1. Linear Secular Theory for Two Massive Planets

The secular interaction energy between two planets is

$$H = -G\frac{mm'}{a'}R \; , \tag{1}$$

where the outer planet's properties are primed, and the disturbing function is, to second order in e and $s \equiv \sin(i/2)$ (Murray & Dermott):

$$\begin{aligned} R &= f_2(e^2 + e'^2) + f_3(s^2 + s'^2) + f_{10}ee'\cos(\varpi - \varpi') & (2) \\ &+ f_{14}ss'\cos(\Omega - \Omega') & (3) \end{aligned}$$

where

$$\begin{aligned} f_2 &= \frac{1}{8}(2\alpha D + \alpha^2 D^2)b^0_{1/2} = \frac{1}{8}\alpha b^1_{3/2} \approx \frac{3}{8}\alpha^2 & (4) \\ f_3 &= -\frac{1}{2}\alpha b^1_{3/2} \approx -\frac{3}{2}\alpha^2 & (5) \\ f_{10} &= \frac{1}{4}(2 - 2\alpha D - \alpha^2 D^2)b^1_{1/2} = -\frac{1}{4}\alpha b^2_{3/2} \approx -\frac{15}{16}\alpha^3 & (6) \\ f_{14} &= \alpha b^1_{3/2} \approx 3\alpha^2 & (7) \\ b^j_s &= \frac{1}{\pi}\int_0^{2\pi} \frac{\cos j\psi}{(1 - 2\alpha\cos\psi + \alpha^2)^s} d\psi & (8) \end{aligned}$$

For the approx, we use small α, where

$$\begin{aligned} b^1_{3/2} &\approx 3\alpha + O(\alpha^3) & (9) \\ b^2_{3/2} &\approx \frac{15}{4}\alpha^2 + O(\alpha^4) & (10) \end{aligned}$$

The equations of motion are Hamilton's equations for the canonical variables

$$\begin{aligned} \Gamma &= m\sqrt{GM_\odot a}(1 - \sqrt{1 - e^2}) & (11) \\ \gamma &= -\varpi & (12) \\ \Delta &= m\sqrt{GM_\odot a(1 - e^2)}2s^2 & (13) \\ \delta &= -\Omega & (14) \end{aligned}$$

Or, defining the complex variables

$$\begin{aligned} z_\gamma &\equiv \sqrt{\Gamma}e^{-i\gamma} & (15) \\ z_\delta &\equiv \sqrt{\Delta}e^{-i\delta} \; , & (16) \end{aligned}$$

the equations of motion are

$$\frac{dz_\gamma}{dt} = -i\frac{\partial H}{\partial z_\gamma^*} \ , \tag{17}$$

and similarly for $z_\delta, z'_\gamma, z'_\delta$. To linear order,

$$z_\gamma = M_\odot^{1/2}(GM_\odot)^{1/4}\kappa e e^{i\varpi} \tag{18}$$

$$z_\delta = M_\odot^{1/2}(GM_\odot)^{1/4}\kappa i e^{i\Omega} \ , \tag{19}$$

where

$$\kappa \equiv \sqrt{\frac{\mu\sqrt{a}}{2}} \ , \tag{20}$$

where

$$\mu \equiv \frac{m}{M_\odot} \tag{21}$$

and hence

$$R \cdot M_\odot(GM_\odot)^{1/2} = f_2(\frac{|z_\gamma|^2}{\kappa^2} + \frac{|z'_\gamma|^2}{\kappa'^2}) + \frac{f_3}{4}(\frac{|z_\delta|^2}{\kappa^2} + \frac{|z'_\delta|^2}{\kappa'^2}) \tag{22}$$

$$+ \frac{f_{10}}{2}\frac{1}{\kappa\kappa'}(z_\gamma z'^*_\gamma + c.c.) \tag{23}$$

$$+ \frac{f_{14}}{8}\frac{1}{\kappa\kappa'}(z_\delta z'^*_\delta + c.c.) \tag{24}$$

Therefore the equation of motion is

$$\frac{d}{dt}\begin{pmatrix} z_\gamma \\ z'_\gamma \end{pmatrix} = i\sqrt{GM_\odot}\frac{\mu\mu'}{a'}\begin{pmatrix} f_2/\kappa^2 & f_{10}/(2\kappa\kappa') \\ f_{10}/(2\kappa\kappa') & f_2/\kappa'^2 \end{pmatrix}\begin{pmatrix} z_\gamma \\ z'_\gamma \end{pmatrix} \tag{25}$$