Preprint typeset using LaTeX style emulateapj v. 03/07/07

Subject headings:

1. RESONANCE FOR A TEST PARTICLE

Consider a Kuiper belt object near an exterior j:j-1 MMR resonance with a circular Neptune. (One can also consider interior resonances by defining $\bar{j} = 1 - j$.) The test particle's energy per unit mass is

$$E = -\frac{GM_\odot}{2a} + \frac{Gm_N}{a} fe\cos(j\lambda - (j-1)\lambda_N - \varpi) \ , \quad (1)$$

where a, e, λ, ϖ are for the test particle, and quantities subscripted with N are for Neptune. f is a Laplace coefficient, which is a function of j and a/a_N. Hamilton's equations follow after replacing the particle's variables with the Poincaré canonical variables (per unit mass):

$$\Lambda = \sqrt{GM_\odot a} \quad (2)$$
$$\lambda \quad (3)$$
$$\Gamma = \sqrt{GM_\odot a}\left(1 - \sqrt{1-e^2}\right) \quad (4)$$
$$\gamma = -\varpi \ . \quad (5)$$

However, instead of using these variables, we shall rescale the momenta and Hamiltonian by the same constant factor $2/\sqrt{a_* GM_\odot}$, where a_* is at *nominal* resonance, defined via

$$\sqrt{\frac{GM_\odot}{a_*^3}} = n_N \frac{j-1}{j} \ . \quad (6)$$

and n_N is Neptune's constant mean motion. The equations of motion will still be Hamilton's equations. We denote rescaled quantities by bars, in which case the Hamiltonian is

$$\bar{H} = n_N \frac{j-1}{j}\left(-\frac{a_*}{a} + 2\mu_N \frac{a_*}{a} fe\cos(.)\right) \ . \quad (7)$$

We henceforth assume $a \approx a_*$ in the cosine coefficient, which is typically ok. (At least, it is usually nearly constant.) The canonical variables for this Hamiltonian are

$$\bar{\Lambda} = 2\sqrt{a/a_*} \quad (8)$$
$$\lambda \quad (9)$$
$$\bar{\Gamma} = 2\sqrt{a/a_*}\left(1 - \sqrt{1-e^2}\right) \quad (10)$$
$$\gamma = -\varpi \quad (11)$$

Next, we make a canonical transformation with the generating function

$$F = p_e(j\lambda - (j-1)\lambda_N + \gamma) + p_a\lambda \quad (12)$$

The new Hamiltonian is h, where

$$\frac{h}{n_N(j-1)/j} = -\frac{a_*}{a} - jp_e + 2\mu_N fe\cos\phi \ , \quad (13)$$

and the new canonical momenta and coordinates are

$$p_e = \bar{\Gamma} \quad (14)$$
$$\phi = j\lambda - (j-1)\lambda_N - \varpi \quad (15)$$
$$p_a = \bar{\Lambda} - j\bar{\Gamma} - 2 \quad (16)$$
$$\lambda \quad . \quad (17)$$

To employ Hamilton's equations, one must write a and e in terms of p_e and p_a. Note that p_a is a constant of motion (Brouwer's constant); we shifted it by -2 to make its value small near resonance. Thus far, our manipulations are exact, aside from the coefficient of the cosine term.

We note the following approximate relations:

$$p_e \approx e^2 \quad (18)$$
$$p_a \approx \frac{\Delta a}{a_*} - je^2 \ , \quad (19)$$

where $\Delta a = a - a_*$.

Inserting into the Hamiltonian

$$a = \frac{a_*}{4}(2 + p_a + jp_e)^2 \ , \quad (20)$$

expanding to second order in p_e and p_a, keeping only the leading term for the cosine coefficient, and dropping constants, we have

$$\frac{h}{n_N(j-1)/j} = -\frac{3j^2}{4}(p_e + p_a/j)^2 + 2\mu_N f\sqrt{p_e}\cos\phi \ . \quad (21)$$

(with momenta scaled by $\sqrt{GM_\odot}$):

$$\Lambda = \sqrt{a}\ ; \qquad \lambda \tag{22}$$
$$\Gamma \approx \sqrt{a}e^2/2\ ; \qquad \gamma = -\varpi \tag{23}$$

for which the (scaled) Hamiltonian is

$$H = \sqrt{GM_\odot}\left(-\frac{1}{2\Lambda^2} + \frac{\mu_N}{a} fe\cos(2\lambda - \lambda_N + \gamma)\right)\ , \tag{24}$$

where

$$\mu_N \equiv \frac{m_N}{M_\odot} \tag{25}$$

We shall later take the coefficient of the $e \times$ *cosine* term to be a constant, even though it is really a function of a. That approximation is typically okay, because the variations $\delta a \sim \delta e^2$, and that term is already O(e).

Now, we change variables (canonically) so the argument of the cosine is a new angle. To do that, we use the generating function

$$F = P_1(2\lambda - \lambda_N + \gamma) + P_2\lambda\ , \tag{26}$$

which yields the transformation laws to the new set $\{P_1, Q_1; P_2, Q_2\}$

$$Q_1 = 2\lambda - \lambda_N + \gamma\ ; \qquad Q_2 = \lambda \tag{27}$$
$$\Lambda = 2P_1 + P_2\ ; \qquad \Gamma = P_1\ . \tag{28}$$

Inverting the latter two yields

$$P_1 = \Gamma \approx \sqrt{a}e^2/2 \tag{29}$$
$$P_2 = \Lambda - 2\Gamma \approx \sqrt{a}(1 - e^2)\ . \tag{30}$$

Clearly, P_2 = const, because the Hamiltonian is only a function of Q_1. Therefore we define a_* via

$$\sqrt{a_*} \equiv P_2\ . \tag{31}$$

The transformed Hamiltonian is then

$$H(\Gamma, Q_1) = \sqrt{GM_\odot}\Big(-\frac{1}{2(\sqrt{a_*} + 2\Gamma)^2} \tag{32}$$
$$-\frac{n_N}{\sqrt{GM_\odot}}\Gamma + \frac{\mu_N}{a} fe\cos Q_1\Big) \tag{33}$$

Now, we rescale Γ and H,

$$p \equiv \frac{2}{\sqrt{a_*}}\Gamma \approx e^2 \tag{34}$$
$$\bar{H} \equiv \frac{2}{\sqrt{a_*}}H\ , \tag{35}$$

to arrive at the following Hamiltonian

$$\bar{H} = -n_*\frac{1}{(1+p)^2} - n_N p + n_* 2\mu_N f\sqrt{p}\cos Q_1\ , \tag{36}$$

where

$$n_* \equiv \sqrt{GM_\odot/a_*^3}\ , \tag{37}$$

and we have replaced the a in the cosine coefficient by a_*. Finally, we expand and drop the constant to yield

$$\bar{H} = (2n_* - n_N)p - 3n_* p^2 + n_* 2\mu_N f\sqrt{p}\cos Q_1 \tag{38}$$

2. J+1:J RESONANCE

Energy per unit mass:

$$E = -\frac{GM_\odot}{2a} + \frac{Gm_N}{a} fe\cos((j+1)\lambda - j\lambda_N - \varpi)\ , \tag{39}$$

Generating function:

$$F = P_1((j+1)\lambda - j\lambda_N + \gamma) + P_2\lambda\ , \tag{40}$$

which yields the transformation laws to the new set $\{P_1, Q_1; P_2, Q_2\}$

$$Q_1 = (j+1)\lambda - j\lambda_N + \gamma\ ; \qquad Q_2 = \lambda \tag{41}$$
$$\Lambda = (j+1)P_1 + P_2\ ; \qquad \Gamma = P_1\ . \tag{42}$$

Inverting the latter two yields

$$P_1 = \Gamma \approx \sqrt{a}e^2/2 \tag{43}$$
$$P_2 = \Lambda - (j+1)\Gamma \approx \sqrt{a}(1 - ((j+1)/2)e^2)\ . \tag{44}$$

Define

$$\sqrt{a_*} \equiv P_2\ . \tag{45}$$

The transformed Hamiltonian is then

$$H(\Gamma, Q_1) = \sqrt{GM_\odot}\Big(-\frac{1}{2(\sqrt{a_*} + (j+1)\Gamma)^2} \tag{46}$$
$$-\frac{n_N}{\sqrt{GM_\odot}} j\Gamma + \frac{\mu_N}{a} fe\cos Q_1\Big) \tag{47}$$

$$\bar{H} = -n_*\frac{1}{(1 + p(j+1)/2)^2} - n_N jp + n_* 2\mu_N f\sqrt{p}\cos Q_1$$
$$\approx ((j+1)n_* - jn_N)p - \frac{3}{4}(j+1)^2 n_* p^2 \tag{48}$$
$$+ n_* 2\mu_N f\sqrt{p}\cos Q_1 \tag{49}$$