motion about the curves is shown in Fig. 3. This series illustrates automatic capture into libration.

separatrix is crossed at which time the action will change in the transition to a librating or oppositely circulating state, but thereafter remain constant through further evolution following the transition. From Eq. (31), we see that the action is just the area within a trajectory and is determined by initial conditions far from resonance. Hence, we can identify the trajectory traversed by the system at any stage of the evolution as that trajectory which encloses an area equal to the initial action integral or that area enclosed by one or both curves of the separatrix (not the initial action) after a transition through that separatrix. The series of level curve diagrams coupled with the adiabatically conserved action allow us to understand in a simple way all of the major characteristics of the origin and evolution of orbital resonances. We do this by considering the consequences of several initial conditions far from resonance and two directions of approach to the resonance.

Figure 4 shows the trajectory evolution on a series of xy phase plane plots for the case where Δ increases from negative values as the resonance is approached. From Eqs. (22) and (25), a negative $\delta(\Delta < 3)$ corresponds to positive α and the resonance is being approached as the inside satellite catches up to the resonance configuration by the more rapid expansion of its orbit; that is, n_1/n_2 is decreasing. We have chosen a relatively small initial eccentricity far from resonance so the action (area inside trajectory) is small. Maintaining the area inside the trajectory corresponding to the motion as Δ increases through the series of diagrams shows that the system evolves smoothly into libration about $\phi = 0°$. We have included the separatrix trajectory in the diagrams after its appearance at $\Delta = 3$. For the small eccentricity chosen as an initial condition, the capture into resonance is certain. Subsequent evolution within the resonance is seen to be a continued increase in the forced eccentricity (determined by the separation from the origin of the stationary point enclosed by the trajectory) as the amplitude of libration is reduced.

Figure 5 shows a similar evolution over the same change in Δ but starting with a much larger eccentricity far from resonance. Now the separatrix forms inside the circulation trajectory and expands as Δ increases until the outer curve of the separatrix is nearly the same as the trajectory. At this point, the trajectory must cross the separatrix and a transition from the state of positive circulation must occur. This is the resonance encounter and the transition from positive circulation can have but one of two outcomes. After following a trajectory very close to the separatrix, the system can either reverse its direction of motion and trace the crescent shaped trajectory of libration (capture into resonance) or continue its negative circulation (escape). The action is not conserved across the transition, but assumes a new essentially constant value, which is equal to the area inside the crescent-shaped separatrix at the time of transition into libration (Fig. 5, $\Delta = 6$) or to the area of the inside curve of the separatrix for transition into negative circulation. The evolution beyond transition can now be followed as before with the proper trajectory once again being determined by the conserved area within. In libration, the initial amplitude is 180°, and it is reduced as the forced eccentricity increases, as shown in Fig. 5. Unlike the situation for a small eccentricity far from resonance where capture into libration was certain, here the capture is probabilistic. A procedure for estimating that probability will be discussed below.

Comparison of Figs. 4 and 5 shows that the criterion for certain capture into resonance is that the initial action far from resonance be less than the area inside the separatrix first formed when $\Delta = 3$ ($\delta = 0$). Since the initial trajectories far from resonance are essentially circles centered at the origin, the criterion for certain capture is

$$J_0 = \pi(x_0^2 + y_0^2) = 2\pi R_0 < J_c = 6\pi \tag{32}$$

where J_0 is directly proportional to e_0^2 and J_c is the area inside the critical separatrix for $\Delta = 3$. The integral for J_c is most easily done using the first form of Eq. (31) involving R and ϕ, where Eqs. (24) and (26) are used to change the integration variable from ϕ to R and $dR/d\tau = 0$ and $d\phi/d\tau = 0$ at $R(\phi = 180°) = R^*$ are used to factor terms in the integrand. From Eq. (32) the criterion becomes

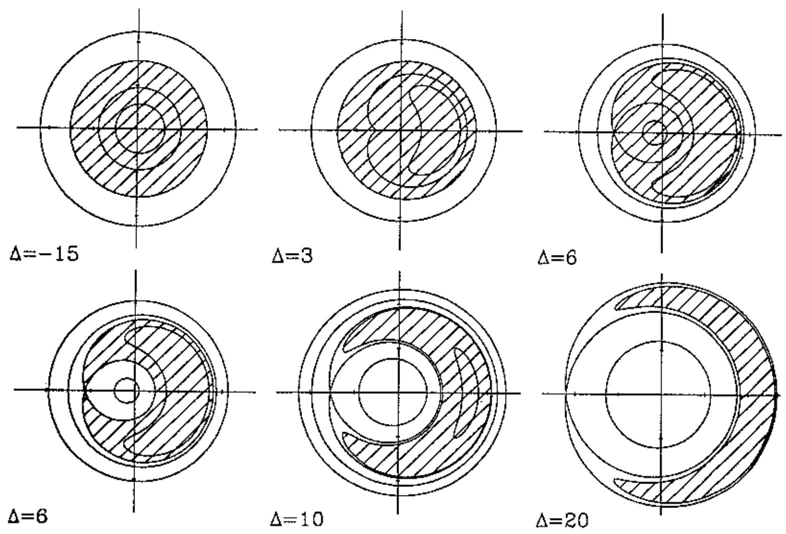

Fig. 5. Same as for Fig. 4 except now the separatrix forms before the resonance is reached because of the higher initial eccentricity. The series shows capture into libration which is now probabilistic. There is a discontinuous change in the action (shaded area) upon transition from circulation to libration.

$$e_{10}^2 < \frac{6}{m_1\sqrt{\mu_1 a_1}}\left(\frac{\varepsilon}{2\beta}\right)^{2/3} \tag{33}$$

for certain capture.

The age of a resonance can be estimated by comparing the current value of Δ and the value of Δ in the past when the area inside the separatrix was equal to the current action. If the current action is less than the area of the critical separatrix at $\Delta = 3$, the transition into resonance was automatic, and the value of Δ when this occurred corresponds to the trajectory passing through the origin of the xy plane. With current and initial values of Δ determined, their difference gives the change in α from Eq. (25), and the age $T = \alpha/\dot{\alpha}$ where $\dot{\alpha}$ follows from the rate of change of $j\dot{n}_1^* - (j+k)\dot{n}_2^*$ due to tides raised on m_0. This age is qualified in one of the applications below where the implicit assumptions on which it is based are questioned.

Before illustrating the calculation of the capture probability, we show the consequence of approaching the resonance from the opposite direction with Δ initially large and positive and decreasing. This corresponds to the case where the orbits are initially too close together for the resonance, but $m_2 >> m_1$ such that m_2's orbit expands faster than m_1's even though it is farther away. This evolution is shown in Fig. 6 where the system starts in negative circula-

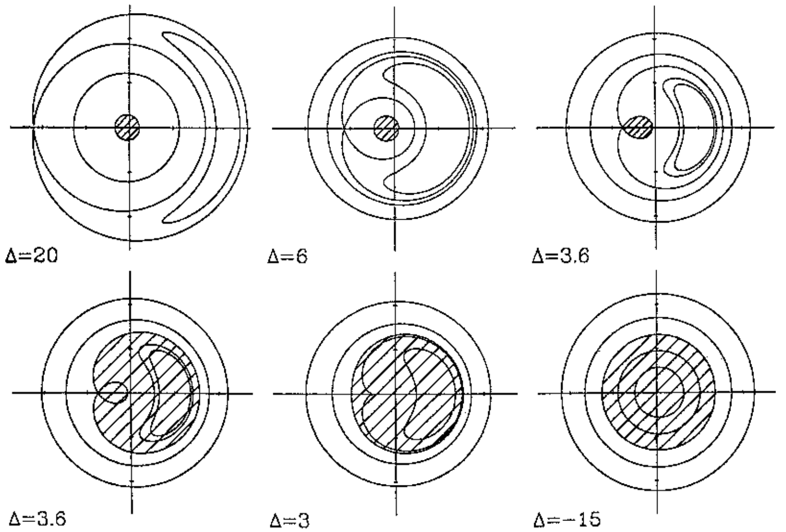

Fig. 6. Same as for Fig. 4, except now the resonance is approached from the opposite direction (orbits initially too close for resonance). This evolution illustrates temporary inverted libration but eventual certain escape from the resonance.

tion with relatively small eccentricity and action. As Δ decreases, the separatrix shrinks and eventually the area enclosed by the inner curve of the separatrix is equal to the initial action (Fig. 6, $\Delta = 3.6$). Again the system has no choice but to traverse a path first very close to the inside curve of the separatrix and then close to the outside curve. This time, however, it must remain in positive circulation as the separatrix continues to shrink. The shrinking area enclosed by the separatrix means the action is only invariant after the transition if the circulating trajectory is left behind as the separatrix shrinks away from it. Capture into the resonance libration from this direction of approach is impossible. This was noted by Sinclair (1972) as well as Yoder (1973) and is discussed in the review by Peale (1976*b*). However, it is more simply understood in this approach due to Henrard and Lemaître (1983). For the initial conditions assumed for Fig. 6, we see that the system is temporarily trapped in a state of inverted libration about $\phi = \pi$. A bound on e_0 for such a temporary inverted libration follows from the condition that the initial action be less than the area of the inner separatrix when that separatrix passes through the origin. The inverted libration is always unstable to continued tidal evolution of the system.

The major consequence of passing through the resonance with decreasing Δ is the substantial jump in the free eccentricity. The new action is the area within the outer separatrix curve at the time of transition. If J_1 and J_2 are the

areas within the inside and outside curves of the separatrix, respectively, $J_1 = 2\pi R_1$, $J_2 = 2\pi R_2$, where R_1 and R_2 are the nearly constant values of R on far opposite sides of the resonance, respectively. The initial eccentricity determines J_1 which in turn determines the value of $\Delta = \Delta_T$ at transition. Given Δ_T, J_2 (the area of the outside separatrix) is determined by an integration described earlier, and we find

$$J_1 + J_2 = 2\pi(R_1 + R_2) = 2\pi\Delta_T \tag{34}$$

and from Eq. (25) with $\Delta_T = 3(\delta_T + 1)$,

$$e_{1i}^2 + e_{1f}^2 = -\frac{2\alpha_T}{\beta L_1} \tag{35}$$

with e_{1i} and e_{1f} being initial and final values of e_1 far from resonance and α_T the value of α at transition. Note that $\Delta_T > 3$ ($\delta > 0$) for this transition, requiring $\alpha_T < 0$ (Eq. 25).

Capture Probability

Sinclair (1972, 1974) determined capture probabilities for several orbital resonances encountered by the satellites of Saturn with numerical techniques, and Yoder (1973) gave the first analytic determination of the probabilities for general resonances (cf. Peale 1976*b*). We shall outline a method developed by Henrard (1982*a*) which led to a second, much simpler analytic determination of the capture probability by Borderies and Goldreich (1983).

A capture probability only applies to the case where $\Delta > 3$ ($P_c = 1$ for $\Delta_T < 3$ at resonance encounter) and a separatrix exists as the system approaches resonance. Transition occurs when the separatrix has expanded to the point where the system makes a last positive circulation very close to the outside curve of the separatrix, and the next motion is a traverse in the opposite sense along a trajectory very close to the inside curve of the separatrix. Let N be the value of the Hamiltonian $K(R, \phi, \Delta)$ (Eq. 24) relative to its value $K(\tilde{R}, \phi, \Delta)$ on the separatrix where $\tilde{R}(\phi)$ corresponds to the separatrix. Since $K(\tilde{R}, \phi, \Delta) = K^*(R^*, \phi^*, \Delta)$ where R^* and ϕ^* are the values at the unstable equilibrium point on the separatrix, we can write

$$N = K(R, r, \Delta) - K^*(R^*, r^*, \Delta). \tag{36}$$

As K and K^* are fixed if Δ is fixed, the total variation in N must be due to the variation in Δ:

$$\frac{dN}{dt} = \left(\frac{\partial K}{\partial \Delta} - \frac{\partial K^*}{\partial \Delta}\right)\frac{d\Delta}{dt}. \tag{37}$$

As the separatrix for which $K = K^*$ approaches the circulation trajectory with Hamiltonian K, necessarily N approaches zero as both K and K^* change. From Eq. (24) a larger circulation trajectory for given Δ when R is large yields a larger K, so N is initially positive and $\mathrm{d}N/\mathrm{d}t < 0$.

For the last circulation trajectory near the outer separatrix curve before the separatrix is crossed, let

$$B_1 = \oint_{c_1} \frac{\mathrm{d}N}{\mathrm{d}t}\mathrm{d}t = \dot{\Delta}\oint_{c_1}\left(\frac{\partial K}{\partial \Delta} - \frac{\partial K^*}{\partial \Delta}\right)\mathrm{d}t = \dot{\Delta}\oint_{c_1}\left(\frac{\partial K}{\partial \Delta} - \frac{\partial K^*}{\partial \Delta}\right)\frac{\mathrm{d}\tilde{R}}{\dot{\tilde{R}}} < 0 \tag{38}$$

be the change in the relative Hamiltonian, where $\mathrm{d}t = \mathrm{d}R/\dot{R} = \mathrm{d}\tilde{R}/\dot{\tilde{R}}$, since the trajectory is very near the separatrix and c_1 is the outside curve. The next motion will be a traverse of the inside separatrix curve in the opposite sense. Call the change in the relative Hamiltonian for this traverse

$$B_2 = \dot{\Delta}\oint_{c_2}\left(\frac{\partial K}{\partial \Delta} - \frac{\partial K^*}{\partial \Delta}\right)\frac{\mathrm{d}\tilde{R}}{\dot{\tilde{R}}} \tag{39}$$

where c_2 refers to the inside separatrix curve. K also increases away from the inside separatrix curve so, if B_2 is also < 0, capture is certain. If $B_2 > |B_1|$, the system gains more relative energy in traverse of inner separatrix than it lost in tracing the outer separatrix and escape is certain. If $B_2 > 0$ but $B_1 + B_2 < 0$ and we assume the possible values of N over the range $0 \le N \le -B_1$ at the start of the last outside circulation to have a uniform probability distribution, the capture probability is

$$P_c = \frac{B_1 + B_2}{B_1} = \frac{2}{1 + \dfrac{\pi}{2\sin^{-1}\left[\dfrac{R_{\max} + R_{\min} - 2R^*}{R_{\max} - R_{\min}}\right]}} \tag{40}$$

where we have used Eqs. (24) and (26) to write

$$B_i = -2\dot{\Delta}\int_{R^*}^{R_i} \frac{\mathrm{d}\tilde{R}}{\sqrt{(\tilde{R} - R_{\min})(R_{\max} - \tilde{R})}} \tag{41}$$

where $R_1 = R_{\max}$, $R_2 = R_{\min}$, the maximum and minimum values of R on the separatrix. Equation (40) was first obtained by Yoder (1973).

We need but evaluate $R_{\max}$, $R_{\min}$ and R^* on the separatrix in terms of Δ. From Eq. (27), $x^* = -\sqrt{2R^*}$ where x^* is the most negative root of Eq. (30). The extremes in R occur on the x axis, so from Eq. (29) with $x_{\max} = \sqrt{2R_{\max}}$, $x_{\min} = \sqrt{2R_{\min}}$, $x_{\max}$ and $x_{\min}$ are roots of $x^4 - 2\Delta x^2 - 8x - 4K$

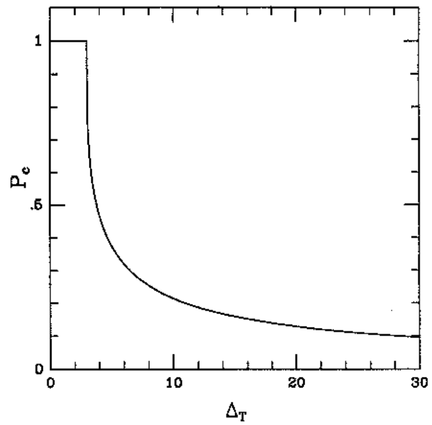

Fig. 7. Probability of capture into a two-body resonance with $n_1 : n_2 = j : j + 1$ as a function of the value of $\Delta = \Delta_T$ at the time of transition.

and x^* is a double root. K can be expressed in terms of Δ and x^* with the help of Eq. (30), $(x - x^*)^2$ factored from the quartic and the remaining quadratic solved for $x_{\max}$ and $x_{\min}$. There results the expression obtained by Borderies and Goldreich (1984):

$$P_c = \frac{2}{1 + \dfrac{\pi}{2\sin^{-1}(sz)^{-3/2}}} \tag{42}$$

where $s = \sqrt{\Delta_T/3}$, $z = \cos(\xi/3) + \sqrt{3}\sin(\xi/3)$ with $\cos\xi = (\Delta_T/3)^{-3/2}$. $P_c(\Delta_T)$ is known and we can evaluate P_c as a function of the initial eccentricity far from resonance by determining the area of the outside separatrix curve as a function of Δ. Then Δ_T is the value of Δ corresponding to an area of the outside separatrix curve equal to the initial action $J = 2\pi R_0$ where $R_0 \propto e_0^2$ by Eqs. (23) and (7). P_c is given as a function of Δ_T for the $j : j + 1$ orbit-orbit resonance in Fig. 7.

An Application

The resonance between Enceladus and Dione is of the type discussed in the illustrative example above with $j = k = 1$. Table I gives the relevant parameters for the two satellites which, when substituted into Eq. (22), yield

$$\begin{aligned} \alpha &= 0°072/\text{day} = 1.45 \times 10^{-8}\ \text{rad s}^{-1} \\ \beta &= 4.05 \times 10^{-44}\ \text{g}^{-1}\ \text{cm}^{-2} \\ \varepsilon &= -3.347 \times 10^{9}\ \text{g}^{1/2}\ \text{cm s}^{-3/2} \end{aligned} \tag{43}$$

TABLE I
Parameter Values for the Enceladus-Dione Resonance

	Enceladus	Dione
n^*	262°.732/day	131°.535/day
a	238,040 km	377,420 km
e	0.0044	0.0022
$\dot{\tilde{\omega}}_s$	0°.410/day	0°.084/day
m/m_0	1.5×10^{-7}	1.9×10^{-6}

where $C = -1.19$ is determined from the coefficient of $\cos(\lambda_1 - 2\lambda_2 + \tilde{\omega}_1)$ in the expansion of Φ for $a_1/a_2 = 0.63$ corresponding to $n_1/n_2 = 2$ (see, e.g., Brouwer and Clemence 1961, p. 490). From Eq. (25), $\delta = -2.0$, and Enceladus and Dione are in a state where no separatrix has formed and the capture into resonance is certain as noted by Sinclair (1972), Yoder (1973) and others. In fact, from Eq. (33) capture would be certain for any $e_{10} < 0.017$ which is much larger than the current value of 0.0044.

The amplitude of libration of $\lambda_1 - 2\lambda_2 + \tilde{\omega}_1$ about 0° is near 1° (Sinclair 1972). From Fig. 4, we see that the trajectories for $\delta \approx -2$ are nearly circular, and the libration amplitude yields the radius of the circle as $\rho = \sqrt{2R_{eq}}$ tan 1° with R_{eq} being the value of R at the stationary point within the trajectory. From Eqs. (23) and (7), $e_1 = 7.38 \times 10^{-3} \sqrt{2R}$ from which $\sqrt{2R} = 0.603$ when $e_1 = 0.0044$ and $\rho = 0.0105$. The action is just $\pi\rho^2$ and if this is conserved, the transition into resonance occurred when the circle of radius ρ passed through the origin, or the stationary point was at $x = \rho$. From Eq. (30), this yields $\delta = -62.5$ at transition and the age of the resonance is the time for δ to go from -62.5 to -2.

From Eqs. (25) and (22), $\mathrm{d}\delta/\mathrm{d}t \propto \mathrm{d}\alpha/\mathrm{d}t = \mathrm{d}n_1^*/\mathrm{d}t - 2\mathrm{d}n_2^*/\mathrm{d}t + \mathrm{d}\dot{\tilde{\omega}}_{1s}/\mathrm{d}t$, where

$$\frac{\mathrm{d}n^*}{\mathrm{d}t} = \frac{-9}{2} k_2 \frac{m}{m_0} \left(\frac{R_s}{a}\right)^5 \frac{n^2}{Q_0} \tag{44}$$

with $k_2 = 0.34$ (Gavrilov and Zharkov 1977), $Q_0 > 1.6 \times 10^4$ (Goldreich and Soter 1966) and R_s being the Love number for Saturn, the dissipation function for Saturn and the radius of the satellite, respectively. Substitution of the values of the remaining parameters from Table I yields $\mathrm{d}\alpha/\mathrm{d}t = -1.7 \times 10^{-23}$ rad s^{-2} and an age of the resonance of 1.6 Gyr.

Qualifications

The Henrard-Lemaître formulation thus describes the origin and evolution of orbital resonances in an orderly and easily understood manner.

The somewhat diverse approaches to the study of such resonances by Allan (1969), Sinclair (1972,1974), Greenberg (1973*a*) and Yoder (1973), for example, are hereby reduced to probably the simplest form and provide a basis for any study of orbital resonances. As is often the case, however, this most elegant and simple model must be applied with caution in interpreting the current configuration and inferred histories of observed orbital resonances. Let us begin with the above example of Enceladus-Dione which appears on the surface to satisfy all the criteria for the model approximations.

Perhaps the most critical assumption is the isolation of a single resonance term in R to simplify the Hamiltonian. The terms nearest in frequency all have $n_1/n_2 \approx 2$, and those frequencies corresponding to the lowest order coefficients are

$$\begin{aligned}
(2n_1^* - 4n_2^* + 2\dot{\Omega}_1) &= -1^{\circ}.496/\text{day} \\
(2n_1^* - 4n_2^* + \dot{\Omega}_1 + \dot{\Omega}_2) &= -1^{\circ}.170/\text{day} \\
(2n_1^* - 4n_2^* + 2\dot{\Omega}_2) &= -0^{\circ}.844/\text{day} \\
n_1^* - 2n_2^* + \dot{\tilde{\omega}}_2 &= -0^{\circ}.254/\text{day} \\
2n_1^* - 4n_2^* + \dot{\tilde{\omega}}_1 + \dot{\tilde{\omega}}_2 &= -0^{\circ}.254/\text{day} \\
n_1^* - 2n_2^* + \dot{\tilde{\omega}}_1 &= 0.
\end{aligned} \tag{45}$$

The frequencies are separated by the different secular motions of Ω_i and $\tilde{\omega}_i$. The identity of two frequencies depends on the existence of the last zero frequency, and this identity would vanish with the resonance. The separation of the frequencies in Eq. (45) is sufficient that all but the resonant term can be considered high-frequency terms whose negligible influence justifies selection of only the single term in R for analysis of the *existing* configuration. However, we are interested in the origin and evolution of the system and the assumption of only a single term in R always dominating the dynamics is not necessarily correct.

First, Sinclair (1972) noticed that the existing resonance would normally be the last one encountered among those in Eq. (45) as n_1 was reduced. The current orbital inclinations are quite small and uncertain. Sinclair (1974) used Struve's (1933) values of $I_1 = I_2 = 4.1 \times 10^{-4}$ rad to determine the values of I_i before an assumed transition from positive to negative circulation. Notice from Figs. 4 and 5 that such a transition reduces the action and hence the eccentricity in the example and would similarly reduce the inclination for an inclination type resonance. So, having escaped capture into the series of inclination resonances implies larger inclinations in the past for the orbits of Enceladus and Dione. These larger inclinations insure a nonzero probability of escape for all the inclination resonances.

We can perform the same analysis as that above using the Henrard-Lemaître formulations for the resonance with θ_2 ($= \lambda_1 - 2\lambda_2 + \tilde{\omega}_2$) as the librating variable. The value of $\varepsilon = 2.87 \times 10^8$ g$^{1/2}$ cm s$^{-3/2}$, $\alpha =$

$-0°.254$/day from Eq. (45) and β has the same value of 4.05×10^{-44} g^{-1} cm^{-2} as it had for the θ_1 $(= \lambda_1 - 2\lambda_2 + \tilde{\omega}_1)$ libration. With $e_2 = 0.0022 = 7.7 \times 10^{-4} \sqrt{2R}$, the current action $J = 25.65$, which is the area of the inside separatrix curve at transition. This determines $\Delta_T = 13.0$ and a capture probability $P_c = 0.18$ which was also found by Sinclair (1974). From nonzero probability of escape for all the resonances, Sinclair (1974) showed the consistency of the current Enceladus-Dione resonance with the tidal hypothesis of origin under the assumption that the resonances are encountered in the order Eq. (45). However, from Eqs. (16) and (7) with $\mathrm{d}\tilde{\omega}_1/\mathrm{d}t = \partial H/\partial W_1 = \dot{\tilde{\omega}}_{s1} - (C'/e_1)\cos\theta_1$, a small e_1 induces a large negative contribution to $\dot{\tilde{\omega}}_1$ when θ_1 is small. For the value of $e_1 \approx 7 \times 10^{-5}$ when libration was established consistent with conserved action, $\dot{\tilde{\omega}}_{1\,\mathrm{res}} \approx -4°.5$/day so the resonance in which we find Enceladus-Dione would be the first one encountered instead of last as given by Eq. (45) (Sinclair 1974; Peale 1976*b*).

The decrease in $n_1^* - 2n_2^*$ within the resonance means each of the other resonances in the 2:1 set were encountered (resonance angles passed through zero frequency) while the system was librating in the existing resonance. The justification of the approximation of keeping a single term in R by other terms being high frequency or having small coefficients is not necessarily valid. However, Sinclair (1983) has pointed out the independence of some of the resonances clustered around the 2:1 commensurability in the sense that each can evolve without significantly affecting the states of any of the others. This follows from the fact that the orbital parameters affected by a given resonance are often essentially unique to that resonance. Such independence has been demonstrated numerically by Wisdom (personal communication, 1983) in a study of the two first-order eccentricity resonances. Second-order resonances involving the sum of the two node longitudes or the sum of the two longitudes of pericenters do affect the same orbital parameters as other first- and second-order 2:1 resonances, so one expects some coupling here. Wisdom's (1983) calculations indicate a weak effect on the evolutions of two overlapping resonances in this case, but the resonances essentially still evolve independently for Enceladus-Dione. The current occupancy of only the first-order eccentricity resonance after apparently having passed through the other 2:1 possibilities thus depends only on the nonzero probability of escape from each of these other resonances, a condition found to be satisfied by Sinclair (1974). The selection of a single term in the disturbing function for the study of a particular resonance is valid even when the frequency is not separated from that of another nearby resonance, provided that the only common angle variables in the two descriptions of the resonances are the mean longitudes.

Another qualification on the use of the simple model is that the eccentricity not be too large or the satellite orbits not be too close together. This is illustrated by the application of the model to the Titan-Hyperion resonance where $3\lambda_1 - 4\lambda_2 + \tilde{\omega}_2$ librates about π with an amplitude of 36° and with $C = 3.26$ in Eq. (16). We use the mean eccentricity of Hyperion's orbit of 0.104

to determine $R = 779e_2^2 = 8.43$ or $x = \sqrt{2R} = 4.11$ as the location of the stationary point (Eqs. 7, 21, 22, 23). From Eq. (30), $\delta = 4.458$ or $\Delta = 3(\delta + 1) = 16.37$. From Eqs. (24) and (26), the zero of $\mathrm{d}\phi/\mathrm{d}\tau$ yields the value of $R = R_m = 8.383$ at the extreme of the libration at $\phi_m = 36°$. Substitution of R_m into Eq. (24) yields the Hamiltonian $K = -73.580$. Equation (29) can now be solved for the extreme values of x with $y = 0$, and from Eqs. (27), (23) and (7), the extremes $0.096 \lesssim e \lesssim 0.111$ with the stationary point at $e = 0.104$. This is almost twice the amplitude of 0.004 found for the fluctuation in e during the 1.75 yr libration of the resonance variable by Woltjer (1928; see also Taylor 1984). [Most of the observed variation in Hyperion's eccentricity $(0.08 \lesssim e \lesssim 0.13)$ results from Titan's large eccentricity and the relative motion of the pericenters.]

Much of the discrepancy can be traced to the expansion in Laplace coefficients and the neglect of higher harmonics of the resonance variable. Although the factor e^n appears in the coefficient of the harmonic $\cos[n(3\lambda_1 - 4\lambda_2 + \tilde{\omega}_2)]$, the factors $C(a_1/a_2)$ grow so rapidly with n that the coefficients are comparable in magnitude to that of the first harmonic. This slow convergence of the expansion is also evident in the determination of the resonance driving of $\tilde{\omega}_2$'s retrograde motion. Henrard (1982*b*) includes terms in the disturbing potential up to the sixth harmonic of the resonance variable and finds $0.103 < e < 0.115$, which is not a sufficient reduction of the discrepancy. The simple theory where a single resonance term is picked from the expansion of the disturbing function in Laplace coefficients is clearly inadequate to describe the Titan-Hyperion resonance, although its application has been dominant in the literature (see, e.g., Peale 1976*b*).

It is noteworthy that the very small amplitude of libration for the current Enceladus-Dione resonance is not necessarily an indication of a small free eccentricity at transition—at least not as small as we indicated above. Dissipation of energy in the variation of tides raised on Enceladus by Saturn necessarily reduces the free eccentricity, and, as we shall see shortly, limits the growth of the forced eccentricity (Sec. III). Tidal energy is dissipated in a synchronously rotating satellite at the rate (Peale and Cassen 1978)

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\frac{21}{2} m_0 n^3 a^2 \left(\frac{R_s}{a}\right)^5 \frac{k_2^s}{Q_s} e^2 \tag{46}$$

where k_2^s, Q_s are the potential Love number and dissipation function for the satellite. The eccentricity is the source of the variation of the amplitude and orientation of the tidal bulge on the satellite leading to the dissipation (see Burns' chapter). The satellite cannot reduce its spin rate in synchronous lock so the energy dissipated must come from the orbital energy $-Gm_0m/2a$, and a must decrease. But orbital angular momentum $m\sqrt{Gm_0a(1-e^2)}$ is conserved (no transfer from spin), so $\mathrm{d}E/\mathrm{d}t = (Gm_1m_0/2a^2)\mathrm{d}a/\mathrm{d}t$ leads to

$$\frac{de}{dt} = -\frac{21}{2}\frac{m_0}{m}\left(\frac{R_s}{a}\right)^5 n\frac{k_2^s}{Q_s}e. \qquad (47)$$

The amplitude of libration vanishes with the free eccentricity, so satellite dissipation could have been the dominant cause of the small libration amplitude. If this is true, the action is no longer a conserved quantity and the area of the trajectory in the xy plane increases as we go back in time. There is also a retardation of the rate of change of $n_1^* - 2n_2^*$ due to satellite dissipation (see Sec. III below). However, if we ignore this for the time being, the larger area enclosed by the circular trajectory in the past means it will intersect the origin a shorter time ago, and the resonance age would be less than the 1.7 Gyr determined above. If satellite dissipation were large enough, even the order in which the resonances in Eq. (45) were encountered could be altered, since a larger free e_1 in the past means $\dot{\tilde{\omega}}_1$ was less negative at encounter.

It is also possible that the departure of the Enceladus-Dione pair from the history inferred from the simple tidal model is much more drastic than the modification implied by the satellite dissipation alone. This follows from the fact that Enceladus is a small, icy satellite and tidal dissipation for the current eccentricity is limited (Yoder 1981*a*). Yet parts of this satellite's surface are almost crater-free and are therefore geologically young (Smith et al. 1982; chapter by Morrison et al.). This requires considerable internal heating to provide the necessary activity for smoothing the surface. Since the current forced eccentricity of 0.0044 is too small for significant tidal heating of a solid Enceladus, and since tidal heating appears to be the only viable means for providing the energy (see the chapter by Schubert et al.), Enceladus' eccentricity must have been considerably higher in the past. A proposed earlier orbital resonance with Tethys has difficulty in generating sufficient eccentricity (Yoder 1981*a*), whereas a fairly recent 2:1 resonance with Janus (one of the coorbitals), although capable in principle of driving the eccentricity to sufficiently large values, is precariously weak because of Janus' small mass.

The unique thing about the latter hypothesis (Lissauer et al. 1984) is that Janus is driven out by torques from density waves generated by Janus in Saturn's A ring at the 7:6, 6:5, etc. orbital resonance positions. The rate of angular momentum transfer is so large that, were Janus locked into the 2:1 resonance with Enceladus, the system would be driven deep into the resonance pushing Enceladus' eccentricity to a large value determined by the dissipative properties in Enceladus (see Sec. III). Enceladus would have approached the 2:1 resonance with Dione far more rapidly than inferred earlier from tides raised on Saturn. Accordingly, the value of $n_1 - 2n_2$ would have been much smaller and the eccentricity much larger than those compatible with the simple 2:1 resonance existing today between Enceladus and Dione. The interaction with Dione would most likely have destroyed the 2:1 resonance between Enceladus and Janus because of this incompatibility and Dione's far

dominant mass. If this happened, Enceladus would have entered the 2:1 resonance with Dione with an eccentricity much too large to be sustained by tides raised on Saturn in the face of the high rate of dissipation in Enceladus tending to reduce it. The eccentricity would have damped down to the current value in only 17 Myr if the breakup of the 2:1 resonance with Janus coincided roughly with the establishment of the 2:1 resonance with Dione. This time is determined by the current separation of Janus from the 2:1 resonance position and the rate at which the ring torques are believed to be transferring angular momentum. This 17 Myr age of the Enceladus-Dione resonance, and the evolution from high eccentricity to low is a drastically different history from the 2.7 Gyr evolution inferred from the simple model. Perhaps a more astounding result which is independent of the possible past existence of a Janus-Enceladus resonance is that the ring torques on Janus and other close ring moons are so large that the rings themselves are in a state of rapid change. The rings have insufficient angular momentum to transfer at the inferred rate for very long and may therefore be less than 200 Myr old (Lissauer et al. 1984).

Although the simple model with a single term in the disturbing potential forms a basis for the discussion of the origin and evolution of the orbital resonances among the major satellites of Saturn, circumstances require much more elaborate models to adequately describe the Titan-Hyperion and the Enceladus-Dione resonances. The former requires at least the inclusion of higher harmonics of the resonance variable to describe the librations or an expression not involving Laplace coefficients, whereas the latter almost certainly requires inclusion of dissipation within Enceladus which in turn may imply a totally different history from that deduced from the simple model. The Mimas-Tethys 2:1 resonance is reasonably well described by the simple model, although Tethys' large mass, which results in a rate of expansion of its orbit near that of Mimas, means the approach to resonance and evolution within the resonance is relatively slow. In this case, the reduction of the Hamiltonian to a form containing a single variable parameter is less appropriate.

The profound effects of dissipation within Enceladus on the dynamical history of the satellite systems of which it is or may have been a part are still speculative. In the Jupiter system, on the other hand, the dissipation of tidal energy within the satellite Io (Peale et al. 1979) has made this body the most thermally active solid object in the solar system (Smith et al. 1979*a*; chapters by Schubert et al. and Nash et al.). We can use the simple resonance model again as a basis for discussing the resonances among the Galilean satellites, but now dissipation in the satellites must be included explicitly. The inclusion of this dissipation allowed an understanding of the origin and evolution of the three-body system of resonances among the Galilean satellites (Yoder 1979*b*), which had eluded scientific minds for 300 yr. It is this history of the Galilean satellites which we describe in the next section.