FINANCIAL MARKETS, CREATION AND LIQUIDATION OF Firms AND AGGREGATE DYNAMICS

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Abstract
This paper models the impact of interest rates in an economy with limited commitment. A firm's temptation to default endogenously induces credit constraints in their optimal long-term relationship with banks. The history of the relationship determines the credit limits generating a non-trivial life-cycle for firms. Collateral takes time to accumulate and raises the scale of operations and firm's survival probabilities.

Firms in many different stages of their life-cycle will coexist in the economy. In this environment, a positive interest rate shock decreases the survival probability of constrained firms while unconstrained firms respond primarily by adjusting their scale of operations. Increases in the interest rate affect the mass and cross-section of active firms, partly through a reduction of the number of new ones. Higher interest rates also tighten constraints of newly created firms, raising the number of periods needed to achieve maturity.

The model generates persistent and asymmetric aggregate output dynamics.

Keywords: Limited Commitment, Long Term Relationships, Firm Heterogeneity Aggregate Fluctuations.

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1 Introduction

This paper develops a model in which interest rate shocks lead to fluctuations in the creation and destruction of firms and in aggregate production. It examines economies in which limited commitment restricts the operations of firms by endogenously imposing limits on their credit. The potential default of entrepreneurs constrains the optimal dynamic contract between banks and firms. The optimal contract provides an explicit link between age, size and the credit available of firms as well as the effect shocks on entry, exit and growth. The response of the economy to interest rate fluctuations depends on the pre-existent cross section distribution of active firms. Shocks propagate affecting the cross section of active firms.

It is widely believed that credit market frictions amplify shocks and propagate them over time. Building on a model of firm dynamics, the paper makes the case that firm heterogeneity can naturally lead to persistent and asymmetric dynamics of aggregate output. The model emphasizes the role of credit market frictions in shaping up the creation and liquidation of firms as well as their growth. The non-trivial life-cycle of firms can generate a a rich cross-section distribution of firms so that, in equilibrium, asymmetries at the firm level can translate into asymmetries at the aggregate level. A large variety of studies employing different methodologies have supported the presence of asymmetries at the aggregate level.

Shocks and frictions in credit markets have received wide attention at all levels of aggregation. Work by T. Dunne, M. Roberts and L. Samuelson [24, 25], D. Evans [28, 29] and B. Hall [33] conclude that young, small firms grow faster but die more frequently than older, larger ones. In our model, firm dynamics plays an important role in the transmission of aggregate shocks. At a more aggregate level some authors argue that smaller firms (M. Gertler and S. Gilchrist [31]) or firms with less access to financial markets (A. Kashyap, J. Stein and O. Lamont, and D. Wilcox [43, 41]) are more sensitive to monetary shocks than their larger or less constrained counterparts. In our model we explore the transmission of interest rate shocks in an endogenous model of credit constraints. The structure of model can examine the behavior of gross flows across firms with different age and size as well as shut-downs and start-ups versus continuing units.

In the present model, new and old firms respond to interest rate shocks depending on their history of idiosyncratic shocks and of aggregate shocks. Over time firms accumulate collateral. When young, firms are likely to have little collateral and therefore to be constrained and small. It takes time for firms to accumulate enough collateral to operate at the unrestricted level. Because young firms face tougher credit limits their scale of operations and profits are lower than mature firms. Thus, the net present value of operating young firms relies heavily on future profits. Changes in interest rates have a stronger impact on the value of young firms than that of mature firms. Young firms respond to movements in interest rates primarily through their decision to continue producing. In contrast, older firms, which are more likely to be unconstrained, respond by adjusting their scale of operations. During recessions led by high interest rates, more young firms will be liquidated while old firms will reduce their scale of operation.

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1 Alternatively, limited commitment imposes solvency and borrowing limits in decentralized securities markets. We show the equivalence of these two statements.

2 Thus, the model is not subject to what Ricardo Caballero [8] has termed the fallacy of composition, i.e. that asymmetries at the micro level do not necessarily lead to asymmetries at the macro level.

3 See for example Durlauf and McCurdy [26], Filardo [27], Hamilton [34], Hansen and Prescott [35] and Sichel [54].
The model predicts persistent and asymmetric responses of aggregate output. When interest rates raise aggregate output declines due to lower creation of new firms, a higher rate of liquidation of firms and a reduction in the operations of existing firms. Furthermore, the model predicts that the destruction of firms is concentrated in the first periods of increasing interest rates, as highlighted by R. Hall[33] on his modeling of recessions. This result is driven by the fact only the most productive firms survive and are therefore more likely to remain active—even if interest rates remain high. In addition, credit constraints do not limit the down-sizing of firms. Then contractions are sharp and short lived.

Expansions (and recoveries) are slower and longer-lived. Periods of lower cost of capital generate enhanced creation of firms and higher optimal scale of operations for each one. As collateral takes time to accumulate, the expansion on aggregate output can only take place as the vintages of young firms accumulate collateral. This effect is enhanced by limitations on the aggregate rate of creation of new firms, which by itself implies that time is needed to build-up the mass of active firms.

It is important to emphasize that in our model, credit constraints arise endogenously from the optimal dynamic contract restricted by limited commitment. We build on the the recent work by R. Albuquerque and H. Hoppenhayn [1]—who in turn extend the previous work by O.Hart and J.Moore [36]—to obtain the non-trivial firm dynamics and heterogeneous firms that respond differently to aggregate shocks. We show that regardless of the non-monotonicity of the bank’s payoff function, in equilibrium the allocations are renegotiation proof. The exit option introduces a non-convexity in the optimal contract design. However, we prove that lotteries are not part of the equilibrium path, and therefore the optimal contract is attained with deterministic contracts. This is to say all randomizations are relevant only outside the path of play. Additionally, we provide a decentralization scheme along the lines of F.Alvarez and U.Jermann[2]. We show how to map the economy into a Radner, sequence of markets equilibrium. Trading in one period securities can replicate the allocations from the optimal infinite horizon contract provided that the right profile of solvency and borrowing constraints are in place. This is important. It formalizes the link of collateral in the decentralized environment with the expected utility of the entrepreneurs in the long term relationship.

As an extension to Rui Albuquerque and Hugo Hoppenhayn [1], our model implies life-cycle firm dynamics that is qualitatively consistent with empirical observations. Yet, for our purposes is unfortunate that most evidence on firm dynamics is based on low frequency data (collected every five years), impeding connections with business cycles. Indeed, the most cited work on asymmetric responses of small and large firms does not distinguish extensive margins (entry, exit) responses from intensive margin responses. For instance, Gertler and Gilchrist [31] study the behavior of large aggregate groups of firms while Kashyap, Lamont and Stein [43] eliminate from the sample those firms liquidated during their sample period. The empirical literature on gross flows (see Davis, Haltiwanger and Schuh [21] and the references therein) highlights the large and volatile gross reallocation flows. The data, unfortunately, is only in terms of jobs and not plants or firms. Interestingly, as J. Campbell and J. Fisher [13] have shown, job flows are larger and more volatile for young firms than for older firms. Yet, a closer look at the data reveals that the asymmetry is exit and entry and not for continuing plants.

Our work is related to recent literature on credit markets and aggregate fluctuations. B. Bernanke and M. Gertler [5] and J. Fisher [30] study how incentive constraints propagate shocks by rationing the credit of a subset of firms. By design, their models cannot address the responses on the margins of creation and destruction of firms and cannot accommodate long firm dynamics. T. Cooley and
V. Quadrini [16] construct a model suitable for entry and exit as well as firm growth, but they take the form of contracts exogenously. The posterior work by these two authors with R. Marimon, [17] study contractual environments very similar to ours, but they rule out endogenous liquidations. The possibility of liquidating the firm not only changes the aggregate response of the economy but also has a direct impact on individual contracts. Indeed, the option of liquidating the firm in some states of the world implies that the firm can grow faster in the states of the world in which it survives. Finally W. den Haan, G. Ramey and J. Watson [22] study models with long term relationships and liquidations. However, their model does not allow for firm growth and emphasize instead the effect of idiosyncratic liquidity shortages of lenders. We study models in which lenders have unrestricted access to financial markets and non-trivial and focus on the firm dynamics generated from entrepreneurs incentives.

The rest of the paper proceeds as follows. In the next section, we lay down the model. The third section provides a detailed characterization of the optimal, infinite horizon contract between a bank and entrepreneur. The implied firm dynamics, including entry and exit decisions are discussed. In the fourth section examines the aggregate dynamics implied by the model and includes a couple of numerical illustrations. The fifth section explains the solvency and borrowing constraints under which a sequence of one-period security markets replicate the allocations of the infinite horizon relationships. The sixth section discusses three issues. First, it discusses three notions of credit constraints. Second, we argue that in our model, contrary to other models with limited enforcement in endowment economies, the least productive units are the most likely to be constrained. Third we examine briefly the possibility of asymmetric effects of interest rate on different firms. The last section concludes. The appendix contains all the proofs.

2 The Environment

Demographics and Preferences:

Time is discrete and the horizon is infinite; periods are indexed by \( t = \ldots, -1, 0, 1, 2, \ldots \). Each period brings along a vintage of many new potential entrepreneurs. All vintages have mass \( \delta \). I shall refer to potential entrepreneurs as agents. In the beginning of every period a fraction \( \delta \in (0, 1) \) of the agents dies. The death probability of each agent is \( \delta \). The total population in the economy is always equal to one. Agents are risk neutral and discount the future geometrically by a factor \( \beta \in (0, 1) \). Thus, their evaluation of own consumption processes \( \{c_t\} \) as of any time \( t_o \) is given by

\[
E \left\{ \sum_{t \geq t_o} (\beta(1-\delta))^t c_t \mid F_{t_o} \right\}
\]

where the probability of death renders an effective discount factor of \( \beta(1-\delta) \), and \( F_{t_o} \) denotes all the information available as of \( t_o \).

The OLG structure of the model effectively operates as an aggregate friction in the creation of firms and jobs as in Caballero Hammour [10] or in the matching function literature, e.g. W. den Haan, G. Ramey and J. Watson [22].

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Technologies:

Entrepreneurs are born with zero endowments. Upon birth, each entrepreneur has access to two mutually exclusive technologies, stochastically identical across agents. The first is a productive technology. Its activation requires a set-up investment cost $K_\alpha > 0$. In each period of activity, it also requires resources (working capital) to produce output. In actual economies, working capital takes the form of materials, inventories of finished goods, hours of labor, etc. Here I bundle all components into a single scalar variable $k$. With an amount of $k$, the technology current production is $zf(k)$ where $\{z^i\}$ is a stochastic process of productivities idiosyncratic to firm $i$, and $f(\cdot)$ is a non-negative, strictly increasing, concave function. I will assume:

A. 1 (F). $f(k) = k^\alpha$, $\alpha \in (0, 1)$,

A. 2 (P). $\{z^i\}$ is a stationary, ergodic, process, that is identical and independently distributed across agents of all vintages. It has finite support $Z = \{z_1, z_2, ..., z_n\}, n < \infty$. The transition are given by $P_2(\cdot, z)$, and the unique invariant c.d.f. by $F_2(\cdot)$. The processes $\{z^i\}$ have positive persistence over time, i.e. if $z_0 < z_1 \in Z$, then, $P_2(\cdot, z_1) \geq f P_2(\cdot, z_0)$, where $\geq f$ denotes first order stochastic dominance.

For simplicity, I assume the option of activating the productive technology is available only in the first period of life for each agent, and it can be operated as long as the agent has not abandoned it or the firm has been liquidated.

The second option is a "backyard" technology. It is always available to the agent. We also use the term underground for this option, which requires no set up costs and produces a constant flow $e > 0$ each period. Once the agent has opted for the underground his utility $U$ is simply

$$U = \frac{e}{1 - \beta(1 - \delta)}$$

In what follows, agents operating the productive technology are called active entrepreneurs.

The Banking Sector

In most of the paper, we will assume a market structure with many (an infinite number of) infinitely lived banks that compete with each other to sign contracts with the entrepreneurs. Banks can commit to honor long term contracts, and competition occurs only at the time of signing up entrepreneurs. Each bank has a sufficiently large array of customers to fully diversify the idiosyncratic shocks. A bank’s sole objective is to maximize the expected present value of each and every contract. For the purposes of this work, differences across banks or frictions in their liabilities and asset portfolios are of no relevance and shall be omitted altogether.

In the beginning of every period all the banks compete to sign up members of the new born cohort. Each of the potential entrepreneurs draws his initial productivity $z_0$ from the distribution $F_2(\cdot)$. The

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4The results of the paper are robust to assuming that the entrepreneurs are born with a small endowment. However, if the initial endowment is large enough, the incentive problems would disappear, as a bond-posting scheme will be available (on the long term relationships) or they do not need to borrow at all (in the decentralized setting).

5The use of the term underground is consistent with our assumption that once an agent is using that technology he cannot be enforced to pay any liability.
realization $z_o$ is also observed by banks before bidding for each entrepreneur. This is important because $z_o$ will determine whether the agent becomes an entrepreneur or not and the contract received.

The Enforceability Problem and the Liquidation Option

Given the previous assumptions, an active entrepreneur always has the option of switching to the underground. The key assumption that separates this economy from a standard frictionless environment is that there is no enforcement device that directly rules out this option from entrepreneurs. On the contrary, I will assume that banks can fully commit to honor their contractual obligations. Therefore, the analysis centers on the incentive problems of entrepreneurs.

There are two possible ways in which entrepreneurs can walk away from the productive technology. First, they can opt out in the beginning of the period. In this alternative they can access the underground technology immediately, attaining an utility of $U$. On the other hand, they can leave in the middle of the period. In this case he will miss the underground technology for that period, but he can seize all the working capital $k$ under his control. We also assume that the defaulting entrepreneurs can steal a fraction $\theta \in [0,1]$ of the liquidation value of the plant. Therefore, the default option depends on $k$:

$$V_d(k) \equiv k + \theta L + \beta (1 - \delta) U$$

Given his inability to commit, these outside options to the entrepreneur will effectively restrict the contracts between banks and entrepreneurs. The first option imposes a minimum on the utility of active entrepreneurs.

Finally, by scrapping the plant they can recover an amount $L$ of the set up cost, where, $L \leq K_o$. Besides the present value of $\{k^i\}$, $L + U$ determines the opportunity cost of maintaining a productive technology in operation. However, as opposed to $L$, $U$ will also have a separate effect as it constrains the set of incentive compatible plans between entrepreneurs and banks. 6

Bond Prices

The opportunity cost of the resources used by banks is determined by the interest rate they face. I assume that the price of one-period real bond $q_t$, for all $t$, follows an exogenous Markov process $\{q\}$. The following is assumed:

A. 3 (Q). $\{q\}$ is a Markov stationary, ergodic process, with finite support $Q = \{q_1, q_2, ..., q_m\} \subset (0, 1), m < \infty$ The transitions are denoted $P_q(\cdot, q)$ and the unique invariant c.d.f. by $F_q(\cdot)$. Moreover, $\{q\}$ has positive persistence in the same sense used in $P$.

In this environment, individual entrepreneurs are affected by a common aggregate shock $q$. Two alternative interpretations can be given. The first is that this is a small economy open to international capital markets. The second is a back-solving interpretation as in Prescott et. al. [23], which views

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6 Alternatively, one could assume that in addition to the working capital, the entrepreneur can also seize a fraction of the value of the plant, $\theta L$. Here, $V_d(k) = k + \theta L + \beta (1 - \delta) U$. This extension does not complicate the analysis at all, but the credit constraints will be tighter in this world. The same remark applies if the agent can access the backyard technology in the same period of default, in which case $V_d(k) = U + k$. The results of these alternative cases can be replicated in the current set-up by manipulating the values of $U$ and $L$. 

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this as an economy where fiscal and/or monetary policy is such that in equilibrium the interest rate follows a Markov process. In Monge [47] I provide the details of one such interpretation.\footnote{Simply put, such modeling strategy lets the "quantities", i.e. government expenses or money supply to follow as complicated stochastic processes as necessary so as the equilibrium interest rates behave in a rather simple way.}

We make the following assumption:

\textbf{A. 4 (I). Entrepreneurs are more impatient than banks: }$\beta \leq \min\{Q\}$

Taking the interest rates an exogenous stochastic process has the obvious disadvantage of omitting the feedback effects of its shocks. The strategy has the value of isolating the effects of shocks in $q$ from the response of government policy on the macroeconomic state of the economy.\footnote{An interesting extension would be to explore the role of intermediate good prices such as in Bergin-Bernhardt and Caballero-Hammour [3, 11]. We believe, however, that the main aggregate qualitative implications of the model will not be altered. Indeed, we suspect that the general equilibrium will actually deepen our finding of asymmetry in the effects across firms.}

\section{Economies with Long Term Contracts}

In this section we assume that the competition of banks is only in terms of signing up the new potential entrepreneurs. To save on notation, the exposition specializes to relationships initiated at period $t = 0$. A dynamic contract specifies, for every period the amount of capital advanced from the bank, $k_t$, and the repayment, $r_t$ from the firm as functions of the history of the relationship. In each period, the idiosyncratic productivity $z_t$ and the economy-wide discount factor $q_t$ are observed prior to setting $(k_t, r_t)$. The allocations of each period will be functions of $h^t \equiv \{z_s, q_s : 0 \leq s \leq t\}$, the history of shocks in the relation. Denote $H^t$ the set of all feasible histories $h^t$, $\forall t \geq 0$, and $H^t$ the product space form by $H$, the power set of $Z \times Q$.

A dynamic contract is a sequence of $H^t$-measurable functions $\sigma \equiv \{\sigma^t : t \geq 0\}$, $\sigma^t : H^t \rightarrow \mathbb{R}_{+} \times \mathbb{R}_{+}$, of the form $\sigma^1_t(h^t) = k_t$ and $\sigma^2_t(h^t) = r_t$, indicating how much working capital is advanced and the amount re-paid by the firm. Contracts must satisfy several constraints. First, the non-negativity of entrepreneur’s consumption imply a form of limited liability constraint:

$$\sigma^2_t(h^t) \leq z_t f(\sigma^1_t(h^t)), \quad \forall h^t \in H^t, \quad \forall t \in \mathbb{N}$$ (2)

The possibility of default also constrains the admissible set of $\sigma$. The transitions $P_q, P_z$ and the functions $\sigma$ will define the payoffs for each party of the relationship. For each $h^t \in H^t, t \in \mathbb{N}$, $P_q, P_z$ define the sequence of probability kernels, $\mu^t(\cdot, h^t)$ on the measurable spaces $(H^s, H^t)$ for $s \geq t$. These kernels define the evolution of the history of each relationship. Indeed, at history $h^t$ the continuation value for the entrepreneur implied by $\sigma$ is given by

$$V^t_{\sigma}(h^t) \equiv \sum_{s \geq t} (\beta(1 - \delta))^{s-t} \int_{H^s} [z_s f(\sigma_1(h^s)) - \sigma_2(h^s)] \mu^t(\sigma^s, h^t)$$ (3)

Thus, the inability of entrepreneurs to commit imposes the constraints

$$V^t_{\sigma}(h^t) \geq U \quad \text{and} \quad V^t_{\sigma}(h^t) \geq V_d(\sigma_1(h^s)) \quad \forall h^t \in H^t, \quad \forall t \in \mathbb{N}$$ (4)
The timing between the investment of working capital, collection of repayments, and the bond markets is needed to define the net payoff for banks. The following picture displays the timing used in this model.

\[ R_n(h^t) \equiv \sum_{s \geq t} (1 - \delta)^{s-t} \int_{H^s} \left( \prod_{j=t}^{s-1} q_j \right) \left[ g_s \sigma_2(h^s) - \sigma_1(h^s) \right] \mu^s(dh^s, h^t), \forall t \]  

(5)

This equation assumes that if the entrepreneur dies, the bank cannot seize, \( L \), the liquidation value of the plant. Banks can always liquidate the firm. Commitment does not force banks to maintain the plants operating in all the eventualities, but instead to honor the utility entitlement to the entrepreneur dictated by the contract in every node of the relationship. Thus, in the case of liquidating the firm, the value of the bank is

\[ L - [V_n'(h^t) - U] \]  

(6)

Notice that the bank needs only to compensate the difference between \( U \) and \( V_n'(h^t) \), which is simply \( V_n'(h^t) - U \) in units of the good and because of the linear utility assumption.

Clearly, an active firm will continue operating if and only if

\[ R_n(h^t) \geq L - [V_n'(h^t) - U] \]  

(7)
Otherwise the firm is scrapped. \( \Sigma \) will denote the set of all incentive compatible allocations with elements \( \sigma \). Notice that, despite the potential non-convexities originated by the scrapping option, we have ignored randomizations in the description of the contract. We will explicitly consider the randomizations in the recursive formulation, and verify that all non-trivial randomizations are relevant only outside the equilibrium allocation. Thus, the description here is without loss of generality.

**Initialization of Contractual Relationships**

An optimizing bank will design \( \sigma \) so as to maximize his profits, but by competition, at the time of activation, agents would not sign unless they receive the best, feasible, incentive compatible contract. Thus, in equilibrium, entrepreneurs will be entitled the best initial utility \( V^0_\sigma(h^0) \). The initial level of utility depends on the idiosyncratic characteristics \( z_0 \) of each entrepreneurs and the economy-wide discount factor \( q \). For each initial history \( h^0 = (z, q) \in Z \times Q \) let

\[
\Omega(h^0) \equiv \{ y \geq U : \sup_{\sigma \in \Sigma} \{ R^0_\sigma(h^0) - K_0 - y \} \geq 0 \}
\]  

(8)

\( \Omega(h^0) \) is the set of promised utilities to the entrepreneurs that permit non-negative payoffs to banks. This set can be empty, in which case the agent does not become active. Otherwise he is activated with an utility entitlement of

\[
V^0_\sigma(h^0) = \sup \{ \Omega(h^0) \}
\]  

(9)

As show below, the sup operator insures that the contract is established is immune to future renegotiations.

**3.1 A Recursive Formulation**

The Markov nature of the shocks \( \{z\} \) and \( \{q\} \) allows the use of recursive methods to solve for the contracting problem. Following the initial insight of Spear and Srinivastava, we can specify the problem in terms of constructing rules for updating the continuation values for the entrepreneur, \( V \). The vector \( (V, z, q) \) is the state of individual relationships, in the sense that it summarizes the history of it and contains all the relevant information for the future evolution of the relationship. The state has two components that evolve exogenously but the stochastic process followed by \( V \) is derived endogenously by the contract design. Given \( (V, z, q) \), the bank must decide the amount of working capital \( k \), the amount repaid by the firm \( r \), and the continuation values for each possible realization in the next period. This is to say, given current state \( (V, z, q) \) the contract specifies for the next period the values \( G_{z', q'}(V, z, q) \) for the continuation utility of the entrepreneur for each and every \( (z', q') \).

**3.1.1 Ongoing Relations**

Let \( C(V, z, q) \) be the cost (in expected present value) of providing an active entrepreneur with a utility level \( V \) when the market discount factor is \( q \) and the productivity of his plant is \( z \). \( C \) is negative the bank obtains positive payoff from the relationship. \( C \) can be positive which implies that the bank has to put net positive resources in the relationship to deliver the value \( V \). The objective of the bank is to minimize this function. First, in case of liquidating the plant the net cost is \( V - U - L \). If the
plant continues, the period cost for the bank is \(-qr + k\) and commits to face the cost for the next period. Hereafter we use the shorthand

\[
E[V'] = \sum_{z', q'} G_{z', q'}(V, z, q) P_z(z', z) P_q(q', q)
\]

(10)

for expected future utility entitlements, and

\[
P(z', q'|z, q) \equiv P_z(z'|z) P_q(q'|q)
\]

(11)

for the conditional probabilities.

For all \((V, z, q) \in [U, \infty) \times Z \times Q\), \(C\) consider the following Bellman Functional Equation:

\[
C(V, z, q) = \min \left\{ V - U - L, \min_{G_{z', q'}, k, r} \left\{ k - qr + q \sum_{z', q'} C(G_{z', q'}, z', q') P(z', q'|z, q) \right\} \right\}
\]

subject to the constraints

\[
\begin{align*}
U & \leq V_{z', q'} & \text{(Participation)} \\
k + \beta(1 - \delta)U & \leq V & \text{(No Default)} \\
z f(k) - r + \beta(1 - \delta)E[V'] & \geq V & \text{(Promise Keeping)} \\
r & \leq z f(k) & \text{(Limited Liability)}
\end{align*}
\]

It is convenient to eliminate from the problem all the intra-temporal decisions. Let \(\pi(z, q), k^u(z, q)\) denote respectively the maximum profits from the technology and the unrestricted optimal use of working capital when enforcement problems are not binding, i.e.

\[
\pi(z, q) \equiv \max_{y \geq 0} \{ qzf(y) - y \} \quad k^u(z, q) \equiv \arg \max_{y \geq 0} \{ qzf(y) - y \}
\]

(12)

Under \(F\) \(\pi\) and \(k^u\) are \(k^u(z, q) = (zq\alpha A)^{1/(1-\alpha)}\) and \(\pi(z, q) = \Theta(zq)^{1/(1-\alpha)}\), respectively, where \(\Theta\) is a positive constant that depends on \((A, \alpha)\).

We are interested in economies where the no default constraint might be binding. In those cases, the firm will use less working capital than the optimal level. We will say that the firm is credit constrained in the intensive margin. More formally, for any \(V \in [U, +\infty)\) the maximum level of capital that is sustainable with no default is

\[
k^F(V) \equiv V - \beta(1 - \delta)U
\]

(13)

It can never be optimal to use \(k\) above the unrestricted optimum; for any \((V, z, q)\) the amount effectively used by continuing firms equals

\[
k(V, z, q) = \min \left\{ k^u(z, q), k^F(V) \right\}
\]

(14)

The enforcement-constrained surplus \(S\) is
\[
S(V, z, q) = \begin{cases} 
\pi(z, q) & \text{if } V \geq V^u(z, q) \\
qz f(k(V)) - k(V) & \text{otherwise}
\end{cases}
\]  
(15)

Several properties of \( S \) will be useful in characterizing the optimal allocation \( \sigma \):

**Proposition 1.** Under the assumption (\( F \)), the function \( S \) is strictly increasing, and supermodular in \((q, z)\). For \( V \leq V_q(k^*(z, q)) \), \( S(V, z, q) \) is strictly increasing, strictly concave, and strictly supermodular in \((V, z, q)\).

Moreover, \( S(V, z, q) \) is globally concave and continuously differentiable in \( V \).

Using \( S \) we can eliminate \( k, r, \) and focus on the choice of the stochastic difference equation \( V_{z'q'} = G_{z'q'}(V, z, q) \) followed by the endogenous state and given by the optimal policy function in the recursive problem. First, the Promise Keeping constraint with equality implies that \( r = z f(k) + \beta(1 - \delta)E[V'] - V \). We verify below that one can take the promise keeping constraint to hold with equality because in equilibrium, the relevant allocations are re-negotiation proof (\( C \) is strictly increasing in \( V \)). Plugging this expression into the Non-Negativity of consumption, it becomes

\[
\beta(1 - \delta) \sum_{z', q'} V_{z'q'}(V, z, q) P(z', q'|z, q) \leq V
\]

The limited liability of the firm impose a ceiling on how fast \( \{V_t\} \) can grow. For each \( V \in [U, +\infty) \), define the feasible set of continuation values \( \Gamma(V, z, q) \) by

\[
\Gamma(V, z, q) = \left\{ y : Z \times Q \rightarrow [U, +\infty) \text{ s.t. } \sum_{z', q'} y(z', q') P(z', q'|z, q) \leq \frac{V}{\beta(1 - \delta)} \right\}
\]  
(16)

a convex and compact set. The net payoff for the bank becomes

\[
k - qr = k - q[z f(k) + \beta(1 - \delta)E[V'] - V]
\]  
(17)

which takes the form of \(-S(z, q, V') + q[V - \beta(1 - \delta)E[V']]\) once the optimal incentive compatible choice of \( k \) is plugged in. These substitutions transform the Bellman Equation to,

\[
C(V, z, q) = \min \left\{ V - U - L, \quad Cc(V, z, q) \right\}
\]  
(18)

\(^9\)In the case twice continuously differentiability, a function \( f(x, y) \) is supermodular (submodular) in \((x, y)\) if \( \frac{\partial^2 f}{\partial x \partial y} \geq (\leq) 0 \). The notions of submodularity and supermodularity are not restricted to differentiable functions a fact that makes them particularly useful in the context of dynamic where the differentiability results are rather limited. (Super)submodularity notions can be used for conditions of complementarity. Indeed, if \( f \) is submodular in \((x, y)\), then an increment in \( y \) reduces the marginal cost of \( x \). See the appendix for a brief discussion of supermodular and submodular functions and Topkis [53] for a complete treatment.
where

\[
Cc(V, z, q) = \min_{y \in \Gamma(V, z, q)} \left\{ -S(z, q, V) + q[V - \beta(1 - \delta)E[y]] + q(1 - \delta) \sum_{y', q'} C(y, z, q, z', q')P(z', q'|z, q) \right\}
\]  

(19)

Finally, note that the option of liquidation introduces a non-convexity in the problem for ongoing relationship. Such non-convexity indicates gains from trade between the bank and the entrepreneur that are not being exploited. Randomizing the exit decisions eliminates those inefficiencies and the problem becomes convex. In turn, convexity greatly pays off in terms of sharp characteristics of the allocations and firm dynamics.

A risk neutral entrepreneur with utility entitlement V accepts any lottery that offers V₀ with probability λ and V₁ with probability 1 − λ as long as λV₀ + (1 − λ)V₁ ≥ V. Given V the admissible set of those lotteries is

\[
\Gamma_E(V) = \left\{ \lambda \in [0, 1], V₁ \geq V₀ \geq U : \lambda V₀ + (1 - \lambda)V₁ \geq V \right\}
\]  

(20)

where, the normalization V₀ ≥ V₁, is obviously with no loss of generality. We can take that the entrepreneur is liquidated if he loses the lottery, and this is without loss of generality because banks and entrepreneurs will only use optimal randomizations. Therefore when the entrepreneur loses he is liquidated at utility value V₀ and if he wins he remains active with an entitlement V₁ ≥ V.

The Bellman Equation becomes that describe the optimal decisions of active firms is:

\[
C(V, z, q) = \min_{(\lambda, V₀, V₁) \in \Gamma_E(V)} \left\{ \lambda(V₀ - U - L) + (1 - \lambda)Cc(V₁, z, q) \right\}
\]  

(21)

where Cc, the cost of continuing the relationship is as given above by equation 18.

Given the value function C, the initialization of relationships is also simplified. Let \( \Omega(z, q) = \{ y \geq U : C(y, z, q) + K₀ \leq 0 \} \). If \( \Omega(z, q) \) is not empty the entrepreneur will become active and with an initial entitlement of \( V₀(z, q) = \sup \{ \Omega(z, q) \} \). Otherwise, the agent will never be an entrepreneur. The \( \sup \) operator is needed because the function C may not be monotone. Taking the \( \sup \) insures the renegotiation proofness of the allocations (see below).

### 3.2 Optimal Contracts and Firm Dynamics

This subsection characterizes the value function C and the optimal policy correspondence G. The detail of the proofs and the notation required for them are relegated to the appendix. First, we can verify that the problem is well defined as there is a unique C that solves the functional Bellman Equation.

**Proposition 2.** Let \( \Gamma_E, \Gamma, S \), be as defined above. If conditions \( F, P, Q \) hold, the B.E. has a unique solution \( C : R_U \times Z \times Q \rightarrow R \). The function \( C \) is globally decreasing in \( z \) and it is strictly decreasing in the regions where the probability of liquidation is less than one. Moreover, \( C \) is globally convex in its first argument.
Proof. See appendix. □

That $C$ is decreasing in $z$ is very intuitive: the cost of providing any level of utility to the entrepreneur is a decreasing function of the productivity the technology under his control, as more resources can be expected from it in the current and subsequent periods. If the optimal decision were to liquidate, the cost is invariant w.r.t. $z$. Exit randomizations suffice to make a $C$ convex function in its first argument. Because $S$ is concave, $Cc$ is convex whenever the function $C$ under the expectation is convex; taking the optimal randomization, the LHS of the equation is convex whenever the function $Cc$ in the RHS is convex.

Convexity is highly desirable analytically. First of all, it implies that $C$ is everywhere subdifferentiable, and almost everywhere differentiable in the first argument. We use the notation $\partial C$ denote the subdifferential of $C$, where $\partial C(V, z, q)$ indicates the set of subgradients of $C$ w.r.t. the first argument at the point $(V, z, q)$. $C$ is differentiable w.r.t. $V$ at $(V, z, q)$ iff $\partial C(V, z, q)$ is a singleton.

As we will see below, for a relevant region, the optimal policy will be at the corners. Then the usual manipulations a la Benveniste-Scheinkman, for the differentiability of $C$ are of limited use. Yet, characteristics of $\partial C$ will help characterizing the optimal policies. Indeed, we can easily verify that:

**Proposition 3.** Given the conditions in the previous proposition, the subdifferentials $\partial C, \partial Cc$ are such that $\partial C(V, z, q) \leq 1$ and $\partial Cc(V, z, q) \leq q$ for all $(V, z, q) \in R_Y \times Z \times Q$. Moreover, if for given $(z, q)$, $\exists V^* \in R_Y$ with $C(V^*, z, q) < -L$, $\partial C(V, z, q)$ is also bounded above by $q$.

Proof. See appendix. □

Therefore the joint surplus $V - C(V, z, q)$ is increasing in $V$, and strictly increasing in neighborhoods where $Cc = C$. This is not surprising as a higher $V$ helps coping with the default problems of entrepreneurs, the only incentive issue in this economy. However, the fixed point $C$ is not necessarily increasing in $V$. This is troublesome because if $C$ is decreasing in $V$, it is not valid to take the promise keeping constraint with equality. Indeed, both parties would be willing renegotiate up the value of $V$ and be better off. We can rule out this possibility, but only after we characterize the initialization of the relationship. This is, while the value function $C$ can have segments where it is decreasing $V$, in equilibrium, those segments are never reached.

The submodularity of $S$ in $(V, z)$ indicates a form of complementarity of the productivity of the technology with the value for the entrepreneur. The larger is the value of the relationship for the entrepreneur, the lesser are the temptations for him to default. Then, higher productivities could be met with larger employment of working capital. The supermodularity of $S$ suffices for the submodularity of $C$ when $\{z\}$ has positive persistence, which holds by assumption $\mathbf{P}$.

---

10To see this, observe that the period return $-S(V, z, q) + q[V - \sum_{V', q'} G_{V', q'}(V, z, q)|P(V', q'|z, q)]$ for the bank it is not necessarily increasing in $V$: On the one hand, for any given policy $V = V'$, the higher the resources consumed by the entrepreneur the higher is the entitlement $V'$; on the other hand, however, lower values of $V$ will constraint more the surplus $S$ from the technology. If the second dominates the period return of the bank can be increasing in $V$. Moreover, even if restriction on $U, f, Z, Q$ can assure that $-S(V, z, q) + qV$ be increasing in $V$, the fixed point $C$ may have decreasing regions w.r.t. $V$ because the feasible set is increasing in $V$ in the set inclusion sense (i.e. $V_0 < V_1 \Rightarrow \Gamma(V_0, z, q) \subseteq \Gamma(V_1, z, q)$).

11For the case of $Q = \beta$ Albuquerque and Hopenhayn [1] prove that the joint surplus $V - C(V, z)$ is increasing in $V$. This does not eliminate the renegotiation problem as it does not imply that the payoff for the bank—the part with full commitment and designing the contract—is decreasing in $V$. 

13
Therefore,

**Proposition 4.** Assume that the conditions $F,Q,P$ hold. Then, the fixed point $C$ is a submodular function in $(V,z)$.

**Proof.** See the appendix. □

The convexity and submodularity of the value function can be used to characterized the optimal allocation $\sigma$ and the implied firm dynamics. First we analyze the continuation decisions (the evolution of $V$ given that the firm survives that period). Then we characterize the entry and exit decisions.

We also verify below that the non-trivial randomization are outside the equilibrium allocation and that the equilibrium allocations are re-negotiation proof.

**Continuation policies**

Let $G_{z',q'}(V,z,q)$ denote the entrepreneur’s utility entitlement profile for the next period, i.e. for each realization $z',q'$ in that period given a current state $(V,z,q)$. The first order conditions – sufficient and necessary for a convex problem– for the optimal $G_{z',q'}(V,z,q)$ are

\[
\beta(1 - \mu_2(V,z,q)) + \mu_1(z',q';V,z,q) \in \partial C(G_{z',q'}(V,z,q),z',q'), \quad \text{for all } z',q'
\]

\[
\mu_1(z',q';V,z,q) \geq 0; \quad G_{z',q'}(V,z,q) \geq U; \quad \text{(and at least one with =)} \quad \text{for all } z',q'
\]

\[
\mu_2(V,z,q) \geq 0, \quad \beta(1 - \delta) \sum_{z',q'} G_{z',q'}(V,z,q) P(z',q'|z,q) \leq V, \quad \text{(and at least one with =)}
\]

where $\mu_1(z',q';V,z,q)$ and $\mu_2(V,z,q)$ are $(\#Z \times \#Q) + 1$ (scaled) Kuhn-Tucker multipliers. The first set of multipliers makes sure that the bank does not assign a value below the underground’s value for the entrepreneur while the last multiplier ensures that the his consumption is non negative. Manipulation of these FOCs provide restrictions on the valid $G_{z',q'}(V,z,q)$. First we can verify that $G_{z',q'}(V,z,q)$ is non-decreasing in $V$:

**Proposition 5.** Let $\sqsubseteq$ denote the higher set partial ordering. The policies $G_{z',q'}(V,z,q)$ are non decreasing in $V$, in the sense that for any $(z,q)$ and $V_0 \leq V_1$, $G_{z',q'}(V_0,z,q) \sqsubseteq G_{z',q'}(V_1,z,q)$ for all $(z',q')$

**Proof.** See appendix. □

This result is hardly surprising as $V$ imposes only an upper bound on $E[V']$. Therefore, if it does not bind for a $V_0$, it does not bind for $V \geq V_0$. In periods when $V$ binds, the entrepreneur has zero consumption and the expected utility entitlement for the next period is as high as possible. This is very intuitive, given the one-sided nature of incentive problem studied here. Initially the value of the relationship for the entrepreneur must grow as fast as possible because it enhances the future profits extractable from the technology. The result obtains regardless of how $Q$ compares with $\beta$, in sharp

\footnote{A $\sqsubseteq B$ if for $x \in A$, $y \in B$ $x \lor y \in B$ and $x \land y \in A$.}
contrast with the results obtained by O.Hart and J.Moore [36] in models with two-sided imperfect commitment. 13

The optimal contract not only must specify the evolution of the entrepreneur’s utility entitlement over time but also across realizations of the idiosyncratic and aggregate shocks. The submodularity
C provides another clear restriction of the optimal continuation profile:

Proposition 6. Let \((V, z, q) \in R_U \times Z \times Q\). \(C\) is submodular, then \(G_{z', q'}(V, z, q)\) is increasing in \(z'\). Moreover, if \(G_{z', q'}(V, z, q) > \U\), then it is strictly increasing.

Proof. See appendix.

The economics of this result is very simple. A higher productivity increases the optimal use of working capital. Yet, a higher advance of credit increases the temptations of the entrepreneur to default. Foreseeing this, it is optimal to increase the utility entitlement in a positive relation with the productivity of the states. Moreover, a higher productivity signals higher productivities thereafter, and a higher value \(V'\) will relax the constraints for setting \(V'', V''', \ldots\). and so on. Symmetrically, low realizations of \(z\) reduce the value of the relationship for the entrepreneur, but this is a result of its optimal design. Indeed, we will see below that when the entrepreneur is liquidated, the bank does not have to compensate the entrepreneur and simply seizes the plant. Before turning to exit decisions, it is convenient to characterize further the continuation policies. Specifically, we will provide conditions under which they are bounded.

By inspection of the first order conditions and of the form of \(Cc\) is not hard to see that if \(\exists q^0 < \beta\), then the bank would find it optimal to prescribe that, regardless of the current state \((V, z, q)\), either \(G_{z', q'}(V, z, q) = \U\) (liquidation) or \(G_{z', q'}(V, z, q)\) to be as large as possible. This is to say, in those states in which the bank is more impatient than the entrepreneur, the bank would postpone the delivery of dividends to the entrepreneurs \textit{ad infinitum}. We rule out this degeneracy by imposing condition \textbf{I}.

The participation constraint of the entrepreneur provides a lower bound \(G \geq \U\). Under assumption \textbf{I} we proceed to verify that \(G\) has a finite upper bound. The boundedness of \(G\) brings about obvious computational advantages.

First, assume that for a given state \((V, z, q)\) the choice of \(G_{z', q'}(V, z, q)\) is interior in the sense that \(V\) is not binding \((\mu_2(V, z, q) = 0)\). In these cases, one can use Benveniste-Scheinkman and obtain

\[
\frac{\partial C(V, z, q)}{\partial \U} = -\frac{\partial S(V, z, q)}{\partial \U} + q
\]

Now, let us assume that the choice of \(G_{z', q'}(V, z, q)\) is such that \(\mu_2(G_{z', q'}(V, z, q), z', q') = 0\). Using the functional form in \(F\) we obtain a unique value \(M^*(z, q)\) that solves the FOC, given by

\[
M^*(z, q) = \beta(1 - \delta)\U + \left(\frac{\alpha z}{1 + \frac{1-\beta}{q}}\right)^{\frac{1}{1-\delta}}
\]

13In Hart-Moore's model the bank cannot commit not to renegotiate the contract with the entrepreneur. In their model, if \(max\{Q\} < \beta\), the optimal contract is the slowest repayment schedule, and if \(\beta < min\{Q\}\), it is the fastest repayment schedule. For \(\beta = Q\) there is an indeterminacy as a continuum of contracts are optimal.
One cannot conclude however that $M^*(z', q')$ is an upper bound of $G_{z', q'}(V, z, q)$ as we have not yet ruled out that it binds in the subsequent choice of $G$. This is to say, it might be the case that for some $z^0, q^0$

$$M^*(z^0, q^0) < \beta(1 - \delta) \sum_{z', q'} M^*(z', q') P(z', q' | z^0, q^0)$$

Yet, the boundedness of $M^*$ leads to an upper bound on $G_{z, q}(\cdot)$. One can find a finite function $D : Z \times Q \to R_{++}$ such that the policies always lie within $[U, D(z', q')]$:

**Proposition 7.** Assume that I holds. Then the continuation policies are bounded. Specifically, $U \leq G_{z', q'}(V, z, q) \leq D(z', q') \leq M(z, q) < \infty$ for all $z', q', V, z, q$. The function $D$ is the unique fixed point that solves

$$D(z, q) = \max \left\{ M^*(z, q), \beta(1 - \delta) \sum_{z', q'} D(z', q') P(z', q' | z, q) \right\}$$

Moreover, $D$ is strictly increasing both arguments and $D(\bar{z}, \bar{q}) = M^*(\bar{z}, \bar{q})$

*Proof.* See appendix. \(\Box\)

Therefore, for the computations one can restrict attention to the domain $[0, M(\bar{z}, \bar{q})]$. Furthermore, one can safely assume that the policy correspondence $G$ is indeed a continuous function, which is guaranteed by the following proposition.

**Proposition 8.** $Cc(\cdot, z, q)$ is strictly convex and strictly submodular in $(V, z)$ if either $\mu_2(V, z, q) > 0$ or $V < M(z, q)$. Moreover, for any $(V, z, q) \in R_U \times Z \times Q$, the policy correspondence $G_{z', q'}(V, z, q)$ is a continuous function of $V$.

*Proof.* See the appendix. \(\Box\)

Therefore, the policy correspondence $G_{z, q}(\cdot)$ is single valued, and that indeed, it varies continuously on the endogenous state $V$. Therefore, the previous monotonicity results, not only can be rewritten with the partial ordering $\leq$ instead of the set ordering $\subseteq$, but can be taken to be strict as long as $G > U$.

**Liquidation Decisions**

We now turn the attention to the *extensive margin* decision of whether to liquidate or continue a relationship. It turns out that, as long as $V > U$, there are only two possibilities: either the entrepreneur continues with probability one or there is a liquidation lottery. This is because either $Cc$ is always below the value of liquidation or they cross at some finite value, $V^*$, a fact that can easily be established because the slope of $Cc$ is uniformly bounded by $q < 1$ while the slope for the liquidation is always 1. Therefore, the envelope of $Cc$ and $V - U - L$ is $Cc$ in its all region or forms from a convex combination.

This suggests that an entrepreneur is never liquidated without a lottery. But we will soon see that exactly the opposite is true. From risk neutrality, it follows that entrepreneurs losing the lottery...
must obtain the lowest utility $U$ and exit. If not, $C$ would have a segment with slope 1 and then a segment with slope $q$ violating convexity.

When $V = U$, the bank recommends exit, and seizes the plant, obtaining the liquidation value $L$. Such process resembles bankruptcy in many respects, as the entrepreneur enters the pool in the underground with no assets left from the firm. More formally,

**Proposition 9.** Fix $(V,z,q)$. If the optimal liquidation probability $\lambda(V,z,q) \in (0,1)$, then $V^0 = U$ and there is a unique value $U < V^1 < M^*(z,q)$ that satisfies

$$ \frac{Cc(V^1,z,q) + L}{V^1 - U} \in \partial Cc(V^1,z,q) $$

The optimal liquidation probability, $\lambda$, is given by $\lambda = \frac{V^1 - V}{V^1 - U}$

**Proof.** See the appendix. \hfill \square

Entrepreneurs "winning" the lotteries obtain a new utility entitlement $V^1$ in a region where $Cc$ is strictly convex. If that were not the case, $Cc$ would have a constant slope of $q$. Generically, this implies that additional gains can be obtained by setting $V^1$ even larger, and consequently, $\lambda$ closer and closer to unity. We haven’t found conditions to rule out degenerate lotteries $V^1 = \infty$ and $\lambda = 1$. They are not present in the computations presented below.

Notice that for each $(z,q)$ either $\lambda(V^1,z,q) = 0$ in all the domain $[U,\infty)$ or there is a region $V \in [U,V^1]$ such that $\lambda(V^1,z,q) > 0$, and for $V \in (V^1,\infty)$ $\lambda(V^1,z,q) = 0$. Indeed, $\lambda(V^1,z,q) = 0$.

**Proposition 10.** The liquidation probability function $\lambda : [U,\infty) \times Z \times Q \rightarrow [0,1]$ is decreasing in its first argument and is strictly decreasing whenever $\lambda \in (0,1)$.

The economics is also simple. The presence of both a fixed cost $U + L$ of operating a plant and the complementarity of $z$, $V$ implies that it is worth to operate the plant as long as the level of operations is large enough. If the current value of $V$ is above $V^1$, the scale of operations is large enough, and no randomization is needed; the plant continues operation with certainty. The lower the value of current $V$, the larger is the risk of liquidation faced by the firm in a fair lottery with a constant winning and losing prizes of $V^1$ and $U$ respectively. Therefore, the model implies that the lower the value of $V$, the lower is the survival probability, and that conditional on surviving, the larger is the growth. Conditional on $z,q$, the lower $V$ the larger is the variance of growth, unconditional on survival.

It seems that to characterize exit we need to examine the function $\lambda(V,z,q)$. This is unnecessary. By inspection of the first order conditions we can verify the following:

**Proposition 11.** Non-Trivial Randomizations are outside the equilibrium allocation. Fix any $(V,z,q) \in R_U \times Z \times Q$, and $V^1(\cdot)$ as defined above. Then, the optimal continuation mapping $G_{Z \times Q}(\cdot)$ can be selected so that $G_{z',q'}(V,z,q) \in \{U\} \bigcup (V^1(z',q'),D(z',q'))$ all $z',q'$.

**Proof.** See the discussion in the text, and the details in the appendix. \hfill \square

Therefore, no randomizations are required. Too see this, notice that in the regions where randomizations occur, the slope of the cost function is constant. The first order conditions equate the marginal reduction in the current cost with the marginal cost of assigning the utility entitlement in
every realization of the next period. If the limited liability constraint does not bind, the first term is just the discount factor $\beta$ while the second is the would be a constant in the region of randomizations. If the discount factor is strictly higher, then the optimal plan requires $G_{z',q'} = U$ while if it is strictly lower, then $G_{z',q'} = V^1(z',q')$. In case of equality, either extreme can be chosen, as the entrepreneur and the bank are indifferent in the timing of the transfers. In those knife-edge cases deterministic rules suffice.

Now, if the limited liability constraint binds, then there is a state $z',q'$ in which the slope of the cost is strictly below the discount factor of the entrepreneur. Then for the cases where $\beta \in \partial C(V^1(z'',q''),z'',q'')$, the optimal solution is $G_{z'',q''} = U$. Even if the limited liability binds, there is a possibility that $\beta(1 - \mu_2(V,z,q) \in \partial C(V^1(z'',q''),z'',q'')$. But again, the deterministic rule that assigns $G_{z',q'} = U$ achieves the optimum.

Clearly, a more binding limited liability prescribes a larger set in which the firm is liquidated in the next period.

**Initialization of Relationships**

Competition prompts banks to offer the best feasible incentive compatible contract to entrepreneurs. Because banks need to finance the fixed cost $K_0 > L$, the *ex-ante* breaking-even condition implies that the bank *ex-post* makes a strictly positive profit from the entering ventures. The function $V_0(z_0,q_0)$ indicates the value of the entrepreneur at the time of signing the contract with the bank.

**Proposition 12.** Let $V_0 : Z \times Q$ as defined above. Given the assumptions $F,P,Q$, then $V_0$ is strictly increasing in $z$. Moreover, given $L$ if $K_0 > L$ is large enough, for $\beta$ close enough to 1 and $\delta$ close enough to 0 then $V_0$ is also strictly increasing in $q$.

Entrepreneurs with a higher initial productivity draw $z_0$ receive a higher value $V_0(z_0,q_0)$ as the present value of the stream of profits is larger, and also because a larger value $V_0$ will enhance the incentive-compatible allocation of credit. The second part requires that $\beta$ and $\delta$ close to 1 and 0, respectively only insofar they limit $E[V_{i+1}]$ and then, in expectation, $C$ is also negative in the next period. In such a case, higher interest rate unambiguously increases the cost and reduces the breaking-even value $V_0(z_0,q_0)$. With higher interest rates not only entering entrepreneurs receive a lower $V_0$, but also some entrepreneurs may no longer become activate.

In periods of high interest rates, the mass of firms entering is smaller. In those periods, the requirements of productivity $z$ to activate are higher. As a result, the entering firms in periods of high interest rates are more likely to survive. On the other hand, the model has the observable implication that conditional on $z$ and given the stationarity of $F_z$, the pool of entrants, entering firms in periods of high interest rates are smaller than firms that enter in periods of lower interest rates. The lower value $V_0$ reduces the survival probabilities of those firms.

We close this section confronting our claim that in equilibrium the allocations are not subject to renegotiation. While, at the time of signing the contract, $V_0(z,q)$ cannot lie in regions where $C$ is decreasing in its first arguments, one might question whether the allocations in the future would entail possible Pareto improvements. If that is the case, the contracting parties will renegotiate a new contract and the studied allocations would no longer be relevant.

Competition implies that the initial entitlement $V_0(z_0,q_0)$ must be located in a neighborhood where $C(\cdot,z_0,q_0)$ is strictly increasing. Otherwise, the entrepreneur is not receiving the best feasible contract.
But it is easy to verify that this fact along with the optimization underlying the function $C$, implies that, unless the utility entitlements are equal to the underground value $U$, they lay in regions where $C$ is also strictly increasing in the first argument. Formally,

**Proposition 13.** Fix $(z,q)$ and assume that $V \geq V_0(z,q)$. Then, for every $z',q' \in Z \times Q$, either $G_{z',q'}(V,z,q) = U$, or $\partial C(G_{z',q'}(V,z,q), z', q') > 0$

*Proof.* See appendix for details.

The economics captured by this proposition is very clear. The optimal contract foresees correctly the future. For the initial expected payoff of the bank to be strictly decreasing in the initial expected payoff of the entrepreneur, then it must be the case that no renegotiation can be feasible in any future date or contingency with positive probability. Otherwise, the contract is not optimally designed, as it can be strictly dominated by modifying it with the Pareto improving amendments at that date and state. Therefore, once the contract is established, the rules specified in at the time of signing the contract. This result validates our using the promise keeping constrained with equality.

We will establish below that these contracts can be decentralized in an environment where entrepreneurs trade in one period securities. In that environment, entrepreneurs carry over a net position of real assets, $a$ his wealth. Given his wealth, productivity of the firm and the realization of interest rate, one can read off what is his expected discounted utility. Clearly, given $z$ there will be a direct connection (a bijection) between $V$ in the economy with dynamics contracts with $a$ the assets in the decentralized trading economy. Thus, $V$ can readily be interpreted as ‘collateral. For the next section, we will take this interpretation as valid. We will verify it in the subsequent section.

## 4 Aggregate Dynamics

As $\{q\}$ is a common component in the state of every ongoing relationship, the state of the aggregate economy is the pair $(q_t, \psi_t)$, where $\psi_t$ is a measure over $(V, z) \in \mathbb{R} \times Z$ the idiosyncratic components of the state of individual relationships.

The total mass of active firms varies according to the entry and exit. Thus, the state space for $\psi_t$ cannot be taken to be a space of probability. However, the relevant space is bounded as at most the entire unit mass of agents can be active. Moreover, from the previous section, we know that the support of the relevant distributions is bounded. Let $Y \equiv [U, +\infty) \times Z$, the set of idiosyncratic states and $(Y, \mathcal{Y})$, the respective measurable product space. Denote $M_1 \equiv \{\psi \in M(Y, \mathcal{Y}), \psi(Y) \leq 1\}$, and $\Lambda \equiv \{\psi \in M(Y, \mathcal{Y}), \psi(Y) = 1\}$ the set of measures bounded by the total mass of living agents and the set of probability measures.

The contracts defined at the individual level will define operators that render the aggregate time series of creation, $(T_C)$, destruction of firms, $(T_L)$, as well as the evolution of the states for the enduring firms, $(T_E)$. Respectively, these operators take the form of the mappings,

\[
T_C : Q \to \Lambda \quad (26)
\]

\[
T_L : Q \times Q \times M_1 \to [0,1] \quad (27)
\]

\[
T_E : Q \times Q \times M_1 \to M_1 \quad (28)
\]
Since \( \{q\} \) is taken to be exogenous, we need only to trace the dynamics of \( \{\psi_t\} \). For simplicity and with no loss of generality, the exposition omits the randomizations in the exit margin. Given the interest rate of the period, \( T_C \) will provide us with a measure over the entrant entrepreneurs. Recall that entry decisions are based on the productivity of the current period and that these productivities are drawn from the unique invariant c.d.f. \( F_z \). Either all, none or only a fraction of the possible entrants become active and that may depend on \( q_t \).

Competition fixes the initial value \( V_0 \). Those agents who are not initialized obtained a promised utility entitlement of \( U \). If \( V^0(z, q) > U \) the agent became active. It’s easy to keep track of the population of entrants. For any pair of Borelian sets \( A \times B \subset (U, \infty) \times Z \), \( T_C(q_t)(A \times B) \) is given by

\[
T_C(q_t)(A \times B) = \sum_{z \in B} \chi[V^0(z, q) \in A] F_z(z)
\]

and therefore the mass of entrants in \( A \times B \) is

\[
\psi^0(q_t)(A \times B) = \delta T_C(q_t)(A \times B)
\]

We adopted the convention to denote by \( G_{z',q'}(V, z, q) = U \) the case when a firm is liquidated. The total mass of firms begin liquidated, when the previous state was \( (q_{t-1}, \psi_{t-1}) \) and current exogenous state is \( q_t \) is given by

\[
T_L(q_t, q_{t-1})\psi_{t-1} = \int_{U \times Z} \left( \sum_{z \in Z} \chi[G_{z',q_t}(V, z, q_{t-1}) = U] P_z(z', z) \right) \psi_t(dV \times dz)
\]

Finally, enduring relationships \( G_{z',q_t}(V, z, q_{t-1}) > U \) will update their state so that for any \( A \in B(U, \infty) \) and \( B \subset Z \)

\[
T_E(q_t, q_{t-1})\psi_{t-1}(A \times B) = \int_{U \times Z} \left( \sum_{z \in B} \chi[G_{z',q_t}(V, z, q_{t-1}) \in A] P_z(z', z) \right) \psi_t(dV \times dz)
\]

The exposition has ignored so far the exogenous death of firms. Since only \((1 - \delta)\) firms actually overcome the exogenous risk the transition of \( \psi \) is actually given by

\[
\psi_t = (1 - \delta) [T_E(q_t, q_{t-1})\psi_{t-1}] + \delta T_C(q_t)
\]

4.1 Stationarity of The Aggregate Economy

Aside for trivial cases, given the presence of aggregate uncertainty, the economy does converge to an invariant distribution of firms. There is a sense, however, in which their limiting behavior is stationary, i.e. independent of calendar time.

Rolling backwards the transition for \( \psi_t \), for any positive integer \( N \)

\[
\psi_t = \delta T_C(q_t, q_{t-1}) + \delta \sum_{j=1}^{N-1} (1 - \delta)^j P_E(q_t : q_{t-j}) T_C(q_{t-(j+1)}) + (1 - \delta)^N P_E(q_t : q_{t-N}) \psi_{t-N}
\]
\[ P_E(q_t : q_{t-j}) : M \rightarrow M \text{ of the form} \]
\[
P_E(q_t : q_{t-j}) = T_E(q_{t-1}, q_{t-2}) \cdots T_E(q_{t-j}, q_{t-j-1}), \quad (35)
\]

for \( j = 0, 1, 2, \ldots \). Because \( T_E \) is mass decreasing so is \( P_E(q_t : q_{t-j}) \), and because \((1 - \delta)^N \rightarrow 0\) as \( N \) grows, the effect of \( \psi_{t-N} \) on \( \psi_t \) washes out over time. Equivalently, the effect of \( \psi_t \) on \( \psi_{t-N} \) vanishes as \( N \rightarrow \infty \). Taking that limit

\[
\psi_t = \delta T_C(q_t) + \delta \sum_{j=1}^{\infty} (1 - \delta)^j P_E(q_t : q_{t-j}) T_C(q_{t-(j+1)}) \quad (36)
\]

Because \( \{q_t\} \) is a stationary process and \( T_C, T_E \) are time invariant operators, it is immediate that the distribution of \( \psi_t \) does not depend on the calendar time. Moreover, given \( \delta > 0 \), the economy is also ergodic, i.e. its initial conditions do not affect its limiting behavior. As all aggregates are defined as time invariant functions of \( \psi_t, q_t \) or \( \psi_{t-1}, q_t, q_{t-1} \), then all relevant series are stationary.

For instance, when there is not aggregate uncertainty, i.e., when \( q_t = q \in (0, 1) \) for all \( t \). In this case, independently of the initial conditions, the economy will converge to a unique invariant distribution. To see this, notice that the measure of entrants \( T_C(q) \) is time invariant and that \( P_E(q_t : q_{t-j}) = T_E^j(q, q), \ j = 0, 1, 2, 3, \ldots \). Then the unique invariant distribution is

\[
\psi(q) = \delta \left[ \sum_{j=0}^{\infty} (1 - \delta)^j T_E^j(q, q) \right] T_C(q) \quad (37)
\]

### 4.2 Numerical Illustrations

In this section we illustrate the aggregate time series that can be generated by the model. As indicated already, it is not hard to generate the extreme cases where no firm is ever activated or where all firms enter and they achieve maturity on entry. We focus instead in intermediate values for the liquidation value \( L \) of the firm and the outside option of the entrepreneur \( U \). In those cases, the realization of the interest rate affects the pool of firms entering and exiting in the period. We report results from two different parameterizations. In the first, interest rates affect the rate of liquidation and creation of firms. In the second, all potential firms enter and remain active until they are exogenously liquidated. Our objective here is to illustrate the qualitative features of the model and not to pursue a serious calibration exercise, which, anyway, seems a rather illusory objective for this model. Table 1 displays the parameter values used.

The model is rather tight \( t \) in the number of parameters. Changing any parameter will change the fixed point \((C, G)\) defining the optimal contract and hence the behavior of entry, exit, and firm growth. For instance, the value \( U \) not only determines the opportunity cost of the entrepreneur but also his outside option. Thus, \( U \) lowers the cost of liquidating plants, but it also raises the credit constraints, increasing the cost of continuing the relationship for the bank. On the other hand, \( L \) clearly increases the liquidation value. But it also increases the continuation value, as it raises the option value of liquidating the firm in the next period. The only \( free \) parameter in the model is the irreversibility of the capital investment on the firm, \( K_0/L \). That ratio determines the initial condition for newly activated firms. A large degree of irreversibility implies that the initial debt (see below)
Table 1: Parameter Values Used

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Entry/Exit sensitive to $q$</th>
<th>Exogenous Entry/Exit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Discount Factor Entrepreneurs</td>
<td>$\beta = (1/1.125)^{1/4}$</td>
<td></td>
</tr>
<tr>
<td>Death Prob./Entry Mass</td>
<td>$\delta = 0.03$</td>
<td></td>
</tr>
<tr>
<td>Entrepreneur's backyard income</td>
<td>$e = 2.5733, \Rightarrow U = 75$</td>
<td></td>
</tr>
<tr>
<td>Fraction of Liquidation Value that can be stolen</td>
<td>$\theta = 0.1$</td>
<td></td>
</tr>
<tr>
<td>Liquidation Value Plants</td>
<td>$L = 250$</td>
<td></td>
</tr>
<tr>
<td>Installation Cost of Plants</td>
<td>$K0 = 1.1 * L$</td>
<td></td>
</tr>
<tr>
<td>Output elasticity to working capital</td>
<td>$\alpha = 0.75$</td>
<td>$\alpha = 0.775$</td>
</tr>
</tbody>
</table>

### Shocks

<table>
<thead>
<tr>
<th>Idiosyncratic Productivities $z$</th>
<th>$\mathbb{Z} \subset [2.5, 5], \quad N_z = 4$</th>
<th>$N_z = 1; \quad Z = 3.2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho_z = 0.9, \quad \sigma_z = 0.12$</td>
<td></td>
<td>$\rho_z = 1, \quad \sigma_z = 0$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Market Discount Factor $q$</th>
<th>$\mathbb{Q} \subset [3.0 .99999], \quad N_q = 3$</th>
<th>$N_q = 3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho_q = 0.62, \quad \sigma_q = 0.00635$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

of the entrepreneur is high. In equilibrium this will imply tighter constraints and a longer stage of accumulating collateral.

A key parameter is $\alpha$, the elasticity of output to working capital. The higher it is, the closer to linear is the production function, and the larger the the unrestricted optimal size of firms. Not surprisingly, in those case the optimal production of firms is very sensitive to interest rates. But, more interestingly, as the implied firm dynamics is longer, it is in those cases where young (and small) firms become disproportionately more sensitive to survival than older (and larger) firms.

We view a period as a quarter and try to set the discount factor $\beta$ consistently. However, we need to impose a slightly large discount rate to allow room for interest rate shocks of some magnitude given the restriction that $1 > q > \beta$. The most plausible assumption for $z$, the idiosyncratic productivity of firms, is that it follows they are highly persistent. Thus, we set $z$ to follow approximately a log normal process with a correlation of 0.9. To compute the contracts, we use also the methods by Tauchen-Hussey. The variance of $z$ and its support $Z$ were selected to our discretion aiming to illustrate the intermediate cases discussed in the first paragraph of this section.

Ideally one would have numerous possible realizations of the idiosyncratic shocks $Z$ to avoid discrete responses of entry and exit to changes in $q$. We also need a numerous realizations for $q$ to better explored the non-linearities of the model. But to keep in control of the computational complexity required to allow long firm dynamics, in this illustrations we opted for $N_z = 4$ and $N_q = 3$. Thus, to solve for the optimal contract we need to solve a dynamic programming problem in which, for each triple $(V, z, q)$, we need to solve for the optimal liquidation lotteries and, in case of continuation, for the updating of $V'$ for each of the $N_z \times N_q = 12$ possible subsequent states $(z', q')$. This problem is solved using a policy function algorithm based on quadrature methods, similar to those discussed in Christiano and Fisher [14]. Appendix C of Monge [47] discusses in detail the algorithms used.\(^{14}\)

In all these exercises we compute the optimal contracts assuming that the real interest rates follow approximately the same process as in the U.S. We take a Markov Chain approximation of the process followed by the ex-post real quarterly FedFunds rate. The Markov Chain is obtained following

\(^{14}\)The Matlab codes used in this version of the paper are available from the author upon request.
Figure 1: Parameterization I: A Typical Realization: Measures of Output, Entry and Exit Rates

Figure 2: Parameterization II: A Typical Realization: Measures of Output (all firms are active)

Tauchen-Hussey [55]. We re-scale its support so that $Q \subset [\beta, 1)$.

Under the parameterization with sensitive entry and exit, half of the potential entrants become active if the interest rate is either at the high or medium level, while less than 30% do if the interest rate is high. Regardless of the interest rate, all the firms with a productivity realization in the two lowest levels are liquidated, and all the firms with the highest productivity remain active. Interest rates determines the liquidation of those firms in the third level. We have chosen a parameter configuration that gives this feature so that output asymmetries are not entirely driven by asymmetries in the series of creation and destruction of firms. Needless to say, the entry and exit asymmetries are interesting in their own right. Those asymmetries are stronger when we parameterize the model so that $L/K_0 \to 1$, i.e. the value of the liquidation option is high (making destruction very sensitive to interest rate) and that $(Z, P_z, F_z)$ is such that entry is very insensitive to interest rates, e.g. only the very best enter in each period. But that is almost trivial to achieve in this model with its built-in assumption of a constant pool of potential entrants. Instead, we try to emphasize the effect of the firms life-cycle.

The second parameterization has only one level of productivity $z$ and implies that, regardless of interest rates, all firms become activated and no firm is endogenously liquidated. The mass of active entrepreneurs is always equal to one. The cross section distribution of firms ages is trivial: a fraction $(1 - \delta) \ast \varphi$ has age $a = 0, 1, 2, \ldots$. The history of $q_t = \{q_s : s \leq t\}$ implies a not so trivial distribution of e.d.u.s $V$. The period realization of $q_t$ determines the initial value $V_0$ for all the entrants.

Under either parameterization, the effect of the initial measure $\psi_0$ vanishes rather quickly. In any event, here we show the realizations of the economy after 500 periods, when only a fraction $(1 - \delta)^{500} = 2.4315 \ast 10^{-007}$ of the initial firms had survived. The top panel shows a typical path for
Figure 3: Impulse Response, sensitive Entry/Exit: Permanent Changes of Interest Rates

Figure 4: Time to Adjust to a Permanent Shock; Sensitive Entry/Exit

different measures of output: 'Net,Active', the output of active firms once the working capital used is substracted and 'Net, all' which adds the output e of each of the $[1 - \psi] \text{"inactive"}$ agents.

A simple eyeball inspection shows that fluctuations are large. With endogenous entry and exit, epochs in which interest rates rise and remain high for three consecutive periods are associated with declines of almost 15% of the aggregate output of active firms and more than 6% of total net output. Clearly, the model does not have problems generating large output responses from interest rates fluctuations. Fluctuations are smaller in case entry is exogenous, even if $\alpha$ is higher and output of active firms is more sensitive to interest rates.

The same casual inspection reveals that periods of expansions are smoother and longer than periods of contractions. Notice that that seems to be the case for both parameter configurations. Thus, aggregate asymmetries are present, even if entry and exit are constant and equal to $\delta$. We argue below that that is indeed the case.

Obviously, as interest rates are the only shock in the economy, all aggregate series of output are strongly related to them. As panel 2 in Figure 1 shows, the series of activation and liquidation are strongly negatively related. Periods of high interest rates yield larger amounts of liquidations and a lower numbers of new firms. The panel shows the fraction of entry and exit with respect to the mass of previously active firms. Given the demographic structure in the model, interest rates shocks automatically lead to an asymmetric behavior for the mass of active firms. In principle all active firms can be liquidated. Interest rates can have a fulminant effect on firms. On the contrary, there is an upper-bound $\delta$ on the mass of newly created firms. We insist that this is only one part of the story.

To explore further the aggregate dynamics, we examine the response to shocks. In this exercise we assume that shocks take one possible realization but contracts are written and implemented under the conditional expectations implied by $P_q$ and the initial state $q_0$. A complication arises from the
fact that the response to a shock depends on the pre-existent cross section distribution $\psi_l$. Additional non-linearities arise as the qualitative response depends on the magnitude of the shock. The latter is particularly evident as twice as a large shock may easily imply more than the double of firms liquidated.

Given this complexity we will limit ourselves to explore two sets of simple exercises. The first ones assume that the economy has faced a constant interest rates for a long period of time and then interest rates switch, permanently, to a new level. The second set of exercises assume also that that the interest rates have been constant for a long period of time, but the shock is temporary. The idea here is to explore, for example, a epoch where interest rates original increase, remains high and then moves back to the original level.

Figure 3 shows the results from this set of exercises. It displays the response of an economy that moves from low ($q(3)$), medium ($q(2)$) and high ($q(1)$) interest rates to the other two levels. All the responses are expressed as ratios of their initial value. As revealed already by previous figures, Figure 3 reveals that indeed, the effect of interest rate fluctuations can be large. The response is orders of magnitude larger if they involve changes in the total mass of active firms, which is the case when the high level of interest rates is involved. But, still, in the cases where the total mass of active firms, i.e. from med to low and from low to med, the response is about 10% of the initial level of output.

More interestingly, notice that it takes a long time for the economy to fully adjust to a permanent, one-and-for-all, change in the interest rate. For example, after one permanent decline, output exhibits positive growth for more than 150 periods. Growth is non-negligible for much longer time if the shocks involves a change in the long term mass of active firms. As long as the process $\{q_t\}$ has positive persistence, the model generates output-growth series with higher persistence.

In addition, the aforementioned non-linearity is evident as the magnitude of the output response from a Hi-to-Med rates is almost 5 times the magnitude of a Med-to-Lo transition. The extensive margin of active firms is responsible of this large non-linearity.

An close inspection of Figure 3 reveals a different asymmetry in the responses from increasing
versus decreasing interest rates in the sense that the adjustment takes less time for a contraction to unravel. Figure 4 shows the asymmetry more clearly. It shows the number of periods required need to attained each fraction of the final response. The figure shows only the cases from Med-to-low and from low-to-med as they do not involve changes in the long term mass of active firms. For instance, after 10 periods only 20% of the expansion of the transition from Med-to-Low has been accomplished. In the same number of periods more than 40% of the contraction associated with the transition from Low-to Med is attained.

An even closer inspection of Figures 3 and 4 show a peculiarity. The response to a Med-to-Low and Low-to-Med is not monotone. The chosen parameterization illustrates a point discussed at the end of the paper: interest rate shocks might have an asymmetric effect across firms. The aggregate effect is small quantitatively small as it is only relevant for mid-aged firms.

The non-linearities induced by the interaction of entry, exit and firm growth can produce interesting dynamics. The second type of exercises illustrate further this point. Figures 5 and 6 display the output response to temporary shocks of the following form: interest rates have remained high (low) for a long period of time; then in period period $t_0$, they go down to either low or medium levels (go up to either high or medium levels) and stay there for two more periods. Then the interest rate goes back to the original level and stays there forever. In all this time agents compute expectations using the initial value $\beta_0$ and the transition $P_0$. This is the closest to the impulse-response functions for interest rate shocks found in the empirical literature.

Not surprisingly, the figures show that the response is higher for larger shocks. More remarkably, there are shape differences between responses to large and small and between declines and increases. A higher interest rate impairs entry and enhances exit. If the shock is small the effects are small and the movement of output is smooth. Yet, if the shock is large, it triggers a sharp contraction of the surviving firms and tighter constraints for maturing firms. As shown by Figure 6, after one period after the shock came back to its original level, i.e. at $t_0 + 5$, output behaves the same with large and with small rise in interest rates. Indeed, both episodes liquidate and impairs the creation of the same mass of firms (not shown here). But, at $t_0 + 4$, when the interest rate went back to its original level, output is lower 1.7% than with the low shock. This reflects the tighter constraints associated with higher previous interest rates. In both cases, as the interest rate goes back to the original level, it will take a long time for the economy to recover the mass of active entrepreneurs that were liquidated.

As illustrated in Figure 5, depending on the parameters, a temporary decline in interest rates leads to a slightly different pattern. Indeed, as interest rate remain low, newly created firms will start growing. But it is only with really low interest rates that a significant mass of additional firms survive, even after interest rates go up. If the interest rates decline only to the medium level, the newly additionally newly created firms during the period will be liquidated when the interest rate goes up again. The response of output will be over with just one period after the interest rates went back to the original level.

Because of lower interest rates more firms entered and less firms were liquidated during the period. And precisely, because of temporary lower interest rates the firms were able to accumulate collateral so that when the interest rates go up again they manage to survive. As with a temporary increase, the effect on the economy last many periods while the increment in the mass of active firms washes out. \footnote{One might complain that there are too many degrees of freedom in the entry margin. That is not the case. The only

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Last but not least, we want to argue that the aggregate asymmetries are not entirely driven by the mass of active firms. The life-cycle of firms can by itself induce an asymmetry at the aggregate level. That is the purpose of including the second parameterization of the model, where all firms are active.

In that version of the model, the asymmetries in expansions and contractions is also present. But in this case, they hinge solely on the length of growth for the firms. To illustrate that point we have chosen a slightly larger value of \( \alpha \). Figure 7 and 8 display the Hodrick-Prescott filtered series of output for the simulated output. In conformity with the interpretation of a period as a quarter, we have set the parameter of the filter, \( \lambda \) to be 1600, the usual value. Similar conclusions obtain with rates of growth.

By construction, the mean of the filtered series is zero. The histograms vividly show that under both parameterizations, large negative deviations are more likely than large positive deviations. This is, the distribution of the absolute value of positive deviations seem to dominate, in the second order stochastic dominance sense, the distribution of absolute value of negative deviations. In general asymmetries are stronger with endogenous. But they can be sizeable even if driven only by firm dynamics.

5 Decentralization With Short-Term Securities

We now show that the allocations attained by long term contracts can be replicated by sequences of one-period contingent contracts. Thus, we can view our environment as a debt-constrained Radner economy. Our arguments follow closely those Alvarez and Jermann [2] and to a lesser Kehoe and Levine [40].\footnote{Kehoe and Levine [40] follow Prescott and Townsend by imposing the incentive constraints in the consumption sets.} There are three additional complications: First, we have to deal with firm-dynamics, margin is the ratio \( K_0/L \). It is precisely to discipline ourselves that we are using \( F_\alpha \), the invariant distribution of \( P_\alpha \).
Second, we have to incorporate liquidation histories. Finally, we need to add borrowing constraints in addition to solvency constraints, given the added possibility of entrepreneurs to appropriate the working capital. Given those, we show how to find the price system and the profile of constraints so that the allocation attained in these less sophisticated trading environments replicates.

In our environment borrowing constraints are needed in addition to solvency constraints. This extra requirement is Extra investments in notation are required by the liquidation option. The steps of this exercise are the following. First, we define the problem of an entrepreneur facing given arbitrary asset prices and solvency and borrowing constraints. Second, from the long term contract we construct a candidate profile for the constraints and the securities prices. In the third, ignoring the participation and default constraints, we verify that the agent find it optimal to choose the allocations from the long term contract, given the candidate prices and constraints. Finally, we verify that the allocation also solves the problem of the agent, even when facing the participation, exit, entry and default decisions. 17

Another decentralization scheme admissible in this environment is to simply interpret the previous environment as one in which the entrepreneur acquires long term, contingent liabilities from an intermediary, but that he can move from one bank to another as long as he pays back the remaining balance in the original relation. Banks break-even in expectation in every period, as the entrepreneur transfer his debt to the new bank which pays back to the original bank. This last interpretation is an immediate by-product of the construction of the candidate budget constraints.

Anyway, the exercises in this section yield an explicit link between the conceptual variable $V$ in a relationship with a much more pedestrian form of collateral, the wealth of the entrepreneur in a decentralized setting.

The Deterministic Case

The essence of the argument can be captured by the case with no uncertainty where $\delta = 0$, and normalize units so that $z_t = 1$ all $t$. The results for the general case are straightforward extensions once the appropriate notation is in place.

Consider the problem of an entrepreneur that can borrow or lend using one-period assets in impersonal markets. The only asset is a one-period bond with price $q$. For now, ignore the occupational choice and the option of default. Assume that for unspecified reasons, the entrepreneur must satisfy solvency and borrowing constraints $\mathcal{A}, \mathcal{B} = \{A_t, B_t\}$. The solvency constraints indicate that in every period $t$, the wealth $a_t$ of the agent must always be at least $A_t$ and the borrowing constraint restricts

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17 Albuquerque and Hopenhayn [1], following Bulow and Rogoff, [7] consider the case when defaulting entrepreneurs can maintain savings in other financial intermediaries and write down contingent contracts. The key difference is that instead, we assume a strong form of concerted action on behalf of financial intermediaries: that defaulting agents are out of the entire financial system. This is, we assume that while financial intermediaries compete with each other in the securities market, they can commit not to lend and also to seize the savings of any entrepreneur who, in any previous period, has defaulted, to any financial intermediary. This form of cooperation among intermediaries could be derived from an infinite horizon game among intermediaries. We can also find support for the assumption from the working of actual civil (common) courts, credit bureaus, etc. in actual economies. Indeed, information sharing arrangements among creditors are pervasive world-wide, e.g. Padilla and Pagano [48], even in developing countries. Indeed, creditors share information not only on credit performance but also on asset ownership by debtors. See f.g. Monge et. al [46] for a country study.
the external borrowing at period $t$ to be at most $\hat{B}_t$. The only asset needed in this case is a one period bond, which price we take as $q$.

The problem $\mathcal{P}(\hat{A},\hat{B})$ of the entrepreneur, given $q$ and $A = \{\hat{A}_t, \hat{B}_t\}$, the arbitrary solvency and borrowing constraints is:

$$\max_{c_t, a_t, k_t} \sum_{t \geq 0} \beta^t c_t \quad (\mathcal{P}(\hat{A},\hat{B}))$$

such that for all $t$

$$a_t \geq A_t \quad \text{(Solvency Const.)}$$
$$0 \leq k_t \leq a_t - A_t + \hat{B}_t \quad \text{(Borrowing Constraints)}$$
$$0 \leq q\alpha_t \leq a_t + q[f(k_t)] - q\alpha_{t+1} + k_t \quad \text{(Sequential Budget Constraint)}$$

The specification of $\hat{A},\hat{B}$ determines whether the agent can become active, the feasible level of operation, his consumption, etc. For example, to become active, the entrepreneur at $t = 0$ it is necessary that $\hat{A}_0 \leq -K_0$ so we can buy the plant, and that $\hat{B}_0 > 0$ to make it operate in that period. Yet, we are not interested in the allocations from arbitrary $\hat{A},\hat{B}$. The question is whether the is a profile, say $A, B$, in which the allocations in this far less sophisticated environment replicate $\sigma$.

Using the long term allocation, $\sigma$ we can read off the implied net asset positions of the entrepreneur in the relation, as $A_t \equiv C_t$, i.e. the value assumed by $C_t$ along the optimal long term contract. The initial value for the sequence is given by $A_0 = -K_0$, i.e. the entrepreneur starts with net debt to finance the plant. The wealth is updated according to the repayments and working capital advances $r_t^\sigma, k_t^\sigma$, following the recursion:

$$A_t = -q r_t^\sigma + k_t^\sigma + q A_{t+1} \quad (38)$$

It is easy to see the infinitely lived contract can be replicated by rolling over a short term liability. Say, in the beginning of the period an entrepreneur owes $A_t < 0$. Then, he can borrow the amount $A_t + k_t$ from another bank, and ask the new bank to pay his debt $A_t$ to the old bank, freeing him from any that liability. In that period the entrepreneur would use the working capital $k_t$, produce $f(k_t)$ and make the repayment $r_t$ to the new bank; those transactions imply that the next period will begin with the agent having a debt (or deposit) of $A_{t+1}$ in the new bank. At that point he can decide to remain or to move on with a different bank, and so on. Banks would be breaking-even in every period. It requires that no bank will lend to entrepreneurs who have previously defaulted. To cope with the incentive to switch to the underground sector and to appropriate of the working capital require the lower limits in the debt rolled over (for participation) and upper limits on the borrowing (default with appropriation of $k$). The obvious candidate for the borrowing constraint is simply $B = \{k_t^\sigma\}$.

The allocation implied by $\sigma$ uniquely solves $\mathcal{P}(A,B,q)$. Suppose that the economy is such that it is optimal to activate the firm in the long term contracts setting. Since $A_0 = C_0 = -K_0$, here the entrepreneur can also become active in the first period of life and by construction he can operate the plant up to $k_0 = B_0 = k_0^\sigma$. This holds for any period $t$; the allocation implied by $\sigma$ is feasible because
by construction it and respects the constraints $A, B$. We can verify that indeed the agent cannot do better (see the appendix) because otherwise the contract used to design $A, B$ could be improved upon, which is a contradiction with the definition of $C$ or the initial value $V_0$. Finally, assume that the entrepreneurs can have the option of leaving without honoring previously accumulated debt or alternatively, leaving the system and appropriating the working capital advanced in the period, exactly as above. Would their participation and default decision be different in this trading environment with $A, B$ with respect to the long term contract environment? Clearly no. While the profile of credit constraints allows infinitely many paths with nodes in which the entrepreneur would run away as the entrepreneur will disregard those paths as suboptimal. This is, for any period $t$, including the activation date $t = 0$, once the agent is placed along the optimal default free allocation from the long term contracts, he will disregard those other alternatives as suboptimal.

The General Case

The case with uncertainty follows naturally from the previous considerations, but requires extra notation. Here, the solvency and borrowing constraints need to be specified as functions of the partial histories of the relationship. The steps of the argument are the same: define the problem of agents facing exogenously given solvency and borrowing constraints; from the long term contract find the candidate prices for the securities traded and the solvency and borrowing constraints; ignoring the participation and default constraints, it would be obvious that the allocation $\sigma$ is optimal, and then argue that incorporating the default and participation options does not affect the arguments.

First, we look at the interpretation of the contract as rolling over a short-term financial liability. Here the entrepreneur in the first period obtains a loan of $K_0$ from one bank to start up the plant, plus $k_0$ for the working capital of that period. The financial system records his balance of $-C(z_0, q_0) - k_0$ with that bank. Uncertainty present, the optimal one period contract will be contingent on the realization at $t = 1$. Clearly, the bank and the entrepreneur can sign a contract in which the balance of the latter is the value $C^*(z^1, q^1)$ attained from the long recursive problem solved in the previous section. This is, a one period contract in which the entrepreneur commits to repay $r\sigma_0(z_0, q_0)$ and the next period is the random variable $C^*([z_1, z_0], (q_1, q_0))$. When the uncertainty of the next period is realized, the financial system records the debt of the entrepreneur. At that point, we is free to contract with that bank with a similar contract, or move on to another bank. If he opts for the latter, then he needs to pay for the balance $C^*([z_1, z_0], (q_1, q_0))$ to the old bank. The new bank pays off the debt to the old one, and now the credit bureau records that the balance of the entrepreneur is with the new bank. And so on. It is possible that $C^*([z_1, z_0], (q_1, q_0)) = -L$; in this case, the value of the plant is exactly the amount owed to the bank. We know from the long term contracts that happens when $G_{z,q}(\cdot) = U$. Then, the plant is seized by the bank to recover the debt. Finally, if the agent dies, no one can recover any resources from the technology and the balance is recorded as zero. By construction, as before, in each period the bank breaks even in expectation.

Along the lines of the previous subsection, we ask whether we can replicate the allocation of the infinite horizon contracts in centralized asset markets with endogenously designed exogenous constraints. A very significant simplification is that in recording the histories of the relationships we can ignore (non-trivial) liquidation randomizations because they are relevant only outside the equilibrium allocation. In addition to $z, q$ shocks, the relationships also face the risk of the death of the entrepreneur,
in which case no resources can be recovered. Moreover, we need to keep track of whether the agent has been previously liquidated, which implies, by assumption, that he cannot operate the productive technology anymore.

Denote \( d_t \) the realization of the death risk; \( d_t = 0 \) indicates that the agent dies at period \( t \) and \( d_t = 1 \) that he survives. Naturally, \( d^t \) denotes the partial history of the death risk and \( D^t \) the set of all partial histories.

The assets used by the entrepreneur to trade are contingent on the realization of the \((z^t, q^t, d^t)\) shocks. Take \( p^* \) as arbitrary the Arrow prices, and \( A, B \) arbitrary solvency and borrowing constraints. Then, \( P(A, B, p^*) \), the problem of the agent becomes:

\[
\max \left\{ \sum_{\{r,a,k\}} \sum_{\{t \geq 0\}} \beta^t c_t(d^t, z^t, q^t) \mu_t(d^t, q^t, z^t | q_0, z_0) \right\} \quad (P(A, B, p^*))
\]

such that, for all \((d^t, q^t, z^t) \in Z_t \times Q^t \times D^t \quad \forall t\)

\[
a_t(d^t, z^t, q^t) \geq A_t(d^t, z^t, q^t) \quad \text{(Solvency Const.)}
\]

\[
0 \leq k_t(d^t, z^t, q^t) \leq a_t(d^t, z^t, q^t) - A_t(d^t, z^t, q^t) + B_t(d^t, z^t, q^t) \quad \text{(Borrowing Constraints)}
\]

\[
a_t(d^t, z^t, q^t) + q_t f(k_t(d^t, z^t, q^t)) - c_t(d^t, z^t, q^t) \geq \sum_{\{z_t+1, q_t+1, d_t+1\}} [p^* (d_{t+1}, z_{t+1}, q_{t+1}|z_t, q_t) a_{t+1}(d_{t+1}, z_{t+1}, q_{t+1}, z^t, q^t)]
\]

\[
c_t(d^t, z^t, q^t) \geq 0 \quad \text{(Limited Liability)}
\]

We have ignored the choices after being liquidated. Following our line of argument, they are considered at the end.

We now construct our candidate price system and solvency and borrowing constraints. The natural candidate for Arrow prices are:

\[
p_t^*(d^{t+1}, z^{t+1}, q^{t+1} | d^t, z^t, q^t) = \begin{cases} 
0 & \text{if } d_{t+1} = 0 \text{ or } d_s = 0 \text{ for some } s \leq t \\
(1 - \delta)q_t P(z_{t+1}, q_{t+1}|z_t, q_t) & \text{otherwise}
\end{cases}
\]

The implied Arrow-Debreu prices are defined by the recursion

\[
P_t^*(q^{t+1}, z^{t+1} | q_0, z_0) = p_t^*(d_{t+1}, z_{t+1}, q_{t+1} | d^t, z^t, q^t) P^*(d^t, q^t, z^t | q_0, z_0)
\]

The price system is suggested by the risk neutrality of the banks and entrepreneurs. But, this price system is valid if and only if the credit constraints insure that active entrepreneur will not default. Obviously, the value of a security that pays in the states of the world in which the entrepreneur defaults must be equal to zero.

The price system indicates that the liabilities of dead entrepreneurs or liabilities of entrepreneurs that pay off only in the states where the entrepreneur dies must necessarily be zero. The valuations
of these securities do not pose any discussion. But notice that we have specified that the liabilities of agents who have been previously liquidated have zero value too. This obeys to the assumption held all over the paper that once the agent is in the underground sector, he does not have any incentives to repay debts. Notice also that since we are assuming $q \geq \beta$, liquidated agents would not want to save, and therefore, the assumption is not with loss of generality.

From the long term contract, the candidate solvency constraints are:

$$A_t(d^t, z^t, q^t) = \begin{cases} 
-K_0 & \text{if } t = 0 \text{ and } V_0(z_0, q_t) > U \\
C(z_t, q_t, G_{z_t,q_t}(z^{t-1}, q^{-1}, V_{t-1}(z^{t-1}, q^{-1}))) & \text{if } d_s = 1 \text{ all } s \leq t \\
 & \text{and for all } s < t \\
& G_{z_t,q_t}(z^{s-1}, q^{-1}, V_{s-1}(z^{s-1}, q^{-1})) > U \\
-L & \text{if } d_1 = \ldots = d_t = 1 \text{ and } \\
& G_{z_t,q_t}(z^{t-1}, q^{-1}, V_{t-1}(z^{t-1}, q^{-1})) = U \\
0 & \text{otherwise}
\end{cases}$$

(41)

and the candidate borrowing constraints are given by

$$B_t(d^t, z^t, q^t) = \begin{cases} 
-k^*(z_t, q_t, V_t(d^t, z^t, q^t)) & \text{if for all } s \leq t, d_s = 1 \text{ and } \\
& G_{z_t,q_t}(z^{s-1}, q^{-1}, V_{s-1}(z^{s-1}, q^{-1})) > U \\
0 & \text{otherwise}
\end{cases}$$

(42)

The $A, B$ profile is very simple and intuitive. First, the solvency constraints allows the entrepreneurs in their first period of life to borrow up to the cost of the plant $K_0$ only if, given the economywide interest rate and the realized idiosyncratic productivity $z$ the entrepreneur can expect a utility higher than $U$. The solvency constraints specify that in those states of the world in which the entrepreneur is going to be liquidated, then he can owe at most the liquidation value of the plant. Entrepreneurs that have been liquidated or died previously cannot negative balances.

Borrowing constraints are necessary because the entrepreneur might be better off by appropriating the working capital at hand and leaving the financial system with all the liabilities unpaid. For active agents the constraint is the incentive compatible upper limit that results from comparing the utility that a maximizing agent can achieve inside the system to the one attained by defaulting. Trivially, agents dead in that period cannot borrow, but more importantly, entrepreneurs who are alive but that have been previously or are currently being liquidated, cannot borrow in positive amounts, as they would not have the incentives to repay.

With the candidate $A, B, p^*$ in the place, the argument is exactly as for the case of certainty. First, by construction the allocation $\sigma$ of the infinite horizon contract is feasible. Second, given the constraints, the allocation is indeed optimal. Too see this, notice that constraints and prices given, $\mathcal{P}(A, B, p^*)$ is a convex problem. Since $\beta < q$ the transversality condition holds trivially, the Euler conditions are sufficient. Given prices $p^*$, we can readily verify that the allocation $\sigma$ satisfy the Euler equations. The details are in the appendix. Obviously, this implies that the entry decisions coincide in both environments.
To finish the argument, we then consider $P(\mathcal{A}, \mathcal{B}, \beta)$ allowing the agent to default and to decide whether to participate. By our assumptions, once the entrepreneur defaults or is liquidated, his problem becomes trivial: consume $c$ every period. This is a logical implication of the assumption that once in the background sector, he cannot be forced to honor any liability.  

The profile $\mathcal{A}, \mathcal{B}$ allows trading strategies that involves nodes in which the entrepreneur will be better off defaulting. Some of those trading strategies require that the agent consume over time below the dictates of $\sigma$ and accumulate assets $a$ above the specification of $\mathcal{A}$, or that accumulates assets. Since $q < \beta$, there are values $a$ high enough so that the entrepreneur would find it optimal to borrow $B_t$ and default. By construction, the incentive problems would arise in excessive savings as the solvency constraints impel the agent to save. However, once the agent is in the path of $\sigma$ we would find it suboptimal to implement any of this strategies. Optimizing entrepreneurs will start and remain in the path $\sigma$.

Conditional on $z,q$, there is a one-to-one positive relationship between the state $V$ in the economy with long term contracts with the equilibrium asset $a$ owned by the entrepreneur in the centralized asset trading environment. Both can legitimately be thought of as collateral: the first one because of the full commitment of the bank to honor his commitments; the second for much more obvious reasons. We refer to them interchangeably as collateral.

6 Discussion

6.1 Which Firms are Credit Constrained?

In this section we discuss the credit constraints implied the limited commitment of the entrepreneurs. In our environment, credit limits can affect firm behavior in the scale of operations (intensive margin) as well as in the entry and exit (extensive margins). Definitions of these notions of credit constraints arise naturally after we compare the allocations with those in an environment with full enforcement. We then link the likelihood of a firm to be constrained in terms of its age and size.

In a world of perfect commitment the no-default constraint disappears. Once they have signed the contract, entrepreneurs cannot opt out. With no default constraint disappear, regardless of their age and history, active firms, would always use the unrestricted level of working capital $k^u(z,q)$ and the period surplus is equal to the unconstrained profits $\pi(z,q)$. The most obvious notion of inefficiency is that active firms cannot use the optimal level because they cannot commit not to default:

**Definition 1.** Given $(V,z,q)$ a firm is **credit rationed in the intensive margin** if remains active but $k(V,z,q) < k^a(z,q)$

But credit constraints can also affect entry and exit decisions. To see this, we need to lay out the value function for ongoing relationships need only to satisfy the promise keeping constraint and the non-negativity of entrepreneur’s consumption. Therefore, the set of feasible promise utilities $\Gamma^u$ is given by

\[
\Gamma^u(V,z,q) = \left\{ y : Z \times Q \to [0, +\infty) \text{ s.t. } \sum_{z', q'} y(z', q') P(z', q' | z, q) \leq \frac{V}{\beta(1 - \delta)} \right\}
\]  

18 Otherwise, given that $\beta < Q$ the entrepreneur would want to borrow everything $c/(1-q)$ and consume it right away. But once in the underground, the next period he would default again and consume his endowment.
It is not hard to see that since \( q \geq \beta \), the full enforcement allocations requires that after the initial period, as and as long as the firm is in operation, the consumption of the entrepreneur must be equal to zero. Entrepreneurs receive transfers from the bank in the first period and they consume all it right away. \(^9\) Using this feature, the value function for ongoing relationships is simply:

\[
C^u(z, q) = \min \left\{ -L, \left\{ -\pi(z, q) + q(1 - \delta) \sum_{z', q'} C^u(z', q')P(z', q' | z, q) \right\} \right\}
\]  

(44)

Of course, there is no need for randomizations on \( V \). The initialization of contractual relationships is done a similar fashion as above. Because \( \pi(z, q) \geq S(V, z, q) \) and \( \Gamma(V) \subset \Gamma^n(V), \forall V \geq U \), it is obvious that \( C^u(z, q) \leq C(V, z, q) \) all \((V, z, q)\). With commitment, entrepreneurs would receive a higher initial entitlement. The second type of inefficiency arises because some firms that would be created with perfect enforcement are not activated because incentive problems along the expected operation of the plan. This is

**Definition 2.** Given \((z, q)\) a firm is credit rationed in the (extensive) creation margin if simultaneously

\[
\exists y^u \geq U \text{ s.t. } y^u - U + C^u(z, q) + K_0 \leq 0 \text{ and } \forall y \geq U, C(y, z, q) + K_0 > 0
\]

(45)

Similarly, for subsets of exogenous states \( Z \times Q \), some firms might be destructed only because of the enforceability problem. This type of credit rationing can be defined as:

**Definition 3.** Given \((V, z, q)\) a firm is credit rationed in the (extensive) destruction margin if

\[
C(V, z, q) = V - U - L > C^u(V, z, q)
\]

(46)

All three types of credit rationing can be present, depending on the processes \( z \) and \( q \) and the values of \( \alpha, K_0, L \) and \( U \). These notions of rationing are not independent. Rationing in the intensive margin is a necessary but not sufficient condition for the existence of rationing in any of the extensive margins.

Which firms are more likely to be constrained? The larger is the value of \( V \) (\( a \)), the larger is the amount of working capital that banks can advance to the entrepreneur without concerns of triggering his default. Thus, conditional on \( z, q \), firms who poses a larger collateral in the initial period are less constrained. In economies in which the limited liability constraint is binding in the first stages of life of the firm, then younger firms are more likely to have a lower level of \( V \) (\( a \)) than more mature firms. Moreover, again conditional on \( z, q \), for a firm to be larger than other is just equivalent to be less constrained. Now, since the optimal usage of working capital for more productive firms is also higher, one could suspect that they are more likely to be constrained in the intensive margin. But this is not necessarily the case because the optimal policy function specifies that \( G_{z', q'}(\cdot) \) is strictly increasing in \( z' \).

Firms with small levels of collateral are also more likely to be constrained in the destruction margin. Remember that the optimal liquidation policy requires no randomization and that when firms are liquidated, all the assets are seized. The more binding the limited liability constraint, i.e. the lower

\(^9\)The contract under full enforcement resembles the very sad situation described by Horacio Quiroga in *Los Mensú*. 

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V (a), the larger is the set of states in which the next period the firm will be liquidated. Clearly, the lower the productivity, the more likely those firms will be shut down. The higher the productivity of the firm, the less relevant the participation constraint becomes because the less attractive is the underground sector. \(^{20}\) Firms with lower productivities are also more prompted to be constrained in the creation margin.

Higher interest rates are associated with more firms constrained in the extensive margins. In our setting the continuation value of firms depends on interest rates while the scrapping value \(L + U\) does not. In the regions where liquidation/creation decisions are made, the value of continuation is decreasing in the interest rate. Therefore, with higher interest rates less firms will have a continuation value above \(L + U\) for continuation of \(K_0 + U\) for creation. Just because of the limited commitment of the entrepreneur, more firms are destroyed and less are created.

### 6.2 Asymmetries in the Effect of \(q\) Shocks

The realization of \(q\) affects not only the pool of new entrants, but also which of the active firms are liquidated. For entry, there are two opposite effects: higher interest implies a higher initial \(z_0\) for a firm to be active; but given \(z_0\) the newly created firms are more credit constrained as the value \(V_0(z_0, q)\) is decreasing in \(q\). It also turns out that the effect on active firms depends on their value \(V\). The liquidation of firms with smaller \(V\) is more responsive to \(q\). This section extends this discussion.

#### 6.2.1 Asymmetric Effect of \(q\) across active firms

The function \(C\) is not monotone in \(q\). In this model, \(q\) shocks are relevant only for their effects on \(Cc\), the continuation value as the scrapping value of the plant is independent of \(q\). On one hand, the timing between the commitment of working capital and the recollection of output implies that with higher interest rates, the net present value of the profits of the technology is reduced. But on the other hand, \(q\) discounts the present value of the consumption (dividends) for the entrepreneur, and hence higher \(q\) implies lower \(C\); the net effect results of the balance of these two forces. Because the state \(V\) has the dual role of determining both profits and consumption streams, \(C\) is not submodular in \((V, q)\). But for larger values of \(V\), the imperfect enforceability is not binding and the effect on the present value of consumption dominates. Imperfect enforceability implies that, for same productivity \(z\), firms with lower \(V\) (younger and smaller) are more likely to liquidate in periods of high interest rates.\(^{21}\)

Even if in equilibrium the contract never arrives to regions in which \(C\) is increasing in \(q\) (increasing in the real interest rate), interest rates affect more the continuation of small firms, because of the transition dynamics. Firms that have accumulated enough collateral can use more working capital and the total expected discounted value of the surplus from the technology can dominate the opportunity

\(^{20}\)Note the difference with the endowment economies of Kelso-Levine and Alvarez-Jermann [40, 2]. In their case, in autarky the entrepreneur retains his endowment, and therefore higher current productivities make the participation constraints more binding. In our case, the autarky is independent of the productivity \(z\) which of the technology, which can only be used if the entrepreneur remains in the financial system.

\(^{21}\)The intuition clearer for the decentralized trading environment. In any event, interest rates reduce the present value of the firms. But there is also a wealth effect of \(q\): if \(a < 0\) higher \(q\) reduces further the expected present value of the consumption for the agent that continues in the financial system. If \(a > 0\) the effect is the opposite and if \(a\) is high enough, the expected utility of the agent can actually increase with higher interest rates.
cost $L + U$, and the plant continues. However, during the first periods the firm is operating at a much lower scale, and the present value of the surplus of the firm might fall below the liquidation value.

To illustrate this assume that $z_t = z$, $q_t = q$ for all $t$, and $\delta = 0$. Here, a relationship either liquidates immediately or remains active forever; continuing relationships converges to a steady state. During the transition, $V_t$ grows as fast as possible until reaching its steady state value. Let $M(q)$ and $S(q)$ denote respectively, the steady state level of $V$ and the period profits extracted from the technology in steady state. Let $V_0 \leq M(q)$. Obviously the cost $Cc(V_0, q)$ for a bank to maintain a relationship with initial value $V_0$ equals the net present value of the profits (NPVP) minus the NPV of the dividends (NPVC), or

$$Cc(V_0, q) = NPVC(V_0, q) - NPVP(V_0, q) \quad (47)$$

Defining $T^*(V_0, q)$ the number of periods needed to achieve the steady state, i.e.

$$T^*(V_0, q) = \min_{t \in \mathbb{Z}} \left\{ \beta^{-t} V_0 \geq M(q) \right\} \quad (48)$$

we can decompose the previous terms in steady state and transition components:

$$NPVP(V_0) = \sum_{t=0}^{T^*(V_0, q)-1} q^t S(V_0 \beta^t) + q^{T^*(V_0, q)} \frac{S(q)}{1 - q} \quad (49)$$

$$NPVC(V_0) = q^{T^*(V_0, q)-1} [q (\beta^{T^*(V_0, q)-1} V_0 - \beta M(q))] + q^{T^*(V_0, q)} \left[ \frac{q M(q)(1 - \beta)}{1 - q} \right] \quad (50)$$

Notice that $T^*(V_0, q)$ is strictly decreasing in $V_0$, and because for all $t \leq T^*(V_0, q)$ $S(V_0 \beta^t) < S(q)$ the NPVP($V_0$) is strictly decreasing in $V_0$.

If, $\frac{S(q)}{1 - q} > L + U$ then those firms that have achieved the steady state remain active at the given rate $q$. Similarly for firms that are close to the steady state. However, it may easily be the case that for a low value $V_0$ NPVP($V_0$) falls way below $L + U$, and they are optimally liquidated.

This is reinforced by the effect on the NPV of the dividends. One can verify that $V_0 - NPVC(V_0)$ is strictly increasing in $V_0$.

The previous analysis extends to the general case. [write briefly]

### 6.3 Effects on Entry: Cleansing vs. Sullying Effects

Is the average productivity of the resources used by the plant higher or lower with higher interest rates? In general, higher interest rates (lower $q$), heighten the inefficiencies of the timing of production. Therefore, the working capital used by all active plants is reduced, implying, that for all $z$ the marginal product of working capital is higher and therefore its average productivity also higher. This intensive margin effect is reinforced by the extensive margins because surviving and newly created firms have also higher $zs$. Therefore, if we are only concerned with the productivity of the working capital used by the firms, productivity is higher with higher interest rates (recessions). Moreover, extensive margin responses induce cleansing in the pool of firms in terms of $z$.

On the other hand, if we also consider the average productivity of the fixed capital, which in the model is given by plants, then it is very likely that the average productivity declines, depending on the relative strengths of the intensive versus the extensive margins effects, which now operate in the
opposite direction. Moreover, there is a *sullying effect* in the sense that with higher interest rates, the entrants are more credit constrained, a fact that reduces the productivity of the plant as well as the chances of future survival.

7 Conclusion

To the extent that contractual limitations induce non-trivial life-cycle dynamics at the firm level, the economy will be populated by a potentially rich demographic structure of firms. In these contexts, some of these firms are young and will be accumulating collateral. Over time, surviving firms will increase their average level of operations and will become more likely to survive. Before firms achieve maturity—a condition where credit constraints cease to limit their operation—they must endure a period of good performance during which the credit constraints are relaxed. The length of the period before maturity determines the age and size structure of the firms. The model in this paper has shown that the induced firm heterogeneity can be relevant for the dynamics of the aggregate economy. In the presence of shocks to the rate of interest, the model shows that aggregate output can be highly persistent and exhibit non-linearities, in the form of asymmetric and state dependent responses to interest rate shocks.

It is important to emphasize that credit constraints, firm dynamics and their cross-section heterogeneity are implications of optimal dynamic contract. In this paper we have applied the work by Rui Albuquerque and Hugo Hopenhayn [1] to include fluctuations in the interest rate faced by lenders. As in their work, our model implies life-cycle firm dynamics which generates firm heterogeneity at the aggregate level.

The paper focused on the sensitivity of the aggregate economy to fluctuations in the interest rates. Those shocks affect firms depending on their collateral, having implications not only on the mass of active firms but also on the cross-section distribution of firms. That collateral takes time to accumulate at individual level translates into a protracted response at the aggregate level. It also implies asymmetries in upward versus downward movements in output. The model implies that the series liquidation of firms is *concentrated* in the sense that it takes the form of spikes in short periods of time. It must be stressed the quantitative relevance of these feature of aggregate dynamics is linked directly to the length of the firm life-cycle. The non-trivial life-cycle of firms imply that asymmetries at the firm level can translate into asymmetries at the aggregate level.

Studying the implications of interest rate shocks is of practical relevant for most of the actual economies. On the one hand, they are an important source of fluctuations for small open economies. On the other, an important part of the implications of fiscal and monetary policy operates via interest rates. We have shown that the implications of movements in the interest rate depends on the state of the economy. The model suggests that the response to rise in interest rates is stronger when more firms are credit constrained and the opposite for a decline in the interest rate. A firm conclusion on this matter would, however, require to explicitly introduce other shocks, e.g. total factor productivity and their joint dynamics with interest rates.
Appendices

A Proofs

A.1 Preliminaries and Notation

Let \( R_U \equiv [0, \infty) \times Z \times Q \rightarrow R \), \( f \) bounded and continuous in the first argument, and the norm \( ||f|| = \max_{x, z, q} \sup_{x \in R_U} |f(x, z, q)| \). Obviously, \( (F, ||\cdot||) \) is a normed linear space. With the metric \( d(f, g) \equiv ||f - g|| \), \( (F, d) \) is a Banach space. We denote the operators \( T, Cc \) that take any function \( f \) in \( F \), and return functions which for any \( (V, z, q) \in \mathbb{R}^U \times Z \times Q \), have the values

\[
Cc(f)(V, z, q) = \min_{y \in \Gamma(V, z, q)} \left\{ -S(V, z, q) + q \left[ V - \beta(1 - \delta) \sum_{z', q' \in Z \times Q} y(z', q')P(z', q'|z, q) \right] + q(1 - \delta) \sum_{z', q' \in Z \times Q} f(y(z', q'), z', q)P(z', q'|z, q) \right\}
\]

and

\[
T(f)(V, z, q) = \min_{(\lambda, V^0, V^1) \in \mathfrak{R}(V, z, q)} \left\{ \lambda(V^0 - U - L) + (1 - \lambda)Cc(f)(V^1, z, q) \right\}
\]

\( l(x, z, q) \equiv -L - U + x \) is the cost of the bank for liquidating the firm.

We make use of convex and submodular functions. Comprehensive treatment of these topics are in Rockafellar [50] and Topkis [53]. We summarize the results used here:

Let the correspondence (multivalued mapping) \( \partial f \) denote the subdifferential of \( f \), where \( \partial f(x) \) denotes the set of subgradients of \( C \) at \( x \). \( f \) is differentiable at \( x \) if \( \partial f(x) \) is a singleton. A convex function \( f \) is almost everywhere differentiable: the left and right derivatives, \( f'_L, f'_R \) satisfy \( f'_L(x) \leq f'_R(x) \) all \( x \in \mathbb{X} \). \( f'_L \) is left-continuous and \( f'_R \) is right continuous. For \( x_1 < x < x_2 \), then \( f'_L(x_1) \leq \partial f(x) \leq f'_R(x_2) \); if \( f \) is strictly convex, the two inequalities are strict.

Let \( \leq \) be a partial ordering defined on a set \( X \). For \( x_1, x_2 \in X \) the operations \( \vee \) and \( \wedge \) are defined as \( x_1 \vee x_2 = \inf \{ x | x_1 \leq x \text{ and } x_2 \leq x \} \) and \( x_1 \wedge x_2 = \sup \{ x | x_1 \leq x \text{ and } x \leq x_2 \} \). A set \( X \) with the partial ordering \( \leq \) is a lattice if for all \( x_1, x_2 \in X \) then both \( x_1 \vee x_2 \) and \( x_1 \wedge x_2 \) are in \( X \). Given two sets \( A, B \subseteq X \), \( B \) is said to be higher than \( A \), denoted \( A \subseteq B \) if for \( x_1 \in A \), \( x_2 \in B \), \( x_1 \vee x_2 \in B \) and \( x_1 \wedge x_2 \in A \). We say that \( B \) is strictly higher, \( A \subset B \) if the previous condition holds and \( A \) and \( B \) are disjoint. A function \( f : X \rightarrow R \) is submodular if for all \( x_1, x_2 \in X \), \( f(x_1 \vee x_2) + f(x_1 \wedge x_2) \leq f(x_1) + f(x_2) \). \( f \) is said to be strictly submodular if for all unordered \( x_1, x_2 \in X \), i.e. \( x_1 < x_1 \vee x_2 \) and \( x_2 \leq x_1 \vee x_2 \) and \( x_1 < x_1 \wedge x_2 \) and \( x_2 < x_1 \wedge x_2 \), then \( f(x_1 \vee x_2) + f(x_1 \wedge x_2) < f(x_1) + f(x_2) \). A function \( f \) is (strictly) supermodular in \( X \) if \(-f \) is (strictly) submodular in \( X \).

A.2 Properties of the Value Function

Proof of Proposition 2, First Part

The cost of liquidation, \(-L - V - U \) is an unbounded function and for any \( f \in F \) \( Cc(\cdot) \) is also unbounded. We use a restricted version of the recursive problem, and later verify that the restriction is not binding.

Consider the Bellman Equation in the text, but impose an arbitrary upper bound, \( B < \infty \) on the admissible set of promise utilities. For any \((V, z, q) \in [0, B] \times Z \times Q \), the value function \( C^B \) in this restricted problem must satisfy the same Bellman Equation, but with the additional restriction that \( G_{z, q}(V, z, q) \leq B \). Define \( T^B \) the functional operator associated with the Bellman Equation is monotone and that because \( q(1 - \delta) < 1 \), discounting holds. Thus \( T^B \) is a contraction on \((F^B, d)\), where \( F^B \) restricts the domain of the functions to \([0, B]\).

Below, we use the assumption of \( q < \beta \) to find a \( B < \infty \) such that for any \( V \in R_U \) the constraint does not bind. Using the unique fixed point of \( T^B \), we can also write the formula for the unique fixed point \( C \) of the operator \( T \), for any \( V \in R_U \).
Convexity holds by definition of optimal lotteries. Since $S$ is strictly increasing in $z$, so is the fixed point. The details are standard and omitted.

**Proposition 3:** $\partial C \leq 1$ and $\partial Cc(V, z, q) \leq q$

We first show the second part. Pick $z, q$ and let $V_0 < V_1$ and $f \in F$. Given $f$ let also $y^0$ be the optimal policies given $(V, z, q)$. $y^0 \in \Gamma(V_0, z, q)$, and therefore, if $Cc^0(f)(V_1, z, q)$ denotes the continuation value at state $(V, z, q)$ but restricted to use the policy $y^0$, then $Cc^0(f)(V_1, z, q) \geq Cc(f)(V_1, z, q)$. Therefore

$$Cc(f)(V_1, z, q) - Cc(f)(V, z, q) \leq Cc^0(f)(V_1, z, q) - Cc(f)(V_0, z, q)$$

$$= -S(V_1, z, q) + qV_1 - [-S(V_0, z, q) + qV_0]$$

$$\leq q(V_1 - V_0)$$

as desired. The first part is immediate. □

**Proposition 4**

We already know that the fixed point is convex. Consider a function $f$ convex and submodular in $(V, z)$. For simplicity assume that $\partial f$ is a singleton $f_1$ everywhere. Fix $V, q_0$, two states $z_0 < z_1$ and consider two other $z^0 \leq z'$ in which there is not liquidation, given next interest rate $q_1$. Then,

$$\partial Cc(f)(V, z_0, q) = -\frac{\partial S(V, z_0, q_0)}{\partial V} + q_0 + q_0(1 - \delta)f_1(G_{z^0, q_1}(V, z_0, q_0), z^0, q_1)$$

$$\geq -\frac{\partial S(V, z_1, q_0)}{\partial V} + q_0 + q_0(1 - \delta)f_1(G_{z^0, q_1}(V, z_0, q_0), z^0, q_1)$$

$$\geq -\frac{\partial S(V, z_1, q_0)}{\partial V} + q_0 + q_0(1 - \delta)\frac{f_1(G_{z^0, q_1}(V, z_1, q_0), z^0, q_1)}{f_1(G_{z^0, q_1}(V, z_0, q_0), z^0, q_1)}$$

$$= \partial Cc(f)(V, z_1, q)$$

The first equality derives by using the envelope condition and the assumption that $G_{z^0, q_1}(V, z_0, q_0) > U$; the second from the fact that $z_1 > z_0$ and that $S$ is submodular. The second inequality requires more argument. First, our assumption that $G_{z^0, q_1}(V, z_0, q_0) > U$ and $G_{z^1, q_1}(V, z_0, q_0) > U$, then optimality requires $f_1(G_{z^0, q_1}(V, z_0, q_0), z^0, q_1) = f_1(G_{z^1, q_1}(V, z_0, q_0), z^0, q_1)$; but since $f$ is submodular, then, $(G_{z^0, q_1}(V, z_0, q_0), z^0, q_1) \leq (G_{z^1, q_1}(V, z_0, q_0), z^0, q_1)$. Therefore, given submodular $f$, the optimal continuation policies are increasing in $z'$; but since $P(z', z)$ is monotone in $z$, the non-negativity constraint of consumption requires that $(G_{z^0, q_1}(V, z_1, q_0), z^0, q_1) \leq (G_{z^0, q_1}(V, z_0, q_0), z^0, q_1)$ all $z'$. Using this, the second inequality follows from the convexity of $f$. The last equality holds again from using the envelope condition, which completes the argument for $Cc(f)$ to be submodular.

A.3 Continuation Policies

From the convexity of $C$ it follows that he first order conditions are necessary and sufficient for the continuation policies. They are given by

$$\beta(1 - \mu_2(V, z, q)) + \mu_1(z', q'; V, z, q) \in \partial C(G_{z', q'}(V, z, q), z', \{q'\}, V) \forall (z', q') \in Z \times Q$$

$$\mu_1(z', q'; V, z, q) \geq 0; G_{z', q'}(V, z, q) \geq U; \text{ and at least one with equality } \forall (z', q') \in Z \times Q$$

$$\mu_2(V, z, q) \geq 0; \beta(1 - \delta) \sum_{z', q'} G_{z', q'}(V, z, q) P(z', q'|z, q) \leq V; \text{ and at least one with equality}$$

where $\mu_1(z', q'; V, z, q)$ and $\mu_2(V, z, q)$ are $\# Z \times \# Q + 1$ (scaled) Kuhn-Tucker multipliers.
Proof of Proposition 5: $G$ is non-decreasing in $V$

Fix $(z, q)$ and let $V_j < V_1$. For all those $z', q'$ s.t. $\mu_1(z', q', V, z, q) > 0$ the proposition holds trivially as $G_{z', q'}(V, z, q) = G_{z', q'}(V_1, z, q) = U$. Also, if $\mu_2(V_0, z, q) = \mu_2(V_1, z, q) = 0$ then $G_{z', q'}(V_0, z, q) = G_{z', q'}(V_1, z, q)$, as the choice is not restricted by $V_0$, $V_1$. Assume now that $\mu_2(V_0, z, q) > 0$ and hence $\beta(1-\delta)\sum_{z', q'} G_{z', q'}(V, z, q)P(z', q'|z, q) = V_0$. Assume to the contrary of the proposition that $\exists z^*, q^*$ s.t. that all the optimal choice for $z^*, q^*$ is strictly at state $(V_0, z, q)$ than given $V_1, z, q$, i.e. $G_{z^*, q^*}(V_1, z, q) \not\subseteq G_{z^*, q^*}(V_0, z, q)$. If that is the case, necessarily $C(\cdot, z^*, q^*)$ cannot have constant slope in the entire region $[\max\{G_{z^*, q^*}(V_1, z, q)\}, \min\{G_{z^*, q^*}(V_0, z, q)\}]$. This implies that $\partial C(G_{z^*, q^*}(V_1, z, q), z^*, q^*) \not\subseteq \partial C(G_{z^*, q^*}(V_0, z, q), z^*, q^*)$

Using the FOC for $(V_0, z, q)$ and $(V_1, z, q)$ is direct to conclude that $\partial C(G_{z^*, q^*}(V_1, z, q), z^*, q^*) \not\subseteq \partial C(G_{z^*, q^*}(V_0, z, q), z^*, q^*)$

all $z', q'$.

Since $\mu_2(V, z, q) > 0$ it cannot be the case that there is a policy selection $G$ such that $G_{z^*, q^*}(V_0, z, q) \leq G_{z^*, q^*}(V_0, z, q)$ for all $z', q'$ and that for some inequality is strict (otherwise the FOC could be satisfied with $\mu_2(V_0, z, q) = 0$). Therefore, it must be the case that $G_{z^*, q^*}(V_0, z, q) \geq G_{z^*, q^*}(V_1, z, q)$ all $z', q'$ and for some $z', q'$ the inequality being strict. But that contradicts the optimality of $G_{z^*, q^*}(V_1, z, q)$ as $G_{z^*, q^*}(V_0, z, q) \in \mathbb{G}(V_1, z, q)$

\[\Box\]

Proof of Proposition 6

Let $(V, z, q)$ be the initial state, fix any $q' \in Q$ and let $z_0 < z_1$. If both $\mu_1(z_1, q'|V, z, q) > 0, \mu_1(z_0, q'|V, z, q) > 0$ the conclusion trivially holds. Consider the case $\mu_1(z_1, q'|V, z, q) = \mu_1(z_0, q'|V, z, q) = 0$. The FOC imply that $\partial C(G_{z_0, q'}((V, z, q), z_0, q')) \subseteq \partial C(G_{z_1, q'}((V, z, q), z_1, q'))$. The proposition holds from the convexity and submodularity of $C$. In the regions where $G_{z_0, q'}(\cdot) > U$, the probability of continuation is positive, and $C\mathbb{C}$ is strictly decreasing in strictly submodular in $V, z$, and therefore the order is strict.

Proof of Proposition 8

The operator in the RHS defines a contraction on the space of bounded functions; hence $D < \infty$ and is unique. Since $M$ is strictly increasing in both arguments and $P(\cdot, \cdot|z, q)$ is monotone, then the fixed point $D$ is also strictly increasing in both arguments too. That $D(\bar{z}, \bar{q}) = M^*(\bar{z}, \bar{q})$ follows immediately from $D(\bar{z}, \bar{q}) > \beta(1-\delta)D(z', q')$

all $z', q'$.

Proof of Proposition 9

For the second part, because $q \geq \beta$, the FOC indicate that either $G_{z^*, q^*}(V, z, q) = M(z', q')$ or that $\mu_2(V, z, q) > 0$. By convexity policies outside these regions cannot intersect with the region where the probability of liquidation is positive, because in the latter region $C$ has a constant slope lower than $q$. The argument for the strict submodularity of $C\mathbb{C}$ on $(V, z)$ follows the same lines.

Proof of Proposition 10

Assuming that $\lambda \in (0, 1)$. Since $C\mathbb{C}$ has a slope lower or equal to $q < 1$, it is optimal to make $V^1$ as large as possible. Then, if $\lambda \in (0, 1)$, then $V^0 = U$ and $\lambda = \frac{\lambda^V}{\lambda^V + \frac{V}{\lambda^V}}$. Substituting this expression, and optimizing with respect to $V^1$ gives the expression in the proposition. The LHS is strictly decreasing as $\partial C\mathbb{C} \leq q$, and the RHS is increasing. Then if there is a finite $V^1$ satisfying the equation, it is unique. The conclusion that such $V^1$ must be necessarily less or equal to $M(\bar{z}, \bar{q})$ follows from the fact that $C$ has constant slope $q < 1$ for $V > M^*(\bar{z}, \bar{q})$. Thus, if no value less or equal to $M(\bar{z}, \bar{q})$ can satisfy the expression, then a degenerate solution $V^1 \to \infty$ and $\lambda \to 1$ is optimal.

End of Proof of Proposition 2

When the upper-bound $B \geq M(\bar{z}, \bar{q})$, then, the optimal continuation policies in the restricted problem coincide with those in the general problem. Then, the unique fixed solution of $C$ is as follows: for any $(V, z, q)$, if $V \in [U, B]$, $C(V, z, q) = C^B(V, z, q)$. For $V > B$,
\[ C(V, z, q) = \min \left\{ -L + V - U, \, Ce^n(V, z, q) + q(V - B) \right\} \]

**Proof of Proposition 12**

If \( \mu_1 > 0 \) then \( G_{z', q'}(V, z, q) = U \), and the result holds trivially. Assume then \( \mu_1 = 0 \). First consider the case with assume \( \mu_2 = 0 \) (the limited liability does not bind). In the first order conditions, the first term is the discount factor \( \beta \) while the second is the would be a constant in the region of randomizations, and strictly increasing outside of it until achieving \( q \). If the discount factor is strictly higher, then the optimal plan requires \( G_{z', q'} = U \) while if it is strictly lower, then \( G_{z', q'}(V, z, q) = V'(z', q') \). In, zero probability even that \( \beta \in \partial C(V'(z', q'), z', q') \), then select either extreme, say \( U \). (and therefore the agent consumes in the present date). If limited liability holds of equality, either extreme can be chosen, as the entrepreneur and the bank are indistinct in the transfers. If the limited liability binds, repeat the argument with \( \beta(1 - \mu_2) \).

**Proof of Proposition 13**

As long as \( K > L \), it is easy to prove that it is never optimal to create a firm that will be liquidated with positive probability in the very first period. Then, necessarily newly created firms satisfy \( Ce(V_0, z_0, q_0) + K_0 = 0 \). The first part of the proposition follows as \( Ce \) is globally strictly decreasing with respect to \( z \), and in the relevant region, strictly increasing in \( V \). Now for a given \( L \), for a large enough \( K_0 > L \), there exist \( 0 < \delta_K \) and \( 0 < \beta_K < 1 \) s.t. for all \( 0 < \delta < \delta_K \) and \( \beta < [\beta_K, 1) \), it is the case that whenever \( Ce(V_0, z, q) = -K_0 \) then

\[ q \left[ V_0 + (1 - \delta) \sum_{z', q'} [C(G_{z', q'}(V_0, z, q), z', q') - \beta G_{z', q'}(V_0, z, q)]P(z', q'|z, q) \right] < 0 \]

This conditions simply says that the loan cannot be fully recovered from the repayment in the first period of the firm. In this case, higher \( q \) unambiguously reduces the value \( Ce(V_0, z, q) \) for all \( V \) and since \( Ce_0, z, q) \) is strictly increasing –in the region of initialization– then the breaking even condition requires \( V_0(z, q) \) to increase.

**Proof of Proposition 14**

Fix \( (V, z, q) \) and assume that \( V \geq V_0(z, q) \). Notice that independently of the value of \( \mu_2(V, z, q) \), for those realizations \((z', q') \) such that \( \mu_1(z', q'|V, z, q) > 0 \) the optimal policy is \( G_{z', q'}(V, z, q) = U \) and the proposition follows directly.

¿From now on, assume that \( \mu_1(z', q'|V, z, q) = 0 \). Consider first the case in which \( \mu_2(V, z, q) = 0 \). Then the first order condition implies that

\[ \beta \in \partial C(G_{z', q'}(V, z, q), z', q') \]

but since \( C \) is convex, then for any \( \epsilon > 0 \)

\[ \partial C(G_{z', q'}(V, z, q) + \epsilon, z', q') \geq \beta > 0 \]

and therefore, a positive increment in the utility entitlement of the entrepreneur induces a strictly positive increment in the cost for the entrepreneur, as claimed in the proposition. Finally, consider the case \( \mu_2(V, z, q) > 0 \). The envelope condition is

\[ \partial Ce(V, z, q) = - \frac{\partial S(V, z, q)}{\partial V} + q(1 - \mu_2(V, z, q)) \]

Since \( Ce \) is convex and \( V \geq V_0(z, q) \) then \( \partial Ce(V, z, q) \geq \partial Ce(V_0(z, q), z, q) > 0 \). Thus, necessarily \( \mu_2(V, z, q) < 1 \). But then, from the FOC and for any \( \epsilon > 0 \)

\[ \partial C(G_{z', q'}(V, z, q) + \epsilon, z', q') \geq \beta(1 - \mu_2(V, z, q)) > 0 \]

and the argument is complete.
Allocations in $\mathcal{P}(\mathcal{A}, \mathcal{B}, p^*)$ replicate $\sigma$

**Deterministic case** Assume that there is an allocation $\{\bar{a}_t, \bar{h}_t, \bar{k}_t\}$ that dominates $\sigma$ and satisfies $\mathcal{P}(\mathcal{A}, \mathcal{B})$. Consider first the case where $q = \beta$. There is a $t_0$ such that $\sum_{t \geq t_0} \beta^t c_t^\sigma < \sum_{t \geq t_0} \beta^t \bar{c}_t$, with $\bar{c}_t > c_t^\sigma$. Let $\bar{e}_t = \bar{c}_t - c_t^\sigma$. By construction $\sum_{t \geq t_0} e_t > 0$. Therefore, it is the case that the optimal long term contract at time $t_0$ could have achieved the utility $\sum_{t \geq t_0} \beta^t c_t$ with the resources $A_t - \sum_{t \geq t_0} e_t = A_t = c_t^\sigma$ which is absurd given the definition of $c_t^\sigma$. But since in the case of $\beta = q$ there are infinitely many solutions to the recursive optimal financial relationships, then there can also be infinitely many optimal solutions to $\mathcal{P}(\mathcal{A}, \mathcal{B})$. Consider now the case $q > \beta$: now the allocation solving the recursive long term contract is unique. By discounting, the far future can be neglected, so, without loss of generality, we can assume that there is a finite $n$, such that $\bar{c}_t = \bar{a}_t, \bar{h}_t, \bar{k}_t t \geq n$. That $\bar{c}$ dominates $\sigma$ means that $\sum_{0 \leq t \leq n} \beta^t c_t^\sigma < \sum_{0 \leq t \leq n} \beta^t \bar{c}_t$. If $\bar{c}_t \geq c_t^\sigma \forall t$ and at least in one period the inequality is strict, we obtain a contradiction with the construction of $\mathcal{A}$. The only possibility is that there are at least two dates $\tau_1 < \tau_2$ such that the in one date the consumption is lower while in the other it is higher. Because $\sigma$ is the fastest repayment, then, $c_{\tau_1} > c_{\tau_2}$. Then, $\bar{a}_{\tau_1+1} = A_{\tau_1+1} - q[c_{\tau_1} - c_{\tau_2}^\sigma] > A_{\tau_1+1}$, so $\bar{c}$ is not feasible in $\mathcal{P}(\mathcal{A}, \mathcal{B})$.

References


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