Problem set #3
Econ 308, Money and Banking
Solutions
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Exercise 1
Let’s compute the duration gap
\[ Dur_{gap} = Dur_a - \left( \frac{1}{A}Dur_r \right) = 2 - \frac{42500}{50000} \times 3.7 = -1.1325 \]

The change in the market value of net worth as a percentage of asset is calculated as:
\[ \%\Delta NW = -Dur_{gap} \times \frac{\Delta r}{1+r} \]
so in our example, \( \%\Delta NW < 0 \) if \( \Delta r < 0 \).
So if interest rate goes down, the net worth decreases. We are not better off then if interest rate goes down.

Exercise 2
1. The borrower needs to pay an interest rate so that expected payoffs are identical if investors invest in either project.

(note: what he really needs is to give a higher expected payoff to attract investors, but he would be better off by giving the smallest one, which is the one for which the expected payoffs are equal. In that case, investors are indifferent between investing in either project since they are risk-neutral).

\[ 1 + i_s = pf + (1 - p)(1 + i_r) \]
i.e.
\[ i_r = \frac{1 + i_s - pf}{1 - p} - 1 \]

\[ \frac{\partial i_r}{\partial s} \geq 0, \quad \frac{\partial i_r}{\partial f} \leq 0, \quad \frac{\partial i_r}{\partial p} \geq 0. \]

2. firm 1, \( i_r = \frac{1 + i_s - pf}{1 - p} - 1 = \frac{1 + 0.05 - (0.1 \times 0.95)}{1 - 0.1} - 1 = 6.1111 \times 10^{-2} \)

firm 2, \( i_r = \frac{1 + i_s - pf}{1 - p} - 1 = \frac{1 + 0.05 - (0.2 \times 0.7)}{1 - 0.2} - 1 = 0.1375 \)

Exercise 3
1.
\[ E(i_t^1/i_0^1 = i_h) = i_h P(i_t^1 = i_h/i_0^1 = i_h) + i_t P(i_t^1 = i_t/i_0^1 = i_t) \]
\[ = 0.065 \times 0.65 + 0.025 \times 0.35 = 0.051 \]

Note: \( i_h^1 \) denotes the yield to maturity of a one period bond starting at period 0.
\[ P(\bar{i}_1 = \bar{i}_h/\bar{i}_0 = \bar{i}_h) \text{ is the probability that the yield to maturity of a one period bond starting at period 1 is } \bar{i}_h \text{ given that you know that the current yield to maturity of a one period bond } \bar{i}_0 \text{ is } \bar{i}_h. \]

Under PET, we have:
\[
(1 + \bar{i}_0)(1 + E(\bar{i}_1)) = (1 + \bar{i}_0)^2
\]

The forward rate for next period is just \( E(\bar{i}_1/\bar{i}_0 = \bar{i}_h) \).

and we find the yield to maturity of a two period bond as:
\[
\bar{i}_0^2 = [(1 + \bar{i}_0^2)(1 + E(\bar{i}_1^2))]^{\frac{1}{2}} - 1
\]

\[
= [(1 + 0.065)(1 + 0.051)]^{\frac{1}{2}} - 1 = 0.57
\]

2. \( E(\bar{i}_1/\bar{i}_0 = \bar{i}_h) = \bar{i}_h P(\bar{i}_1 = \bar{i}_h/\bar{i}_0 = \bar{i}_h) + \bar{i}_P(\bar{i}_1 = \bar{i}_h/\bar{i}_0 = \bar{i}_h) = 0.065 \times 0.15 + 0.025 \times 0.85 = 0.31 \)

\[
\bar{i}_0^2 = [(1 + \bar{i}_0^2)(1 + E(\bar{i}_1^2))]^{\frac{1}{2}} - 1
\]

\[
= [(1 + 0.025)(1 + 0.031)]^{\frac{1}{2}} - 1 = 2.80 \times 10^{-2}
\]

3. \( E(\bar{i}_1/\bar{i}_0 = \bar{i}_h) = \bar{i}_h P(\bar{i}_h/\bar{i}_0 = \bar{i}_h) + \bar{i}_h P(\bar{i}_h/\bar{i}_0 = \bar{i}_h) + \bar{i}_h P(\bar{i}_h/\bar{i}_h) \bar{i}_h P(\bar{i}_h/\bar{i}_h)
\]
\[
= 0.065 \times (0.65^2 + 0.15 \times 0.35) + 0.025 \times (0.85 \times 0.35 + 0.35 \times 0.65) = 0.44
\]

or more rapidly, we have:
\[
E(\bar{i}_1/\bar{i}_0 = \bar{i}_h) = E(\bar{i}_1/\bar{i}_0 = \bar{i}_h) P(\bar{i}_h/\bar{i}_h) + E(\bar{i}_1/\bar{i}_0 = \bar{i}_h) P(\bar{i}_h/\bar{i}_h) = 0.051 \times 0.15 + 0.031 \times 0.85 = 0.34
\]

The yield to maturity of a three period bond is:
\[
\bar{i}_0^3 = [(1 + \bar{i}_h)(1 + E(\bar{i}_1/\bar{i}_0 = \bar{i}_h))(1 + E(\bar{i}_2/\bar{i}_0 = \bar{i}_h))^{\frac{1}{3}} - 1
\]
\[
= [(1 + 0.065)(1 + 0.051)(1 + 0.044)]^{\frac{1}{3}} - 1 = 5.3308 \times 10^{-2}
\]

\[
\bar{i}_0^3 = [(1 + \bar{i}_h)(1 + E(\bar{i}_1/\bar{i}_0 = \bar{i}_h))(1 + E(\bar{i}_2/\bar{i}_0 = \bar{i}_h))^{\frac{1}{3}} - 1
\]
\[
= [(1 + 0.025)(1 + 0.031)(1 + 0.034)]^{\frac{1}{3}} - 1 = 3.8313 \times 10^{-2}
\]

**Exercise 4**

1. According to the PET, when interest rates are high and are expected to decline (recession), the yield curve is downward sloping. When interest rates are expected to increase (boom), the yield curve is upward sloping.

2. Do it!