1 The CAPM (30 pts)

A simple economy has the following characteristics. The risk-free rate is with \( r_f = 0.085 \); The return of the market portfolio, which is composed of many, many assets, has mean \( \mu_m = 0.15 \) and standard deviation \( \sigma_m = 0.1 \).

You are considering investing in three assets, denoted by \( i = 1, 2, 3 \). You know that the mean returns are \( \mu_1 = 0.15, \mu_2 = 0.05 \) and \( \mu_3 = 0.07 \) while the standard deviation of the returns are \( \sigma_1 = 0.2, \sigma_2 = 0.1, \sigma_2 = 0.25 \). You also know that the correlations of the two assets with market portfolio and between them is given by \( \rho_{12} = 0, \rho_{13} = -0.2, \rho_{23} = 0, \rho_{1m} = 0.2, \rho_{2m} = 0.4, \rho_{3m} = -0.7 \).

1. Calculate the betas for the two assets \( \beta_i, i = 1, 2, 3 \).
2. Calculate the betas \( \beta_p \) of the following portfolios \( p \).
   (a) A trust fund (called A) that holds proportions 50\%, 50\% on assets 1 and 2 respectively.
   (b) A trust fund (called B) that holds proportions 40\% on the market portfolio, 40\% on asset 2 and 20\% on asset three.
   (c) A portfolio that holds 60\% on B and 40\% on A.
3. Explain the following:
   (a) Are the predictions of the CAPM valid for the excess returns of the assets?
   (b) Given the variances and covariances of the assets, find the values of \( \mu_i, i = 1, 2 \) for which the CAPM would hold for assets 1 and 2.
   (c) Alternatively, given the variances and means of the returns of the assets, find the covariances of the assets with the market \( (\sigma_{im}, i = 1, 2, 3) \) for which the CAPM would hold. (Consider only the betas for the three assets.)

2 Extensions of the CAPM (10 pts)

Explain the extension of the CAPM proposed by Jaganathan and Wang that is discussed in the paper by Jaganathan and MacGrattan. What is the measure of the returns of the additional asset in their definition of market portfolio?
3 Moral Hazard (30 pts)

You are a risk neutral banker and you are analyzing the loan application from an entrepreneur who wants to start a project of size of $k = 1$. The entrepreneur has initial wealth $a < 1$, so he needs a loan is $1 - a$. The opportunity cost of those resources is given by $(1 + r)(1 - a)$, where $r$ is the interest rate. You take the interest rate as given.

The entrepreneur has the option of exerting effort $e = 1$ or not $e = 0$. His effort cannot be verified, and has to be exerted before knowing the outcome of the project, which could be a success or a failure. If the project is a success it yields 2. If it fails, it yields nothing (not even the initial investment is recovered). If $e = 1$ the probability of success is $p \in (0, 1)$, while if effort is $e = 0$, the probability of success is just $p/2$. Thus, effort is productive. The utility of the entrepreneur is given by

$$U = \text{prob}(\text{success}) \cdot u(\text{Wealth}_{\text{success}}) + \text{prob}(\text{failure}) \cdot u(\text{Wealth}_{\text{failure}}) - v(\text{effort})$$

where $v(0) = 0$ and $v(1) = v_1 > 0$. Thus, in this economy, effort is costly. For this problem, assume that the entrepreneur is risk neutral, hence $u(x) = x$. We assume that $2p - v_1 > (1 + r) > p$ so the project is worth undertaking if and only if the entrepreneur will exert effort.

1. First, consider an entrepreneur with wealth $a \geq 1$, an entrepreneur that is rich enough so he can self-finance the project. Explain why, given our assumptions, the entrepreneur would want to take the project and exert effort.

2. For now on, consider an entrepreneur with $a < 1$ (hence would need a loan). The lending contract specifies that if the project is a success, the entrepreneur has to pay a part of the proceeds $X \leq 2$ to the bank. If the project fails, the entrepreneur will have zero wealth and hence will pay zero. Write down the condition required for the entrepreneur to want to exert effort.

3. Given that you know what determines the level of effort, write down the expected payoff to the bank for each level of $X$.

4. Assuming that you operate in a competitive industry, write the condition that determines whether you would break-even in expectation in a given loan.

5. What is the maximum loan that you would grant? i.e., what is the minimum wealth for which an agent can become an entrepreneur.

4 Adverse Selection and Collateral (30 pts)

There are only two types of entrepreneurs, a safe type, which comprises a fraction $\alpha$ of the population and a risky type, which comprises the remaining fraction, $1 - \alpha$. Entrepreneurs need to borrow all the investments required for the project, which is $k = 1$. Projects are either successes or failures. Failures yield zero output (does not recover the invested capital). Safe entrepreneurs have a high probability of success, $p_s$, in which case the project returns $R_s$. Risky entrepreneurs have lower probability of success, $p_r < p_s$ but have higher returns $R_r > R_s$ in case of success. Indeed, we assume that the expected value of the returns of the two types is the same, i.e. $p_sR_s = p_rR_r$. Each entrepreneur knows his type, but other agents, including bankers, do not know. Everyone knows the composition in the population, they just don’t know who is who.

Entrepreneurs own a house, with value $H$. Entrepreneurs value the services provided by their housing as well as the consumption goods bought with their wealth. The expected utility function is of the form

$$U = \text{prob}(\text{success}) \cdot [\sqrt{\text{Wealth}_{\text{success}} + h_{\text{success}}}] + \text{prob}(\text{failure}) \cdot [\sqrt{\text{Wealth}_{\text{failure}} + h_{\text{failure}}}]$$

where $h_{\text{success}}, h_{\text{failure}}$ stand for the amount of housing left to the entrepreneur in the respective realization of the project. Houses are perfectly divisible goods and can be used as collateral. This is, the contract can specify an amount $C \leq H$ that the bank can take of the house if the entrepreneur fails. Thus, $h_{\text{success}} = H$ and $h_{\text{failure}} = H - C$. Inactive entrepreneurs have an utility level of $H$
There is only a bank in this economy and it can obtain resources at a constant, exogenously given, interest rate \( r \). Thus, the opportunity cost of the resources lent to entrepreneurs is \( 1 + r \) for each project. The owners of the bank are risk neutral agents.

1. Imagine that you work for the only bank in the economy. Assume that the owners of that want to specialize in loans to safe types. Write down the condition on the repayment and collateral policies \((X, C)\) required so that risky types will not want to apply for a loan.

2. Now imagine that the owners of the bank also want to serve the risky types. Write down the conditions for the contract policies \((X_0, C_0)\), \((X_1, C_1)\) such that risky types will apply for loans of with policies \((X_0, C_0)\) and safe types will apply for the loans with the policies \((X_1, C_1)\).

3. What conditions are required for which it is profitable for the bank to offer the loan contracts \((X_0, C_0)\), \((X_1, C_1)\)?