III. THE GALILEAN SATELLITE SYSTEM

The possible importance of dissipation of tidal energy in the satellites to the dynamical evolution of the Galilean satellite system is supported by the effect of that dissipation on the thermal evolution of Io, the innermost member. The value of the forced eccentricity can be found within the framework of the theory developed in Sec. II, even though more than a single term must be kept in the disturbing function for the general analysis we develop below. The effect of the libration of $\lambda_1 - 2\lambda_2 + \tilde{\omega}_2$ is to force Europa's eccentricity but not Io's, since conjunctions of Io and Europa occur near Europa's apocenter and e_2 is forced to that value to keep $\tilde{\omega}_2$ in step with the conjunctions (see, e.g., Peale et al. 1979). The libration of $\lambda_2 - 2\lambda_3 + \tilde{\omega}_2$ also forces Europa's eccentricity, and this additional forced eccentricity for Europa leads to a decrease in $n_1 - 2n_2$ to maintain $\lambda_1 - 2\lambda_2 - \tilde{\omega}_2$ in libration. The continued libration of $\lambda_1 - 2\lambda_2 - \tilde{\omega}_1$ forces e_1 to a larger value, but its magnitude is determined completely by the value of the two-body parameter $n_1 - 2n_2$; that is, only the resonance term with argument $\lambda_1 - 2\lambda_2 + \tilde{\omega}_1$ is used for determining the forced value of e_1 and we can use the theory of Sec. II above.

Table II (Yoder and Peale 1981) lists the important parameters for the Galilean satellites. In addition, $n_1 - 2n_2 = n_2 - 2n_3 = 0\overset{\circ}{.}739$/day from which we obtain from Eq. (22) for the libration of $\lambda_1 - 2\lambda_2 + \tilde{\omega}_1$

$$\begin{aligned}\alpha &= 0\overset{\circ}{.}900/\text{day} = 1.82 \times 10^{-7}\ \text{rad s}^{-1} \\ \beta &= 3.716 \times 10^{-47}\ \text{g}^{-1}\ \text{cm}^{-2} \\ \varepsilon &= -1.989 \times 10^{12}\ \text{g}^{1/2}\ \text{cm s}^{-3/2}. \end{aligned} \tag{48a}$$

From Eq. (25)

$$\delta = -2.823 \tag{48b}$$

and the positive real root of Eq. (30) is $x = 0.358$, which yields the eccentricity corresponding to the stable stationary point of $e_1 = 0.0043$ from Eqs. (7), (23) and (27). This is the mean eccentricity which must be maintained for Io's orbit given the observed value of α in Eq. (48). Substitution of this value of e_1 into Eq. (46) yields a rate of tidal heating of Io of (Peale et al. 1979)

$$\frac{dE_1}{dt} = \frac{1.9 \times 10^{21}}{Q_1}\ \text{erg s}^{-1} \tag{49}$$

where a solid, homogeneous body with Love number $k_2 \approx 3\ \rho g R/19\mu$ is assumed with rigidity $\mu = 6.5 \times 10^{11}$ dyne cm^{-2} (that of the outer layers of the Moon [Nakamura et al. 1976]), and the remaining parameters are from Table

TABLE II
Orbital Parameters for the Galilean Satellites[a]

	Io	Europa	Ganymede	Callisto
$M/M_J \times 10^5$	4.684 ± 0.022	2.523 ± 0.025	7.803 ± 0.030	5.661 ± 0.019
n(°/day)	203.4890	101.1747	50.3176	21.5711
$\dot{\tilde{\omega}}_s$(°/day)	0.161	0.048	0.007	0.002
$\dot{\Omega}_s$(°/day)	−0.134	−0.033	−0.007	−0.002
a(km)	422,000	671,400	1,071,000	1,884,000
e_{forced} (2:1)	0.0041	0.0101	0.0006	
e_{free}	$(1 \pm 2) \times 10^{-5}$	$(9.2 \pm 1.9) \times 10^{-5}$	0.0015	0.0073
$\sin I_{\text{free}}$	$(7.0 \pm 1.9) \times 10^{-4}$	0.0082	0.0034	0.0049
R(km)	1816 ± 5	1569 ± 10	2631 ± 10	2400 ± 10
ρ(g cm^{-3})	3.53	3.03	1.93	1.79

[a]Table after Yoder and Peale (1981).

II. Io's tidal heating is also considered in the chapters by Schubert et al. and Nash et al.

The value of Q_1 for rocks on Earth is typically near 100 (Knopoff 1964), and the value for Mars for tides raised by Phobos is comparable (Shor 1975; Smith and Born 1976; cf. Burns' chapter). Substitution of $Q_1 = 100$ in Eq. (49) yields a rate of dissipation in Io which is three times that in the Moon from radiogenic sources and a rate per unit volume in the center about 10 times the lunar average. As the Moon appears to be molten or nearly molten in the center (Nakamura et al. 1976; Ferrari et al. 1980; Yoder 1981*b*; Stevenson and Yoder 1982), it is likely that the center of Io was melted by tidal dissipation. Once melted in the center, a less rigid Io suffers a greater amplitude of tidal flexing leading to a higher dissipation per unit volume in the surrounding shell. The total increase in dissipation exceeds the loss due to the reduction in the volume of solid material in which the dissipation is occurring. Peale et al. (1979) demonstrated that solid-state convection would be unable to rid the satellite of this heat as fast as it was generated by the tides and therefore could not prevent a thermal runaway from an initial state of melting in the center to a state where only a relatively thin shell of solid material remained near the surface. (Although it is likely that the inner core would solidify eventually [Schubert et al. 1981 and chapter herein], the high rate of dissipation in the shell persists as long as it is decoupled from the solid core by a liquid layer [Cassen et al. 1982].) That Io had indeed been heated extensively by tides was dramatically verified by the observation of extensive active volcanism in Voyager 1 images (Smith et al. 1979*a*).

Although the stability of the Galilean resonances, including the libration of $\lambda_1 - 3\lambda_2 + 2\lambda_3$ about 180°, was understood at the time of Laplace (this latter resonance bears his name), the almost immeasurably small amplitude of libration of the Laplace angle in particular defied explanation in the framework of the assembly of the resonances by differential tidal expansion of the orbits (see, e.g., Sinclair 1975). In a major dynamical feat, Yoder (1979*b*) showed that inclusion of the dissipation of tidal energy in Io made almost all the observational constraints on the system consistent with an origin and evolution of the resonances from tides raised on Jupiter. (This analysis was later elaborated and extended [see Yoder and Peale 1981].) The dissipation of tidal energy in Io is completely dominant in rapidly reducing the amplitudes of libration to the small values currently observed, and one need no longer infer special damping conditions at the time of origin of the satellite system. An added bonus is the establishment of rather tight constraints on the rate of tidal dissipation in both Io and Jupiter. The upper bound on the Q of Jupiter has led to new dynamical processes inferred for the interiors of the giant planets (Stevenson 1983).

We can develop a formulation in canonical variables for the interaction of the four satellites just as we did for two (see, e.g., Yoder and Peale 1981), but it is more expedient to change variables to a noncanonical set of orbital

elements a, e, I, $\tilde{\omega}$, Ω, λ. If α_i and β_i are conjugate coordinates and momenta used in Sec. II for a single satellite and the new variables are represented by γ_j, we have for Hamiltonian H

$$\frac{d\gamma_j}{dt} = \sum_{k=1}^{6} [\gamma_j, \gamma_k]_{\alpha,\beta} \frac{\partial H}{\partial \gamma_k} \tag{50}$$

where

$$[\gamma_j, \gamma_k]_{\alpha,\beta} = \sum_{i=1}^{3} \left(\frac{\partial \gamma_j}{\partial \alpha_i} \frac{\partial \gamma_k}{\partial \beta_i} - \frac{\partial \gamma_j}{\partial \beta_i} \frac{\partial \gamma_k}{\partial \alpha_i} \right) \tag{51}$$

are the Poisson brackets. There follows (see, e.g., Plummer 1918, p. 142):

$$\begin{aligned}
\frac{da}{dt} &= -\frac{2}{m}\sqrt{\frac{a}{\mu}}\frac{\partial H}{\partial \lambda} \\
\frac{de}{dt} &= \frac{\sqrt{1-e^2}}{me\sqrt{\mu a}}\frac{\partial H}{\partial \tilde{\omega}} - \left(\frac{(1-e^2) - \sqrt{1-e^2}}{me\sqrt{\mu a}} \right) \frac{\partial H}{\partial \lambda} \\
\frac{dI}{dt} &= \frac{\tan \frac{1}{2} I}{m\sqrt{\mu a(1-e^2)}} \left(\frac{\partial H}{\partial \lambda} + \frac{\partial H}{\partial \tilde{\omega}} \right) + \frac{1}{m \sin I \sqrt{\mu a(1-e^2)}} \frac{\partial H}{\partial \Omega} \\
\frac{d\lambda}{dt} &= \frac{2}{m}\sqrt{\frac{a}{\mu}}\frac{\partial H}{\partial a} + \frac{(1-e^2) - \sqrt{1-e^2}}{me\sqrt{\mu a}}\frac{\partial H}{\partial e} - \frac{\tan \frac{1}{2} I}{m\sqrt{\mu a(1-e^2)}}\frac{\partial H}{\partial I} \\
\frac{d\omega}{dt} &= \frac{-\sqrt{1-e^2}}{me\sqrt{\mu a}}\frac{\partial H}{\partial e} - \frac{\tan \frac{1}{2} I}{m\sqrt{\mu a(1-e^2)}}\frac{\partial H}{\partial I} \\
\frac{d\Omega}{dt} &= \frac{-1}{m \sin I \sqrt{\mu a(1-e^2)}}\frac{\partial H}{\partial I}
\end{aligned} \tag{52}$$

where

$$H = -\sum_{i=1}^{4} \frac{Gm_0 m_i}{2a_i} + \Phi \tag{53}$$

with

$$\Phi = -\sum_{\substack{i,j=1 \\ i<j}}^{4} \frac{Gm_i m_j}{r_{ij}} - \sum_{k=2}^{4} Gm_0 m_k \left[\frac{1}{r_{0k}} - \frac{1}{r_k} \right] \tag{54}$$

following from the generalization of Eq. (6) from two to four satellites. Appropriate subscripts may be placed on the variables in Eq. (52) for each satellite. Equation (54) is expanded to lowest order in m_i/m_0 and subsequently to a series like Eq. (12), from which the important terms may be selected. To the disturbing potential Φ must be added the contribution of the nonspherically symmetric parts of Jupiter's gravitational field leading to conservative secular motions. The effects of tidal dissipation in both Jupiter and in the satellites must also be included. Perturbations by the Sun are small and have been omitted although, as we shall point out later, they may have an important influence on the interpretation of the libration amplitude of the Laplace angle.

The selection of the low-frequency (resonant) terms in the expanded version of Φ eliminates Callisto completely except for its contribution to the conservative secular motions. Next, the current values of $n_1 - 2n_2 = n_2 - 2n_3 = 0^\circ\!.739$/day coupled with the secular motions, $\dot{\Omega}_{si} \gtrsim -0^\circ\!.134$/day (the subscript s indicates secular) means all the inclination resonances associated with the 2:1 commensurability (see Eq. 45) have not been encountered and are sufficiently far away to be considered high frequency. Further simplification is obtained by keeping only those resonant terms in Φ which are first order in the eccentricity, which is reasonable since these terms are also multiplied by one factor of m_i/m_0 and the eccentricities are relatively small. This last simplification eliminates the terms with argument $\lambda_1 - 4\lambda_3 + 3\tilde{\omega}_1$, since the coefficient has a factor e_1^3. The infinite number of terms in Φ has now been reduced to only those terms involving the arguments $\lambda_1 - 2\lambda_2 + \tilde{\omega}_1$, $\lambda_1 - 2\lambda_2 + \tilde{\omega}_2$, $\lambda_2 - 2\lambda_3 + \tilde{\omega}_2$, $\lambda_2 - 2\lambda_3 + \tilde{\omega}_3$ and the secular terms.

The general assumption is that the orbits of the Galilean satellites were originally in a nonresonant configuration with Io being pushed away from Jupiter most rapidly such that it approaches the 2:1 resonance with Europa. Those terms in Φ involving λ_3 are thus initially high frequency and are not included here. Although two resonant terms are important in Φ, the librations $\lambda_1 - 2\lambda_2 + \tilde{\omega}_1$ and $\lambda_1 - 2\lambda_2 + \tilde{\omega}_2$ are semi-independent, since most of the variation of each resonance variable is in the respective $\tilde{\omega}_i$, and we may consider the capture of each variable into resonance whether or not the other variable is librating at the time. (See the discussion of the Enceladus-Dione resonance in Sec. II.) The fact that the δ's for both resonance variables are now negative (Eq. 48b) and were more negative in the past, shows that both variables were captured into their respective librating states with certainty. Hence, we can consider the subsequent evolution of the Io-Europa system with both resonance variables librating.

From Eqs. (52) and (53) with $n_i = \sqrt{\mu_i/a_i^3}$

$$\frac{dn_i}{dt} = \frac{3}{m_i a_i^2}\frac{\partial\Phi}{\partial\lambda_i}$$

$$\frac{de_i}{dt} = \frac{1}{m_i e_i\sqrt{\mu_i a_i}}\frac{\partial\Phi}{\partial\tilde{\omega}_i}$$
$$\frac{d\tilde{\omega}_i}{dt} = \frac{-1}{m_i e_i\sqrt{\mu_i a_i}}\frac{\partial\Phi}{\partial e} \tag{55}$$

where higher-order terms in the equations of variation are omitted, consistent with the order of the terms kept in Φ, $\partial\Phi/\partial\Omega = \partial\Phi/\partial I = 0$ and the equations for λ and I are not needed. With

$$\Phi = \frac{-Gm_1m_2}{a_2}\{C_1e_1\cos(\lambda_1 - 2\lambda_2 + \tilde{\omega}_1) + C_2e_2\cos(\lambda_1 - 2\lambda_2 + \tilde{\omega}_2)\} \tag{56}$$

for the resonance part with $C_1 = -1.19$ and $C_2 = 0.428$ for $a_1/a_2 = 0.63$ (see, e.g., Brouwer and Clemence 1961, p. 490), we need but add the effects of tidal dissipation and the secular motions.

The disturbing potential from which the tidal effects are found is given by (Kaula 1964; see Burns' chapter)

$$\Phi_T(m, b) = \frac{-k_2Gm^2R^5}{a^{*3}a^3}\sum_{m'=0}^{2}\frac{(2-m')!}{(2+m')!}(2-\delta_{0m'})$$
$$\times\sum_{p=0}^{\ell}\sum_{q=-\infty}^{\infty}[F_{2m'p}(I)][F_{2m'p}(I^*)][G_{2pq}(e)][G_{2pq}(e^*)]$$
$$\times\cos(v^*_{2m'pq} - \epsilon_{2m'pq} - v_{2m'pq}) \tag{57}$$

where m is the tide-raising body, k_2 and R are the Love number and radius, respectively, of the body on which the tide is raised and

$$v_{2m'pq} = (2-2p+q)\lambda - q\tilde{\omega} - (2-2p-m')\Omega - m'\psi \tag{58}$$

with ψ being an angle defining the rotation of the tidally distorted body. The starred variables refer to m as the body raising the tide and the unstarred variables refer to m as the body reacting to the tidal distribution of mass. Generally, secular tidal effects arise only when disturbed and disturbing bodies are the same, so starred and unstarred coordinates are equated after the derivatives relative to the unstarred variables are taken. There is a secular contribution to the perturbations of one satellite by the tide raised on m_0 by another if the orbital periods are commensurate, but these are small and will be neglected. The angle ϵ_{2mpq} is the phase lag of the response of the tidally distorted body relative to the phase of the tide-generating potential and is numerically equal to $1/Q$. The $F_{2mp}(I)$ are closed functions of cos I and $G_{2pq}(e)$ are infinite series, which are both tabulated in Kaula (1964) and in Table I of

Burns' chapter. For our purposes here we can set $I = 0$ in which case $F_{201}(0) = -1/2$, $F_{220}(0) = 3$ and all remaining F's are zero. The $d\Phi_T/d\lambda$ term must be retained in the de/dt equation since those terms in Eq. (57) with $q = 0$ are not factored by e.

Substitution of Eqs. (56) and (57) into Eq. (55) yields

$$
\begin{aligned}
\frac{dn_1}{dt} &= 3n_2^2 \frac{m_2}{m_0}\frac{a_2^2}{a_1^2}(C_1e_1\sin\theta_1 + C_2e_2\sin\theta_2) \\
&\quad - c_1n_1^2[1-(7D_1-12.75)e_1^2] \\
\frac{dn_2}{dt} &= -6\frac{m_1}{m_0}n_2^2(C_1e_1\sin\theta_1 + C_2e_2\sin\theta_2) \\
&\quad - c_2n_2^2[1-(7D_2-12.75)e_2^2] \\
\frac{de_1}{dt} &= \frac{m_2}{m_0}n_1\frac{a_1}{a_2}C_1\sin\theta_1 - \frac{c_1n_1}{3}(7D_1-4.75)e_1 \\
\frac{de_2}{dt} &= \frac{m_1}{m_0}n_2C_2\sin\theta_2 - \frac{c_2n_2}{3}(7D_2-4.75)e_2 \\
\frac{d\tilde\omega_1}{dt} &= n_1\frac{m_2}{m_0}\frac{a_1}{a_2}\frac{C_1}{e_1}\cos\theta_1 + \dot{\tilde\omega}_{1s} \\
\frac{d\tilde\omega_2}{dt} &= n_2\frac{m_1}{m_0}\frac{C_2}{e_2}\cos\theta_2 + \dot{\tilde\omega}_{2s}
\end{aligned}
\tag{59}
$$

where $\theta_1 = \lambda_1 - 2\lambda_2 + \tilde\omega_1$ and $\theta_2 = \lambda_1 - 2\lambda_2 + \tilde\omega_2$ and

$$
\begin{aligned}
c_i &= \frac{9}{2}\frac{k_2^J}{Q_J}\left(\frac{R_J}{a_i}\right)^5\frac{m_i}{m_0} \\
D_i &= \frac{k_2^i}{k_2^J}\left(\frac{m_0}{m_i}\right)^2\left(\frac{R_i}{R_J}\right)^5 f\frac{Q_J}{Q_i}.
\end{aligned}
\tag{60}
$$

The Love numbers $k_2^i \approx 3\rho_i g_i R_i/19\mu_i$ for the satellites are those for a homogeneous, incompressible sphere, and f is an enhancement factor to account for added dissipation if a satellite interior is partially molten. Equations (59) represent the variations with dissipative effects included, both in Jupiter and in the satellites. The D_i are measures of the ratio of the dissipation within the satellites to that in Jupiter. The satellite dissipation tends to increase n_i and decrease e_i. The numerical constant in the coefficient of e_i corresponds to the lowest-order term in eccentricity for dissipation in Jupiter, and it leads to an increase in e_i and a decrease in n_i. The $c_in_i^2$ are just the rates of change of n_i from tidal torques from m_0 for m_i in a circular orbit.

We can simplify the dissipative terms by noting that $k_2^{(1)} \approx 0.036$ for $\mu = 5 \times 10^{11}$ dyne cm^{-2} and that $k_2^J = 0.38$ (Gavrilov and Zharkov 1977) from which

$$
D_1 > 0.478\frac{Q_J}{Q_1} \tag{61}
$$

since $f > 1$. But $Q_J > 6 \times 10^4$ from the proximity of Io to Jupiter after 4.6 Gyr (Goldreich and Soter 1966), so with $Q_1 = 0(100)$ by analogy with terrestrial rocks, $D_1 > 300$ (we shall argue later that $D_1 \approx 4000$), and $7D_1 >>$ the numerical constant in the coefficients of e_1. This means the effect of dissipation in Jupiter in changing e_1 and in the e_1 effect on n_1 is small compared to the effect of dissipation in the satellite and may be neglected. Next $c_2n_2^2 \approx 0.026\ c_1n_1^2$ and we also make little error in neglecting the dissipation in Europa altogether.

Equations (59) are most easily solved by changing variables first to $h_i = e_i\sin\tilde\omega_i$, $k_i = e_i\cos\tilde\omega_i$ to eliminate the singularity when $e_i = 0$ and finally to $p_i = k_i - \iota h_i$, $q_j = k_i + \iota h_i$, where $\iota = \sqrt{-1}$. The dissipative terms are written in terms of the new variables as follows:

$$
\begin{aligned}
\left.\frac{dh_1}{dt}\right|_{\text{diss}} &= \frac{de_1}{dt}\sin\tilde\omega_1 + e_1\cos\tilde\omega_1\frac{d\tilde\omega_1}{dt} \\
&= -\frac{7}{3}c_1n_1D_1e_1\sin\tilde\omega_1 + \dot{\tilde\omega}_{s1}e_1\cos\tilde\omega_1
\end{aligned}
\tag{62}
$$

where only the secular $\dot{\tilde\omega}_{s1}$ enters here since the resonance-controlled motion is accounted for by terms with argument θ_i. There results

$$
\begin{aligned}
\frac{dp_1}{dt} &= -\iota n_1\frac{a_1}{a_2}\frac{m_2}{m_0}C_1\exp(\iota V_1) - \left[\frac{7}{3}c_1n_1D_1 + \iota\dot{\tilde\omega}_{s1}\right]p_1 \\
\frac{dq_1}{dt} &= \iota n_1\frac{a_1}{a_2}\frac{m_2}{m_0}C_1\exp(-\iota V_1) - \left[\frac{7}{3}c_1n_1D_1 - \iota\dot{\tilde\omega}_{s1}\right]q_1 \\
\frac{dp_2}{dt} &= -\iota n_2\frac{m_1}{m_0}C_2\exp(\iota V_1) - \left[\frac{7}{3}c_2n_2D_2 + \iota\dot{\tilde\omega}_{s2}\right]p_2 \\
\frac{dq_2}{dt} &= \iota n_2\frac{m_1}{m_0}C_2\exp(-\iota V_1) - \left[\frac{7}{3}c_2n_2D_2 - \iota\dot{\tilde\omega}_{s2}\right]q_2
\end{aligned}
\tag{63}
$$

where $V_1 = \lambda_1 - 2\lambda_2$. The equations for n_1 and n_2 are unchanged except for dropping the small numerical constant in the coefficients of e_i.

The equations are solved by successive approximations justified as follows. Before the system entered the resonances, the free eccentricities of Io and Europa were damped at rates given by Eq. (47) with time constants of

$$\begin{aligned}\tau_1 &= 3.3\times10^4\,Q_1\ \text{yr}\\ \tau_2 &= 1.4\times10^6\,Q_2\ \text{yr}.\end{aligned}\tag{64}$$

With $Q_i = 100$ both eccentricities must have been extremely small by the time the 2:1 commensurability was approached. The automatic transitions into libration as described in Fig. 4 occurred when the δ's were very negative with the trajectory in the phase plane being a tiny circle. There could have been only a negligibly small fluctuation in $n_1^* - 2n_2^*$ even when the libration in θ_i was 90° immediately after libration began since the librations are accommodated by the variations in $\tilde{\omega}_i$ (see, e.g., Peale 1976*b*). As the forced eccentricity grows (the small circular trajectory moving to the right of the origin in Fig. 4), librations could begin to cause fluctuations in $n_1^* - 2n_2^*$, but the librations are reduced to negligible amplitude by this time as can be perceived from Fig. 4 with a tiny circular trajectory about the stationary point on the positive x_1 axis. Hence, throughout the history of the evolving two-body resonance $\dot{V}_1 = n_1^* - 2n_2^*$ ($n_i^* = n_i + \dot{\lambda}_{si}$; Eq. 16) is only slowly varying, and we can use the zero order solution $n_i^* =$ constant in Eqs. (63).

Equations (63) now separate from the n_i equations and, with $\nu_1 = \dot{V}_1 = n_1^* - 2n_2^*$, have the particular solutions

$$\begin{aligned} p_1 &= -n_1\frac{a_1}{a_2}\frac{m_2}{m_0}\frac{C_1}{(\nu_1+\dot{\tilde{\omega}}_{s1})}\exp\iota(\nu_1 t+\xi_1)\\ p_2 &= -n_2\frac{m_1}{m_0}\frac{C_2}{(\nu_1+\dot{\tilde{\omega}}_{s2})}\exp\iota(\nu_1 t+\xi_2)\end{aligned}\tag{65}$$

where q_i are the complex conjugates of Eq. (65) and

$$\xi_i = \frac{7}{3}\frac{c_i n_i D_i}{\nu_1+\dot{\tilde{\omega}}_{si}}\,.\tag{66}$$

We have used $1+\iota\xi_i \approx \exp\iota\xi_i$ and neglected ξ_i^2 in the denominator. The solution of the homogeneous equation

$$p_1 = p_{10}\exp\left[\frac{-7}{3}c_1 n_1 D_1 - \iota\dot{\tilde{\omega}}_{s1}\right]t\tag{67}$$

is a transient with the very short time constant given by Eq. (64), and it can be ignored. Note that the damping of the free eccentricity represented by Eq. (67) coincides with the damping of the libration amplitude of the resonance variable. The stationary solution in terms of the forced eccentricity and the center of libration follows from Eq. (65), since $p_1 = e_1\cos\tilde{\omega}_1 - \iota e_1\sin\tilde{\omega}_1$. We obtain

$$\begin{aligned} e_{12} &= -n_1\frac{a_1}{a_2}\frac{m_2}{m_0}\frac{C_1}{\nu_1+\dot{\tilde{\omega}}_{s1}}\\ e_{21} &= n_2\frac{m_1}{m_0}\frac{C_2}{\nu_1+\dot{\tilde{\omega}}_{s2}}\\ \tilde{\omega}_1 &= -(V_1+\xi_1)\\ \tilde{\omega}_2 &= -(V_1+\xi_2)+\pi\end{aligned}\tag{68}$$

where e_{ij} indicates the value of e_i forced by m_j. The effect of satellite dissipation is to cause conjunctions to occur slightly past the pericenter of Io's orbit and past the apocenter of Europa's orbit. That is, $V_1 + \tilde{\omega}_1 = \lambda_1 - 2\lambda_2 + \tilde{\omega}_1 = -\xi_1$ so when $\lambda_1 = \lambda_2$, $\lambda_1 = \tilde{\omega}_1 + \xi_1$. There is an additional phase shift in the same direction from the secular decrease in n_1 causing it to arrive late at the conjunction point. This latter phase shift can be determined from the condition that $(\mathrm{d}/\mathrm{d}t)(n_1 - 2n_2 + \dot{\tilde{\omega}}_i) = 0$ but the effect is relatively small compared to the satellite dissipation and will be neglected.

Now $e_i\cos\tilde{\omega}_i$ and $e_i\sin\tilde{\omega}_i$ obtained from Eq. (65) are known functions of time which, when substituted into the equations for $\mathrm{d}n_i/\mathrm{d}t$ yield

$$\begin{aligned}\frac{\mathrm{d}n_1}{\mathrm{d}t} &= -c_1 n_1^2(1-14D_1e_{12}^2) + 7\,c_2 n_1^2\frac{a_1}{a_2}\frac{m_2}{m_1}D_2 e_{21}^2\\ \frac{\mathrm{d}n_2}{\mathrm{d}t} &= -c_2 n_2^2(1+7D_2e_{21}^2) - 14\frac{m_1}{m_2}\frac{a_2}{a_1}n_2^2 c_1 D_1 e_{12}^2.\end{aligned}\tag{69}$$

We neglect the dissipation in Europa compared to that in Io and combine these to give

$$\frac{\mathrm{d}\nu_1}{\mathrm{d}t} = \frac{\mathrm{d}n_1}{\mathrm{d}t} - 2\frac{\mathrm{d}n_2}{\mathrm{d}t} = -c_1 n_1^2(1-34.5D_1e_{12}^2) + 2c_2 n_2^2.\tag{70}$$

We have used Eqs. (66) and (68) to obtain Eqs. (69) and (70). In Eq. (70) we see that $\mathrm{d}\nu_1/\mathrm{d}t$ vanishes when $e_{12}^2 = (34.5D_1)^{-1}$, a consequence of the dissipative effects in Io tending to reduce e_1 balancing the effect of tides raised on Jupiter tending to increase e_1 by driving n_1 closer to the exact commensurability. Once $\dot{\nu}_1 = 0$, e_{12} and e_{21} have equilibrium values determined by Eq. (68) which change only on the slow time scale of the orbit expansion. Transfer of angular momentum between m_1 and m_2 is not efficient until n_1 is reasonably close to $2n_2$ and e_{12} is close to its equilibrium value. The time required for e_{12} to approach equilibrium is thus comparable to that for n_1 to approach $2n_2$ from some unknown initial value. The value of $D_1 \approx 4200$ determined below leads to $e_{12} = 0.0026$ and from Eq. (68) $e_{21} = 0.0014$ at equilibrium.

After equilibrium is reached, the system expands with the outward acceleration of Europa maintaining its mean motion at a fixed factor near 0.5 times the mean motion of Io. This stable state is maintained until Europa encounters the 2:1 resonance with Ganymede. Just as dissipation in Io tends to repel Europa by the secular transfer of angular momentum, dissipation in Europa would tend to transfer angular momentum to Ganymede in the 2:1 resonance. However, if Europa were acting alone, the equilibrium eccentricity for the Europa-Ganymede interaction would be three times the current value of 0.01. Long before this can happen, the frequency $\nu_2 = n_2^* - 2n_3^*$ of the outer pair approaches $\nu_1 = n_1^* - 2n_2^*$ of the inner pair. The vanishing of the difference $\nu_1 - \nu_2$ describes the presently observed three-body Laplace relation.

Once the 2:1 resonance with Ganymede is approached, we must add the appropriate resonance terms in Φ for the Europa-Ganymede interaction. Corresponding Io-Ganymede terms are omitted as they are third order in e, so Io and Ganymede interact almost entirely by using Europa as an intermediary.

The disturbing function becomes

$$\Phi = \frac{Gm_1m_2}{a_2}[C_1e_1\cos(\lambda_1 - 2\lambda_2 + \tilde{\omega}_1) + C_2e_2\cos(\lambda_1 - 2\lambda_2 + \tilde{\omega}_2)] + \frac{Gm_2m_3}{a_3}[C_1e_2\cos(\lambda_2 - 2\lambda_3 + \tilde{\omega}_2) + C_2e_3\cos(\lambda_2 - 2\lambda_3 + \tilde{\omega}_3)] \tag{71}$$

and the variations in n_i, e_i, $\tilde{\omega}_i$ follow from Eqs. (55) and (57). The procedure is identical to that used before, and we find

$$\begin{aligned}
e_2\cos\tilde{\omega}_2 &= -n_2\frac{m_1}{m_0}\frac{C_2\cos(V_1 + \xi_2)}{\nu_1 + \dot{\tilde{\omega}}_{s2}} - n_2\frac{m_3}{m_0}\frac{a_2}{a_3}\frac{C_1\cos(V_2 + \xi_3)}{\nu_2 + \dot{\tilde{\omega}}_{s2}}\\
e_2\sin\tilde{\omega}_2 &= n_2\frac{m_1}{m_0}\frac{C_2\sin(V_1 + \xi_2)}{\nu_1 + \dot{\tilde{\omega}}_{s2}} + n_2\frac{m_3}{m_0}\frac{a_2}{a_3}\frac{C_1\sin(V_2 + \xi_3)}{\nu_2 + \dot{\tilde{\omega}}_{s2}}\\
e_3\cos\tilde{\omega}_3 &= -n_3\frac{m_2}{m_0}\frac{C_2\cos(V_2 + \xi_4)}{\nu_2 + \dot{\tilde{\omega}}_{s3}} + e_{30}\exp[-Kt]\cos\dot{\tilde{\omega}}_{3s}t\\
e_3\sin\tilde{\omega}_3 &= n_3\frac{m_2}{m_0}\frac{C_2\sin(V_2 + \xi_4)}{\nu_2 + \dot{\tilde{\omega}}_{s3}} + e_{30}\exp[-Kt]\sin\dot{\tilde{\omega}}_{3s}t
\end{aligned} \tag{72}$$

where $V_2 = \lambda_2 - 2\lambda_3$, ξ_i is given for $i = 3, 4$ by Eq. (66) with $\nu_2 = n_2 - 2n_3$ replacing ν_1, and e_{30} is an initial free eccentricity. The expressions for $e_1\cos\tilde{\omega}_1$ and $e_1\sin\tilde{\omega}$, are still given by the real and imaginary parts of Eq. (65), and we have retained the (not so) transient solution for m_3 with $K = 7/3\ c_3n_3D_3$ as the inverse of the time constant also given by Eq. (47). This time

constant is $2.5 \times 10^{-5}\ \mu_3Q_3$ yr which perhaps should have led to a more damped e_{30} if the rigidity μ_3 were that of ice (4.8×10^{10} dyne cm^{-2}). But e_{30} is observed to be about 0.0015 compared to a forced $e_{32} \approx 0.0007$ and $\lambda_2 - 2\lambda_3 + \tilde{\omega}_3$ is not librating; that is, the circular trajectory of radius $\propto e_{30}$ in Fig. 4 encloses the origin as the center of the circle is a distance $\propto e_{32} < e_{30}$ away.

If we neglect ξ_i and $c_in_i^2$ for $i > 1$, substitute Eq. (72) into the new expressions for dn_i/dt and write the forced values of the eccentricities as

$$\begin{aligned}
e_{12} &= -n_1\frac{m_2}{m_0}\frac{a_1}{a_2}\frac{C_1}{\nu_1 + \dot{\tilde{\omega}}_{s1}}\\
e_{21} &= n_2\frac{m_1}{m_0}\frac{C_2}{\nu_1 + \dot{\tilde{\omega}}_{s2}}\\
e_{23} &= -n_2\frac{m_3}{m_0}\frac{a_2}{a_3}\frac{C_1}{\nu_2 + \dot{\tilde{\omega}}_{s2}}
\end{aligned} \tag{73}$$

where $e_2 = e_{21} + e_{23}$ is now forced by both Io and Ganymede, we find

$$\begin{aligned}
\frac{dn_1}{dt} &= -3n_1^2n_2\frac{m_2m_3}{m_0^2}\frac{a_1}{a_3}\frac{C_1C_2}{\nu_2 + \dot{\tilde{\omega}}_{s2}}\sin(V_1 - V_2) - c_1n_1^2(1 - 14D_1e_1^2),\\
\frac{dn_2}{dt} &= n_2^3\frac{m_1m_3}{m_2^0}\frac{a_2}{a_3}C_1C_2\left(\frac{6}{\nu_2 + \dot{\tilde{\omega}}_{s2}} + \frac{3}{\nu_1 + \dot{\tilde{\omega}}_{s2}}\right)\sin(V_1 - V_2)\\
&\quad - 14C_1n_2^2\ \frac{m_1}{m_2}\frac{a_2}{a_1}D_1e_{12}^2,\\
\frac{dn_3}{dt} &= -6n_3^2\frac{m_1m_2}{m_0^2}\frac{C_1C_2n_2}{\nu_2 + \dot{\tilde{\omega}}_{s2}}\sin(V_1 - V_2).
\end{aligned} \tag{74}$$

With $\varphi = \lambda_1 - 3\lambda_2 + 2\lambda_3 = V_1 - V_2$, we can write from Eq. (74)

$$\frac{d^2\varphi}{dt^2} + An_2^2\sin\varphi = -c_1n_1^2(1 - 44.8D_1e_{12}^2) \tag{75}$$

$$\frac{d\nu_1}{dt} + (A_1 - 2A_2)n_2^2\sin\varphi = -c_1n_1^2(1 - 34.5D_1e_{12}^2) \tag{76}$$

where

$$\begin{aligned}
A &= A_1 - 3A_2 + 2A_3\\
A_1 &= 3C_1C_2\frac{a_2}{a_1}\frac{m_2m_3}{m_0^2}\frac{n_2}{\nu_2 + \dot{\tilde{\omega}}_{s2}}
\end{aligned}$$

$$A_2 = -3C_1C_2 \frac{a_1}{a_2} \frac{m_1 m_3}{m_0^2}\left(\frac{2n_2}{\nu_2 + \dot{\omega}_{s2}} + \frac{n_2}{\nu_1 + \dot{\omega}_{s2}}\right)$$

$$A_3 = 6C_1C_2\left(\frac{a_2}{a_3}\right)^2 \frac{m_1 m_2}{m_0^2} \frac{n_2}{\nu_1 + \dot{\omega}_{s2}}. \tag{77}$$

The left-hand side of Eq. (75) is that obtained by Laplace and reveals the pendulumlike stability of the libration of φ. The right-hand side of Eq. (75) includes the effects of dissipative tides raised on Jupiter by Io and of tidal dissipation in Io only. Inclusion of dissipation in the remaining satellites has only a small effect on the outcome.

The encounter of Europa with the 2:1 commensurability with Ganymede has introduced fluctuations into ν_1 as indicated by Eq. (76). It is these fluctuations which hold the key to the capture of φ into libration. We can eliminate $\sin\varphi$ between Eqs. (75) and (76) to yield

$$\frac{d\nu_1}{dt} = \frac{A_1 - 2A_2}{A} \frac{d\dot{\varphi}}{dt} - 0.32\, c_1 n_1^2 (1 - 12.6 D_1 e_{12}^2). \tag{78}$$

Integration of Eq. (78) gives

$$\nu_1 = 0.68\, \dot{\varphi} + \nu_{10} \tag{79}$$

where $\langle\nu_1\rangle = 0.68\, \langle\dot{\varphi}\rangle + \nu_{10}$ is slowly varying.

At resonance $\langle\dot{\varphi}\rangle = 0$ and $\dot{\varphi}$ represents fluctuations about zero with $\langle\nu_1\rangle = \nu_{10}$. For the approach to resonance we can define $\delta\dot{\varphi}$ as the periodic part of $\dot{\varphi}$ which is a fluctuation about some mean value. The fluctuating part of ν_1 is thus

$$\delta\nu_1 = 0.68\delta\dot{\varphi}. \tag{80}$$

Equation (75) contains ν_1 and ν_2 through A and through e_{12}. It is the fluctuations in A and e_{12} through ν_1 and ν_2 which define the capture scheme. The fluctuations in ν_2 follow from the definition of $\dot{\varphi} = \nu_1 - \nu_2$

$$\delta\nu_2 = \delta\nu_1 - \delta\dot{\varphi} = -0.32\delta\dot{\varphi}. \tag{81}$$

Now $\delta A/A \approx 0.04\, \delta\dot{\varphi}/(\nu_2 + \dot{\omega}_{s2})$ whereas $\delta(e_{12}^2)/e_{12}^2 = -1.36\, \delta\dot{\varphi}/(\nu_1 + \dot{\omega}_{s1})$, so we can neglect the former and write

$$e_{12}^2 = \langle e_{12}^2\rangle\left[1 - \frac{1.36\delta\dot{\varphi}}{\nu_1 + \dot{\omega}_{s1}}\right] \tag{82}$$

and Eq. (75) becomes

$$\ddot{\varphi} + An_2^2 \sin\varphi = -c_1 n_1^2(1 - 44.8 D_1 e_{12}^2) - \sigma\delta\dot{\varphi} \tag{83}$$

where

$$\sigma = 60.9 \frac{c_1 n_1^2 D_1 e_{12}^2}{\nu_1 + \dot{\omega}_{s1}} \tag{84}$$

and where e_{12}^2 is now the average value with the fluctuations explicitly displayed by $\delta\dot{\varphi}$.

Initially $\dot{\varphi} = \nu_1 - \nu_2 < 0$ since $\nu_2 = n_2 - 2n_3$ must be large. So $\langle\dot{\varphi}\rangle = 0$ can only be approached if $-c_1 n_1^2(1 - 44.8 D_1 e_{12}^2) > 0$ or $e_{12} > (1/44.8 D_1)^{1/2}$. But this is assured if the equilibrium in the Io-Europa 2:1 resonance is reached before encounter with Ganymede since $e_{12} = (1/34.5 D_1)^{1/2}$ in that case.

Equation (83) is that of a pendulum with an applied torque good for either circulation or libration of the Laplace angle φ. As the system passes through resonance, the mean value of $\dot{\varphi}$ is essentially zero and $\delta\dot{\varphi} = \dot{\varphi}$. Replacing $\delta\dot{\varphi}$ by $\dot{\varphi}$ in Eq. (83) yields a form identical with that derived for spin-orbit coupling by Goldreich and Peale (1966), and we can use their expression for the capture probability to obtain (Yoder 1979*b*; Yoder and Peale 1981)

$$P_c = \frac{2}{1 + \dfrac{\pi}{4} \dfrac{\ddot{\varphi}_T}{\sigma|An_2^2|^{1/2}}} = \frac{2}{1 + \left(\dfrac{D_1}{3700}\right)^{3/4}} \tag{85}$$

where $\ddot{\varphi}_T = -c_1 n_1^2(1 - 44.8 D_1 e_{12}^2)$ is the secular rate of change in $\langle\dot{\varphi}\rangle$ from the tidal expansion of the orbits and $e_{12} = (1/34.5 D_1)$ at the time of capture is used to obtain the final form with Eqs. (73) and (77) giving $\nu_1 + \dot{\omega}_{si}$ in terms of D_1 in A and σ. Capture into the resonance is certain if $D_1 < 3700$.

The $\dot{\varphi}$ term will damp the librations in φ and we see directly how the dissipation in Io can provide the previously elusive means of accounting for the extremely small amplitude of the Laplace libration. The tides will continue to reduce n_1 and, with the addition of Ganymede to the resonant system, Eq. (78) shows that ν_1 is also further reduced as e_{12} increases toward a new equilibrium value of $(12.6 D_1)^{-1/2}$. In libration $\langle d\dot{\varphi}/dt\rangle = 0$ in Eq. (78). If the current value of $e_1 = 0.0041$ is the equilibrium value, $D_1 = 4600$. A larger equilibrium e_{12} would result in a smaller D, and we see from Eq. (85) that

$$P_c \geq 0.9. \tag{86}$$

Henrard (1983) has refined the mathematical model used here and finds $P_c = 1$, with similarly slight changes in other parameters. That the value of e_1 is indeed currently at equilibrium follows from the analysis of the damping of the libration in φ.

We pointed out in the discussion of Enceladus that the action, here represented by

$$J = \oint \dot{\varphi} \mathrm{d}\varphi \tag{87}$$

is no longer conserved when there is dissipation in the satellites, and we see from Eq. (87) that J vanishes with the amplitude of libration. This suggests we follow the decrease in J to analyze the tidal damping of the libration in φ. This approach is especially convenient because it avoids altogether any problems with the infinite period on the separatrix which has to be dealt with if we monitor $\varphi_{\max}$ directly. If we write the libration energy as

$$E = \frac{1}{2}\dot{\varphi}^2 - An_2^2 \cos\varphi \tag{88}$$

then $J = J(E, A)$, and

$$\begin{aligned} \frac{\mathrm{d}J}{\mathrm{d}t} &= \left[\frac{\mathrm{d}E}{\mathrm{d}t}\frac{\partial}{\partial E} + \frac{\mathrm{d}A}{\mathrm{d}t}\frac{\partial}{\partial A}\right]\oint\dot{\varphi}\mathrm{d}\varphi \\ &= \left[\left(\frac{\partial E}{\partial t} + \frac{\partial E}{\partial A}\frac{\mathrm{d}A}{\mathrm{d}t}\right)\frac{\partial}{\partial E} + \frac{\mathrm{d}A}{\mathrm{d}t}\frac{\partial}{\partial A}\right]\oint\dot{\varphi}\mathrm{d}\varphi \end{aligned} \tag{89}$$

where $\partial E/\partial t$ is with A held constant and follows by multiplying Eq. (83) by $\dot{\varphi}$,

$$\frac{\mathrm{d}J}{\mathrm{d}t} = \left[\left(-\sigma\dot{\varphi}^2 + \ddot{\varphi}_T\dot{\varphi} - n^2\cos\varphi\frac{\mathrm{d}A}{\mathrm{d}t}\right)\frac{\partial}{\partial E} + \frac{\mathrm{d}A}{\mathrm{d}t}\frac{\partial}{\partial A}\right]\oint\dot{\varphi}\mathrm{d}\varphi. \tag{90}$$

Solving Eq. (88) for $\dot{\varphi}$, we find $\partial\dot{\varphi}/\partial E = 1/\dot{\varphi}$ and $\partial\dot{\varphi}/\partial A = n^2\cos\varphi/\dot{\varphi}$. Making these substitutions in Eq. (90) and averaging over a period τ yields

$$\left\langle\frac{\mathrm{d}J}{\mathrm{d}t}\right\rangle = -\sigma J \tag{91}$$

and

$$J(t) = J(0)\exp\int_0^t -\sigma(t)\mathrm{d}t \tag{92}$$

where $\tau = \oint\dot{\varphi}^{-1}\mathrm{d}\varphi$, σ and $\ddot{\varphi}_T$ are assumed constant during the averaging over the libration period.

An explicit solution of Eq. (91) follows if we change the integration variable from t to $z = \gamma_1/\bar{\gamma}_1$ where $\gamma_1 = \nu_1 + \dot{\tilde{\omega}}_{s1}$ and $\bar{\gamma}_1$ is the value of γ_1 at the equilibrium $e_{12} = (12.6D_1)^{-1/2}$ (Yoder and Peale 1981). There results

$$\left|\frac{A(t)}{A(0)}\right|^{1/2}\frac{\pi}{4}\sin^2\frac{\varphi_m}{2} = \left[\frac{z_0^2(1-z^2)}{z^2(1-z_0^2)}\right]^{7.6} \tag{93}$$

for small libration amplitude φ_m, where the Laplace resonance was established at $t = 0$ when $z = z_0 = 1.63$. From Eq. (77) $A(t)/A(0) = [z_0 - (\dot{\tilde{\omega}}_{s1} - \dot{\tilde{\omega}}_{s2})/\gamma_1]/[z - (\dot{\tilde{\omega}}_{s1} - \dot{\tilde{\omega}}_{s2})/\gamma_1]$ and Eq. (93) yields $z = 1.047$ for a current amplitude of $\varphi_m = 0^\circ.066$ (Lieske 1980). From a solution of Eq. (78) with $\langle\ddot{\varphi}\rangle = 0$,

$$t = \frac{\bar{\gamma}_1}{0.32c_1n_1}\left[z_0 - z + \frac{1}{2}\ln\left(\frac{1+z}{1+z_0}\frac{1-z_0}{1-z}\right)\right] \tag{94}$$

which yields an age of the Laplace resonance of (Yoder and Peale 1981)

$$t = 1600\ Q_J\ \mathrm{yr} \tag{95}$$

for $z = 1.047$. If $\varphi_m = 0^\circ.066$ is not a remnant amplitude of the damping process, Eq. (95) is not applicable and the age of the resonance cannot be determined in terms of Q_J. The amplitude may be forced by a solar perturbation (Yoder and Peale 1981).

The more important result of the damping analysis is that $\varphi_m \to 0$ as $z \to 1$, i.e., as the system approaches equilibrium. The value of $\varphi_m = 0^\circ.066$ and $z = 1.047$ with $e_{12} = 0.0041$ gives $e_{12} = (13D_1)^{-1/2} = 0.0043$ at equilibrium from which $D_1 = 4200$. If $e_{12} = 0.0041$ is the equilibrium value ($\varphi_m = 0^\circ.066$ not a remnant amplitude), $D_1 = 4600$. Even if $\varphi_m = 0^\circ.066$ is only an upper bound on the free amplitude, Eq. (95) places an upper bound on Q_J of

$$Q_J \lesssim 3 \times 10^6 \tag{96}$$

since $t < 4.6$ Gyr.

The upper bound on Q_J given in Eq. (96) is also supported by the observation of the extensive thermal activity on Io. Surely the tidal dissipation in Io is at least as high as the energy generated within from radioactive decay, since, for example, the Moon most likely has a similar content of radioactive elements as Io, yet shows no current thermal activity. Cassen et al. (1979*b*) have estimated the lunar heat source from radioactivity to be 6.9×10^{-8} erg g^{-1} s^{-1} at the present time with nearly four times this amount 4.6 Gyr ago. This implies a current deposition of radiogenic heat of 6×10^{18} erg s^{-1}, and from Eq. (49) $Q_1 \lesssim 300$ if tidal dissipation is to exceed this. From Eq. (60) with $D_1 = 4200$, $k_2^{(1)} = 0.036$, $f = 1$, $Q_J \lesssim 2.6 \times 10^6$.

This upper bound on Q_J is considerably below some estimates of Q_J from first principles (Goldreich and Nicholson 1977; Hubbard 1974; Burns' chapter), and has led to the suggestion that tidal dissipation in Jupiter may not have caused the assembly of the resonances. The resonances would then have to be assembled by unspecified processes at the time the Jupiter system itself was formed (see, e.g., Greenberg 1982*a*). However, the observed dissipation in Io would result in an increase in ν_1 and an eventual destruction of the resonances if there were no transfer of angular momentum from Jupiter tending to decrease ν. (Only the term containing D_1 would remain in Eq. 78.) The only way a primordial origin could be compatible with observation is for the system to have started deeper in the resonance with larger e's and smaller ν's, and we would now be watching the system relax toward smaller values of eccentricity (Peale and Greenberg 1980).

However, a linear stability analysis (Yoder and Peale 1981) has shown the Laplace resonance to be unstable at the current stationary values of the angles for $e_1 > 0.012$, so the system could probably not have started any deeper in the resonance than this. An origin deeper within the resonance might have been possible if the stationary values of the angles change and oscillations about these angles remain stable for $e_1 > 0.012$ (Greenberg 1984*b*,1985), and schemes for storage of the system on the other side of the resonance can be contrived. However, the latter schemes are highly implausible and none has been shown even to be possible (Yoder and Peale 1981). An initial $e_1 > 0.012$ would in any case be rapidly reduced to values less than this bound by the expected high rate of dissipation in Io. If 0.012 is assumed to be a firm upper bound on e_1, the total energy available for heating Io is somewhat less than the change in the orbital energy between the initial configuration with $e_1 = 0.012$ and the present configuration with $e_1 = 0.0041$ under the condition of conserved angular momentum. This latter energy is 5.5×10^{35} erg compared with about 2×10^{36} erg available from radioactive decay in Io (Yoder and Peale 1981). As tidal dissipation must exceed the radiogenic source of heat today and would have been larger in the past, one can infer that the tidal dissipation has been considerably more than that necessary to relax the system from its most extreme resonance configuration to that we see today. The only way to accommodate this dissipation is for Jupiter to supply sufficient torque to retard the relaxation. Otherwise the system would have relaxed to smaller eccentricities than we now observe (Peale and Greenberg 1980; Yoder and Peale 1981). The current eccentricity is therefore close to the equilibrium value whether or not the resonance was primordial and the upper bound $Q_J \lesssim 3 \times 10^6$ still applies.

Stevenson (1983) has found a mechanism of dissipation of tidal energy in Jupiter involving a phase change of helium which yields Q_J within the bounds imposed by the dynamics of the satellite system.

The lower bound $Q_J > 6.6 \times 10^4$ established by the proximity of Io to Jupiter after 4.5 Gyr is apparently in conflict with the measured heat flow

from the satellite, which has been estimated to be near 1500 erg cm^{-2} s^{-1} averaged over the surface (Matson et al. 1981*a*; Sinton 1981; Morrison and Telesco 1980; chapter by Schubert et al.). One can calculate the dissipation in Io as a function of Q_J independent of the properties of the satellite provided that an equilibrium configuration is assumed (Lissauer et al. 1984). For a given Q_J, the rate at which the torque does work on Io is known and the rate of increase in the orbital energy in the preserved configuration can be calculated. The rate of work done exceeds the rate of increase in the orbital energy by the rate at which energy is dissipated in Io. For minimum Q_J this dissipation corresponds to a surface flux density of about 800 erg cm^{-2} s^{-1} or only about half of that estimated from observation. It has been pointed out by Johnson et al. (1984) that the observations of Io in the infrared from which the heat flow had been estimated were done while Io was eclipsed by Jupiter and were all therefore of the same hemisphere. Since the estimates of total heat flux had been made on the assumption of a uniform distribution of sources, a lack of sources on the far hemisphere would reduce these estimates to values near the above maximum *average* heat flux determined by the dynamical constraints. Further observations of Io at a variety of phases do show a concentration of hot spots in the previously observed hemisphere (Io eclipsed when viewed from Earth) (Johnson et al. 1984), but continuing heat flux estimates from all sources show time fluctuations about 1500 erg cm^{-2} s^{-1} with no apparent trend in the mean value (McEwen et al. 1985). This implies that the heat flux observed over the last six years must considerably exceed the long-term average flux.

Yoder's (1979*b*) solution of the long-standing problem of the origin and subsequent damping of the Laplace libration is one of the most outstanding accomplishments of dynamical analysis during the past decade. It was probably not anticipated that the solution of this particular problem in dynamical evolution would also lead to such narrow bounds on the dissipative properties of a giant gaseous planet.

The understanding of the origin and evolution of the resonances among the Galilean satellites has been a fascinating exercise with the introduction of the dissipation of tidal energy in the satellites providing a solution to a long-standing enigma about the route to the currently completely damped configuration. We turn now to a system in which dissipation appears to be more important in drastically altering the dynamical configuration in the future than it has been in establishing the current state. The interest in the coorbital satellites of Saturn discussed in the next section is generated by their being bodies of comparable mass locked in 1 : 1 orbital resonance and having such a large amplitude of libration that one satellite approaches quite close to the other, alternately in front and behind. An elegantly simple modification of the restricted three-body problem provides an adequate description of the system and allows a determination of the masses and ultimately the densities of both satellites.

IV. THE COORBITAL SATELLITES OF SATURN

The 1 : 1 orbital resonance has received an enormous amount of attention in the literature as part of a more general study of the restricted three-body problem (see, e.g., Szebehely 1967) with application to the Trojan asteroids. These asteroids librate about the stable stationary points of the restricted three-body problem, the L_4 and L_5 points 60° ahead and behind Jupiter with average mean motions identical to that of Jupiter. Three more Trojan-type objects were found among the satellites of Saturn by groundbased observers at the time of the Voyager 1 and 2 flybys—one librating about Dione's L_4 point (leading by 60°) and one each librating about Tethys' L_4 and L_5 points.

The coorbital satellites, Janus and Epimetheus, orbit at a mean distance of 151,000 km from the center of Saturn. This latter example of a 1 : 1 resonance is unique in the sense that the satellite masses are comparable (M_1/M_2 = 1/5 to 1/3), and the amplitude of libration is so large that the satellites can come quite close to each other. The orbit of Epimetheus in a frame rotating with Janus has the shape of a horseshoe enveloping both the L_4 and L_5 Lagrange points instead of a Trojan-like path confined to a region near one or the other point. Reduction of groundbased observations and Voyager 1 and 2 orbit determinations yield $(M_1 + M_2)/M = (3.9 \pm 1.2) \times 10^{-9}$ and $M_2/(M_1 + M_2) = 0.216 \pm 0.009$, where M is the mass of Saturn (C. F. Yoder and S. Synnott, personal communication, 1985). Estimates of the volumes of the satellites by P. Thomas (unpublished, 1984; see chapter herein) then give densities of 0.85 (± 0.3) g cm^{-3}. The small mass ratio allows a simple analytic approximation which adequately describes the motion of the coorbital pair and can be used to infer some aspects of their dynamical evolution (Yoder et al. 1983).

Following Yoder et al. (1983), we can write the equations of motion of Janus as perturbed as Epimetheus as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(r_1^2 \frac{\mathrm{d}\theta_1}{\mathrm{d}t}\right) = \frac{\partial F_{12}}{\partial \theta_1} \tag{97}$$

$$\frac{\mathrm{d}^2 r_1}{\mathrm{d}t^2} - r_1\left(\frac{\mathrm{d}\theta_1}{\mathrm{d}t}\right)^2 = -\frac{GM}{r_1^2} + \frac{\partial F_{12}}{\partial r_1} \tag{98}$$

where

$$F_{12} = GM_2\left[\frac{1}{\Delta} - \frac{r_1^2}{r_2^2}\cos(\theta_1 - \theta_2)\right] \tag{99}$$

$$\Delta^2 = r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta_1 - \theta_2). \tag{100}$$

The subscripts 1 and 2 correspond, respectively, to Janus and Epimetheus. The polar coordinates r and θ are referred to the center of mass of Saturn, and the satellite orbits are assumed coplanar. The perturbations from other satellites, the rings, and the Sun are neglected for the time being. The influence of the oblateness of Saturn does not qualitatively change the character of the motion and would unnecessarily clutter the analysis. A similar set of equations is appropriate for the effect of Janus on Epimetheus, where F_{21} is not simply proportional to F_{12} because of the indirect term resulting from the use of a noninertial frame. We can write the equations as perturbations from circular reference orbits with a_0 and n_0 representing the mean distance from Saturn and mean motion, respectively, and δn_i and δr_i defined by

$$\frac{\mathrm{d}\theta_i}{\mathrm{d}t} = n_0 + \delta n_i \tag{101}$$

$$r_i = a_0\left(1 - \frac{2}{3}\frac{\delta n_i}{n_0}\right) + \delta r_i. \tag{102}$$

In Eq. (102) the variation in r_i is separated into that part resulting from Kepler's third law with the orbits remaining circular and the additional increment related to an induced eccentricity. The advantage of this separation will be evident below. Substitution of Eqs. (101) and (102) into (97) and (98) and expansion to first order in δr_1 and δn_1 yields

$$-\frac{1}{3}a_0^2\frac{\mathrm{d}(\delta n_1)}{\mathrm{d}t} + 2n_0 a_0 \frac{\mathrm{d}(\delta r_1)}{\mathrm{d}t} = \frac{\partial F_{12}}{\partial \theta_1} \tag{103}$$

$$\frac{\mathrm{d}^2(\delta r_1)}{\mathrm{d}t^2} - \frac{2}{3}a_0 n_0^{-1}\frac{\mathrm{d}^2(\delta n_1)}{\mathrm{d}t^2} - 3n_0^2\delta r_1 = \frac{\partial F_{12}}{\partial r_1}. \tag{104}$$

The relative magnitudes of the terms in Eqs. (103) and (104) for the variations in δr_1 and δn_1 due to the mutual interactions depend on the small parameter $\varepsilon = [(M_1 + M_2)/M]^{1/2}$. Although the satellites interact only when they are close to each other, the average rate of change of the increments δn_i or δr_i is the magnitude of the change divided by the time between interactions, which is half the libration period. At the time of the Voyager 1 observations, Epimetheus was gaining on Janus by 0°254/day, leading to a time between encounters of about 4 yr. In terms of ε, this time is of order $2\pi/n_0\varepsilon$ with ε^2 being of order M_1/M given above. Using this representation of the time in Eqs. (103) and (104) and noting that F_{12} is of order $\varepsilon^2 a_0^2 n_0^2$, one finds that δn is of order εn_0 and δr is of order $\varepsilon^2 a_0$. Hence, a formulation correct to order ε^2 is obtained by omitting δr in Eq. (103) and replacing both r_1 and r_2 by a_0 in F_{ij}. In this approximation $M_1 F_{12} = M_2 F_{21}$ and we can write the equation for the variation of the difference angle $\phi = \theta_1 - \theta_2$,

$$-\frac{1}{3}\frac{\mathrm{d}^2\phi}{\mathrm{d}t^2} = \varepsilon^2 n_0^2 \frac{\partial}{\partial\phi}\left[\frac{1}{2\left|\sin\frac{\phi}{2}\right|} - \cos\phi\right] \tag{105}$$