Problem set #4  
Econ 308, Money and Banking  
Solutions  
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Exercise 1  
1.(a) $\pi^e = P_L \pi_L + P_M \pi_M + P_H \pi_H$  
   $= 0.3 \times 0.01 + 0.5 \times 0.04 + 0.2 \times 0.08 = 0.39$

(b)  
   $i_{r,L} = \frac{i_n - \pi_L}{1 + \pi_L} = \frac{0.1 - 0.01}{1 + 0.01} = 8.91 \times 10^{-2}$
   
   $i_{r,M} = \frac{i_n - \pi_M}{1 + \pi_M} = \frac{0.1 - 0.04}{1 + 0.04} = 5.76 \times 10^{-2}$
   
   $i_{r,H} = \frac{i_n - \pi_H}{1 + \pi_H} = \frac{0.1 - 0.08}{1 + 0.08} = 1.85 \times 10^{-2}$

(c)  
   $i^e_r = P_L i_{r,L} + P_M i_{r,M} + P_H i_{r,H}$
   $= 0.3 \times 8.91 \times 10^{-2} + 0.5 \times 5.76 \times 10^{-2} + 0.2 \times 1.85 \times 10^{-2}$
   $= 0.05923$

(d) The true formula is:

   $i_r = \frac{i_n - \pi^e}{1 + \pi^e}$

   i.e.

   $i_r (1 + \pi^e) = i_n - \pi^e$

   $i_r (1 + \pi^e) = i_n - \pi^e$

   $i_r = i_n - \pi^e - i_r \pi^e$

   so the approximation $i_r = i_n - \pi^e$ is a good approximation if $i_r \pi^e \approx 0$.

2. We still use our favorite formula:

   $i_r = \frac{i_n - \pi^e}{1 + \pi^e}$

   and rearrange it to get:

   $\pi^e = \frac{i_n - i_r}{1 + i_r} = \frac{0.12 - 0.05}{1 + 0.05} = 6.66 \times 10^{-2}$
Exercise 2

\[ E(\hat{i}_1^{i} / i_0^{i} = \hat{i}_h) = i_h P(\hat{i}_1^{i} = \hat{i}_h / i_0^{i} = \hat{i}_h) + \hat{i}_i P(\hat{i}_1^{i} = \hat{i}_i / i_0^{i} = \hat{i}_i) \]

\[ = 0.06 \times 0.65 + 0.02 \times 0.35 = 0.46 \]

The forward rate for next period is \( f_{1,2} = E(\hat{i}_1^{i} / i_0^{i} = \hat{i}_h) + l_p = 0.046 + 0.015 = 0.061 \)

Under liquidity premium theory,

\[ (1 + \hat{i}_1^{i})(1 + f_{1,2}) = (1 + \hat{i}_0^{i})^2 \]

and we find the yield to maturity of a two period bond as:

\[ \hat{i}_0^{i} = \left[ (1 + \hat{i}_1^{i})(1 + f_{1,2}) \right]^\frac{1}{2} - 1 \]

\[ = \left[ (1 + 0.06)(1 + 0.061) \right]^\frac{1}{2} - 1 \]

\[ = 0.060 \]

2. \( E(\hat{i}_1^{i} / i_0^{i} = \hat{i}_i) = i_h P(\hat{i}_1^{i} = \hat{i}_h / i_0^{i} = \hat{i}_h) + \hat{i}_i P(\hat{i}_1^{i} = \hat{i}_i / i_0^{i} = \hat{i}_i) \)

\[ = 0.06 \times 0.2 + 0.02 \times 0.8 = 0.28 \]

\( f_{1,2} = E(\hat{i}_1^{i} / i_0^{i} = \hat{i}_i) + l_p = 0.028 + 0.015 = 0.43 \)

\[ \hat{i}_0^{i} = \left[ (1 + \hat{i}_1^{i})(1 + f_{1,2}) \right]^\frac{1}{2} - 1 \]

\[ = \left[ (1 + 0.02)(1 + 0.043) \right]^\frac{1}{2} - 1 \]

\[ = 0.031 \]

3. \( E(\hat{i}_2^{i} / i_0^{i} = \hat{i}_h) = i_h P(\hat{i}_2^{i} / i_h) + i_h P(\hat{i}_h / \hat{i}_h) + i_i P(\hat{i}_h / \hat{i}_h) + i_i P(\hat{i}_i / \hat{i}_h) + i_i P(\hat{i}_i / \hat{i}_i) \)

\[ = 0.06 \times (0.65^2 + 0.2 \times 0.35) + 0.02 \times (0.80 \times 0.35 + 0.35 \times 0.65) = 0.39 \]

\( f_{2,3} = E(\hat{i}_2^{i} / i_0^{i} = \hat{i}_h) + l_p = 0.039 + 0.015 = 0.54 \)

\[ \hat{i}_3^{i} = \left[ (1 + \hat{i}_h)(1 + f_{1,2})(1 + f_{2,3}) \right]^\frac{1}{2} - 1 \]

\[ = \left[ (1 + 0.06)(1 + 0.061)(1 + 0.054) \right]^\frac{1}{2} - 1 \]

\[ = 0.058 \]
or more rapidly, we have:

\[
E(i_2^1/i_0^1 = \hat{i}_i) = E(i_2^1/i_0^1 = \hat{i}_k)P(\hat{i}_k/\hat{u}_i) + E(i_2^1/i_0^1 = \hat{i}_i)P(\hat{u}_i/\hat{u}_i)
\]
\[
= 0.046 \times 0.2 + 0.028 \times 0.8 = 0.316
\]

so \( f_{2,3} = E(i_2^1/i_0^1 = \hat{i}_i) + l_p = 0.0316 + 0.015 = 0.466 \)

\[
i_3^1 = [(1 + \hat{i}_i)(1 + f_{1,2})(1 + f_{2,3})]^{\frac{1}{\gamma}} - 1
\]
\[
= [(1 + 0.02)(1 + 0.043)(1 + 0.0466)]^{\frac{1}{\gamma}} - 1
\]
\[
= 0.036
\]

4. It helps us to predict that when short-term interest rates are low, yield curves are more likely to slope upward whereas when interest rates are high, yield curves are more likely to be inverted.
Exercise 3

ftbpF139pt97.375pt0ptFigure
Source: cmfn.com
This shows the yield curves for the US this week. It has the typical upward slope.
According to the PET, it would suggest that short-term interest are expected to rise in the future.
According to the liquidity premium theory, even if short-term interest rates stay the same on average, long-term interest rate are typically above short-rate because of the liquidity premium.

Exercise 4
1. The entrepreneurs wants to maximize his income, i.e. he wants to maximize \((1-\phi)\pi(z,r)\).
But since \(\phi\) is fixed, maximizing \((1-\phi)\pi(z,r)\) is equivalent to maximize \(\pi(z,r)\).
The choice variable of the entrepreneur, taken \(r\) as given, is the level of capital used in production.
So he will choose \(k\) to max his profit, i.e. he solves the following pb:

\[
\max_k zk^\alpha - (1 + r)k
\]

The first-order condition gives:

\[
\alpha zk^{\alpha-1} - (1 + r) = 0
\]
i.e.

\[
k^* = \left(\frac{1 + r}{\alpha z}\right)^{\frac{1}{\alpha - 1}}
\]

So the total demand for funds when the interest rate is \(r\) is \(N_r \times k^* = \left(\frac{1 + r}{\alpha z}\right)^{\frac{1}{\alpha - 1}}\)
so

\[
\pi(z,r) = z \left(\frac{1 + r}{\alpha z}\right)^{\frac{1}{\alpha - 1}} - (1 + r) \left(\frac{1 + r}{\alpha z}\right)^{\frac{1}{\alpha - 1}}
\]

2. The optimization problem for the lhs is as followed:

\[
U(a) = \max_{c_o,c_1} \sqrt{c_o} + \beta \sqrt{c_1}
\]
\text{s.t. } c_o + s = a
\]
\[
c_1 = s(1 + r) + \phi \pi(z,r)
\]
Substituting the constraints into the objective function, the optimization problem becomes:

\[ U(a) = \max_{0 \leq s \leq a} \sqrt{a-s} + \beta \sqrt{s(1+r)} + \phi \pi(z,r) \]

The first order condition is:

\[ \frac{-1}{2\sqrt{a-s}} + \beta \frac{1+r}{2\sqrt{s(1+r)} + \phi \pi(z,r)} = 0 \]

\[ \frac{1}{\sqrt{a-s}} = \beta \frac{1+r}{\sqrt{s(1+r)} + \phi \pi(z,r)} \]

\[ \frac{\sqrt{s(1+r)} + \phi \pi(z,r)}{1+r} = \beta \sqrt{a-s} \]

\[ s(1+r) + \phi \pi(z,r) = \beta^2 (a-s) \]

\[ s^* = \frac{-\phi \pi(z,r) + \beta^2 a (1+r)^2}{(1+r)(1+\beta^2 + \beta^2 r)} \]

So the total supply for funds for a given interest rate \( r \) is

\[ N_k \times s^* = \frac{-\phi \pi(z,r) + \beta^2 a (1+r)^2}{(1+r)(1+\beta^2 + \beta^2 r)}. \]

3.

\[ \frac{\partial k^*}{\partial z} = \frac{\partial}{\partial z} \left( \frac{1+r}{\alpha z} \right)^{\frac{\alpha}{\alpha-1}} = -\frac{(1+r)^{\frac{\alpha}{\alpha-1}}}{(\alpha-1)z} > 0 \]

because \( 0<\alpha < 1 \).

\[ \frac{\partial s^*}{\partial z} = \frac{\partial}{\partial z} \left( -\phi \left( z \left( \frac{1+r}{\alpha z} \right)^{\frac{\alpha}{\alpha-1}} - (1+r) \left( \frac{1+r}{\alpha z} \right)^{\frac{\alpha}{\alpha-1}} + \beta^2 a (1+r)^2 \right) \right) \]

\[ = -\phi \left( -\frac{(1+r)^{\frac{\alpha}{\alpha-1}}}{(\alpha-1)z} + \frac{(1+r)^{\frac{\alpha}{\alpha-1}}}{(\alpha-1)z} + r \left( \frac{1+r}{\alpha z} \right)^{\frac{\alpha}{\alpha-1}} \right) \]

\[ = -\phi \left( \frac{(1+r)^{\frac{\alpha}{\alpha-1}}}{(\alpha-1)z} \right) \]

so \( \frac{\partial s^*}{\partial z} \geq 0 \) if \( \frac{1+r}{\frac{1+r}{\alpha z}} \geq z \)
and \( \frac{\partial s^*}{\partial z} \leq 0 \) if \( \frac{1+r}{(1+\alpha z)^{\frac{1+r}{\alpha z}}} \leq z \)

but note that you will not operate a company if you have a negative profit.

so you want \( \pi(z, r) = z \left( \frac{1+r}{\alpha z} \right)^{\frac{1+r}{\alpha z}} - (1+r) \left( \frac{1+r}{\alpha z} \right)^{\frac{1+r}{\alpha z}} \geq 0 \)

\( \Leftrightarrow \left( \frac{1+r}{(1+\alpha z)^{\frac{1+r}{\alpha z}}} \right) \leq z \)

So we actually have

\[ \frac{\partial s^*}{\partial z} \leq 0 \]

(this makes sense, the sign of \( \frac{\partial s^*}{\partial z} \) is determined by the sign of \( -\frac{\partial \pi(z, r)}{\partial z} \). It would be strange to have that profit declines when there is a positive technological shock. Regarding the algebra, we may find \( \frac{\partial \pi(z, r)}{\partial z} < 0 \) but only for values of \( \pi(z, r) < 0 \).

So since dividends are more profitable with a positive technological shock, hhs are willing to save less.

4. In equilibrium, supply = demand

so \( N_h s^* = N_c k^* \)

so \( r^* \) should satisfy the following equation:

i.e

\[
\left( \frac{1+r}{\alpha z} \right)^{\frac{1+r}{\alpha z}} = -\phi \left( \frac{z \left( \frac{1+r}{\alpha z} \right)^{\frac{1+r}{\alpha z}} - (1+r) \left( \frac{1+r}{\alpha z} \right)^{\frac{1+r}{\alpha z}}} {(1+r) \left( 1 + \beta^2 + \beta^2 r \right)} + \beta^2 a (1+r)^2 \right)
\]

for instance, if we take \( \alpha = 0.5, z = 1 \) and \( \beta = 0.8 \), we have:

\[
\left( \frac{1+r}{0.5} \right)^{\frac{1+r}{0.5}} = -0.8 \left( \frac{\left( \frac{1+r}{0.5} \right)^{\frac{1+r}{0.5}} - (1+r) \left( \frac{1+r}{0.5} \right)^{\frac{1+r}{0.5}}} {(1+r) \left( 1 + 0.8^2 + 0.8^2 r \right)} + 0.8^2 0.5 (1+r)^2 \right)
\]

and \( r^* = 0.26835 \)

Total capital used by the firm is then

\[ k^* = \left( \frac{1+r^*}{\alpha z} \right)^{\frac{1+r}{\alpha z}} \]

5. If \( a \) increases, this shifts the supply of capital to the right, which will decreases the equilibrium interest rate.

If \( z \) increases, we have seen that demand increases. Supply decreases, so the effect on \( r^* \) is increases.

If \( \phi \) increases, demand is unaffected, but supply decreases, which increases the interest rate.