Exercise 1
1.a
w denotes the fraction of the portfolio on the risky asset that is allocated in asset 1.
Let’s look at this risky portfolio return and variance.
We have

\[ E(r_p) = wE(r_1) + (1 - w)E(r_2) \]
\[ = w \times 12.5\% + (1 - w) \times 14\% \]

and

\[ \sigma_p^2 = w^2\sigma_1^2 + (1 - w)^2\sigma_2^2 + 2w(1 - w)\sigma_1\sigma_2\rho_{12} \]
\[ = w^2 (3\%)^2 + (1 - w)^2 (4.5\%)^2 - 2w(1 - w) (3\%) (4.5\%) \times 0.2 \]

\[ \sigma_p = \sqrt{\sigma_p^2} \]

Using those formulas for the different values of w, we find:

b.
Let’s draw the relationship between expected return and standard deviation.

c.
Now, we look at a portfolio with risk free asset and risky assets.
The objective is to find the weight \( w^* \) of asset 1 (as a share of the risky portfolio) that results in the highest reward-to-variability portfolio (tangency point between the CAL and the efficient frontier), i.e. we want to maximize with respect to w the following:

\[ S_p = \frac{E(r_p) - r_f}{\sigma_p} \]
\[ = \frac{wE(r_1) + (1 - w)E(r_2) - r_f}{\sqrt{w^2\sigma_1^2 + (1 - w)^2\sigma_2^2 + 2w(1 - w)\sigma_1\sigma_2\rho_{12}}} \]
\[
\frac{\partial S_p}{\partial w} = 0
\]

\[
\begin{align*}
\omega^* &= \frac{(E(r_1) - r_f)\sigma_2^2 + (E(r_2) - r_f)\sigma_1^2 - (E(r_1) - r_f + E(r_2) - r_f)\text{cov}(r_1, r_2)}{(E(r_1) - r_f)\sigma_2^2 + (E(r_2) - r_f)\sigma_1^2 - (E(r_1) - r_f + E(r_2) - r_f)\sigma_1\sigma_2\rho_{12}} \\
&= \frac{(E(r_1) - r_f)\sigma_2^2 + (E(r_2) - r_f)\sigma_1^2 - (E(r_1) - r_f + E(r_2) - r_f)\sigma_1\sigma_2\rho_{12}}{(12.5 - 10)(4.5)^2 + 0.2(14 - 10)(3)(4.5)} \\
&= \frac{(12.5 - 10)(4.5)^2 - (14 - 10)(3)^2}{0.59} \\
&= 0.39
\end{align*}
\]

This \( \omega^* \) gives us the optimal composition of risky assets. And we can find the highest slope CAL. Everybody wants to allocate its portfolio on this "line" no matter the level of risk aversion.

Risi aversion determines which point on this line an agent wants to choose (i.e. which share of risk-free asset he wants to hold).

d.

\[
\begin{align*}
S_p &= \frac{E(r_p) - r_f}{\sigma_p} \\
&= \frac{0.59E(r_1) + (1 - 0.59)E(r_2) - r_f}{\sqrt{0.59^2\sigma_1^2 + (1 - 0.59)^2\sigma_2^2 + 2 \times 0.59(1 - 0.59)\sigma_1\sigma_2\rho_{12}}} \\
&= \frac{0.59 \times 12.5 + (1 - 0.59)14 - 10}{\sqrt{0.59^2(3)^2 + (1 - 0.59)^2(4.5)^2 - 2 \times 2 \times 0.59(1 - 0.59)(3)(4.5)}} \\
&= 1.36
\end{align*}
\]

2. A=100.
(Note: there is a typo, it should be 100 and not 1000 for the problem to make sense).

Let's find the optimal portfolio. Denote \( y \) the position in the risky portfolio. The agent wants to maximize his utility:

\[
Max \ (1 - y)r_f + yE(r_p) - 0.005 \times A \times y^2\sigma_p^2
\]

Taking the first-order condition, we get:

\[
y^* = \frac{E(r_p) - r_f}{2 \times 0.005 \times A \times \sigma_p^2}
\]

So the agent wants to invest 60% in the risky asset and 40% in the risk-free asset.
3. \( A=150 \)  
(The note for 2 still holds)

\[
y^* = \frac{E(r_p) - r_f}{2 \times 0.005 \times A \times \sigma^2_p} = 0.4
\]

So the agent wants to invest 40% in the risky asset and 60% in the risk-free asset.

To find those values of \( y \), I use (as they do in the book) the percentage (i.e.
I use \( E(r_p) = 13.11 \) in the formula and not 0.1311. Note that this makes a
difference when we divide by \( \sigma^2_p \). Let’s take \( \sigma_p = 2.38 \%), i.e and look at the
ratio \( \frac{E(r_p)}{\sigma^2_p} \)

\[
\frac{13.11}{2.38^2} \neq \frac{13.11/100}{(2.38/100)^2} = \frac{13.11 \times 100}{2.38^2} \text{!!!!!!!!!!}
\]

What matters is how \( \sigma^2_p \) is defined in the utility function. So what is implicit
is the book is that in the form of \( U() \) they give, when \( \sigma_p = 3\% \), you use 3.
They should be precise about that, because otherwise, what they do is wrong.

In the exercise, I assumed that \( U() \) is defined like that (to be consistent with
the book) but I just want you to be aware of the difference.

**Exercise 2**

1. We have

\[
\beta_i = \frac{cov(r_i, r_m)}{\sigma^2_m} = \frac{\sigma_i \sigma_m \beta_{im}}{\sigma^2_m}
\]

So

\[
\beta_1 = \frac{0.1 \times 0.15 \times 0.3}{(0.15)^2} = 0.2
\]

\[
\beta_2 = \frac{0.2 \times 0.15 \times 0.7}{(0.15)^2} = 0.93
\]

\[
\beta_3 = \frac{0.075 \times 0.15 \times (-0.3)}{(0.15)^2} = -0.15
\]

2. 
(a)
\[
\beta_M = \frac{\text{cov}(r_M, r_m)}{\sigma_m^2} \\
= \frac{0.6\text{cov}(r_1, r_m) + 0.4\text{cov}(r_2, r_m)}{\sigma_m^2} \\
= 0.6\beta_1 + 0.4\beta_2 \\
= 0.492
\]

(b) Identically, we have:
\[
\beta_N = \frac{\text{cov}(r_M, r_m)}{\sigma_m^2} \\
= 0.5\beta_{MP} + 0.2\beta_2 + 0.3\beta_3 \\
= 0.5 \times 1 + 0.2 \times 0.93 + 0.3 \times (-0.15) \\
= 0.641
\]

Note that the beta of the market portfolio is just:
\[
\beta_{MP} = \frac{\text{cov}(r_M, r_m)}{\sigma_m^2} = \frac{\sigma_M^2}{\sigma_m^2} = 1
\]

(c)
\[
\beta_P = \frac{\text{cov}(r_P, r_m)}{\sigma_m^2} \\
= 0.6\beta_N + 0.4\beta_M \\
= 0.34
\]

(d)
The CAPM implies that:
\[
E(r_i) - r_f = \beta_i [E(r_m) - r_f]
\]
i.e.
\[
E(r_i) = r_f + \beta_i [E(r_m) - r_f]
\]
Let’s compare if it gives us a good prediction of the expected return of the asset.
\[
E(r_1) = 0.075 + 0.2 [0.175 - 0.075] \\
= 0.095 \neq 0.175
\]
\[
E(r_2) = 0.075 + 0.93 [0.175 - 0.075] \\
= 0.168 \neq 0.25
\]
\[ E(r_3) = 0.075 - 0.15[0.175 - 0.075] \]
\[ = 0.06 \neq 0.075 \]

The prediction of the CAPM are not very good.

(e) Ok, we just found them!