Supplement to

Efficient Use of Information and Social Value of Information

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Abstract

In this document we motivate the restriction $\alpha < 1$ used in the paper (where $\alpha \equiv -U_{kK}/U_{kk}$ is the slope of the best response). We first explain why this restriction is essentially equivalent to uniqueness of equilibrium under complete information (Section A1). We then discuss the role of this restriction for comparative statics (Section A2).

A1. Uniqueness

To understand the role of $\alpha < 1$ for equilibrium determinacy, it is useful to consider the following two modifications of our model.

(i) There is a *finite* number of players, $J \geq 2$. The payoff for player i is

$$u_i = U(k_i, K_{-i}, \sigma_{-i}, \theta),$$

where $K_{-i} \equiv \frac{1}{J-1}\sum_{j\neq i} k_j$ is the mean, and $\sigma_{-i} \equiv [\frac{1}{J-1}\sum_{j\neq i}(k_j - K_{-i})^2]^{1/2}$ the standard deviation, of the actions taken by i's opponents. As in the paper, U is quadratic and its partial derivatives satisfy $U_{k\sigma} = U_{K\sigma} = U_{\theta\sigma} = 0$, $U_\sigma(k_i, K_{-i}, 0, \theta) = 0$ for all (k_i, K_{-i}, θ), $U_{kk} < 0$ and $U_{k\theta} \neq 0$.

(ii) Actions are *bounded*: $k \in [-M, +M]$, for some $M \in (0, \infty)$.

Our model can be viewed as the limit when both J and M become infinite. We will build on this connection in a moment. We first show that, for any finite J and M, the following is true.

Proposition A1. *Let $\alpha \equiv -U_{kK}/U_{kK}$. Under complete information, the equilibrium is unique for all θ if and only if*

$$-(J-1) < \alpha < 1.$$

When instead $\alpha \leq -(J-1)$ or $\alpha \geq 1$, there is a non-empty set $\mathcal{Q}$ such that multiple equilibria exist whenever $\theta \in Q$; moreover, when $\alpha > 1$, $Q = [-M, +M]$, so that the region of multiplicity is independent of J, increases with M, and converges to the entire real line as $M \to \infty$.

Proof. Consider the function $f(k,\theta) \equiv U_k(k,k,0,\theta)$. f is a linear function with $f_k = U_{kk} + U_{kK} = (1-\alpha)U_{kk}$ and $f_\theta = U_{k\theta} \neq 0$. When $\alpha = 1$, f is flat in k, while it is monotonic in θ. It follows that there exists a (unique) $\hat{\theta}$ such that $U_k(k,k,0,\theta) = 0$ for any k when $\theta = \hat{\theta}$. This implies that any symmetric strategy profile is an equilibrium when $\theta = \hat{\theta}$, which proves the claim for the case $\alpha = 1$. In the rest of the proof, we thus consider $\alpha \neq 1$, in which case f is monotonic in k and hence the condition $U_k(k,k,0,\theta) = 0$ necessarily admits a unique solution $\kappa = \kappa(\theta) \in \mathbb{R}$, for any θ. Following the same steps as in the proof of Proposition 1, we then have that the best response of agent i is

$$k_i = \begin{cases} M & \text{if } (1-\alpha)\theta + \alpha K_{-i} > M \\ (1-\alpha)\theta + \alpha K_{-i} & \text{if } (1-\alpha)\kappa(\theta) + \alpha K_{-i} \in [-M,+M] \\ -M & \text{if } (1-\alpha)\theta + \alpha K_{-i} < -M \end{cases} \tag{1}$$

Without any loss of generality, we henceforth normalize $\kappa(\theta) = \theta$.[1]

The rest of the proof proceeds in two steps. Step 1 shows that there exist multiple equilibria for a non-empty interval of θ whenever either $\alpha > 1$ or $\alpha \leq -(J-1)$, and that this interval converges to the entire real line as $M \to \infty$ when $\alpha > 1$. Step 2 shows that $-(J-1) < \alpha < 1$ is, not only necessary, but also sufficient for uniqueness, which completes the proof of the proposition.

Step 1. First consider $\alpha > 1$. For any $\theta \in [-M, +M]$, we have that

$$(1-\alpha)\theta + \alpha(-M) \leq -M < M \leq (1-\alpha)\theta + \alpha M,$$

which implies that, along with $k_i = \theta$ for all i, the following are also equilibria: $k_i = M$ and $k_i = -M$ for all i. Hence, $\alpha > 1$ implies multiplicity for any $\theta \in \mathcal{Q}$, with $\mathcal{Q} = [-M, +M]$.

[1]One can always re-scale θ so that $\kappa(\theta) = \theta$. To see this, let $\tilde{\theta} \equiv \kappa(\theta) = \kappa_0 + \kappa_1\theta$ and $\tilde{U}(k,K,\sigma,\tilde{\theta}) \equiv U(k,K,\sigma,\tilde{\theta}/\kappa_1 - \kappa_0/\kappa_1)$; then, by construction, $\tilde{U}_k(k,k,0,\tilde{\theta}) = 0$ if and only if $k = \tilde{\theta}$.

Next, consider $\alpha < -(J-1)$. Pick any $\hat{J} \in \{1, ..., J-1\}$, let

$$\underline{\theta}(\hat{J}; M) \equiv \frac{M}{1-\alpha}\left[\frac{(J-1+\alpha)+\alpha(J-2\hat{J})}{J-1}\right], \quad \bar{\theta}(\hat{J}; M) \equiv \frac{M}{1-\alpha}\left[\frac{-(J-1+\alpha)+\alpha(J-2\hat{J})}{J-1}\right],$$

and note that $J-1+\alpha < 0$ ensures that $\underline{\theta}(\hat{J}; M) < \bar{\theta}(\hat{J}; M)$. The following is then true for any $\theta \in [\underline{\theta}(\hat{J}; M), \bar{\theta}(\hat{J}; M)]$:

$$(1-\alpha)\theta+\alpha\frac{\hat{J}M+(J-\hat{J}-1)(-M)}{J-1} < -M < M < (1-\alpha)\theta+\alpha\frac{(\hat{J}-1)M+(J-\hat{J})(-M)}{J-1}. \quad (2)$$

The fraction $\frac{\hat{J}M+(J-\hat{J}-1)(-M)}{J-1}$ in the furthest left side of (2) is simply the average action of player i's opponents when $\hat{J}$ players other than i play M and the remaining $J-1-\hat{J}$ play $-M$. Similarly, the fraction $\frac{(\hat{J}-1)M+(J-\hat{J})(-M)}{J-1}$ on the furthest right side is the average action of player i's opponents when $\hat{J}-1$ players other than i play M and the remaining $J-\hat{J}$ play $-M$. Combined with (1), condition (2) implies that the best response for player i is to play $k_i = -M$ in the former case and $k_i = M$ in the latter. It follows that, for any $\hat{J} \in \{1, ..., J-1\}$ and any $\theta \in [\underline{\theta}(\hat{J}; M), \bar{\theta}(\hat{J}; M)]$, there exists an asymmetric equilibrium in which $\hat{J}$ agents play $k = M$ and $J-\hat{J}$ play $k = -M$. Because, for any $\hat{J} \in \{1, ..., J-1\}$, $-M < \underline{\theta}(\hat{J}; M) < \bar{\theta}(\hat{J}; M) < M$, for any $\theta \in [\underline{\theta}(\hat{J}; M), \bar{\theta}(\hat{J}; M)]$, there also exists a symmetric equilibrium in which $k_i = \theta$ for all i. We conclude that, when $\alpha < -(J-1)$, there are multiple equilibria whenever $\theta \in \mathcal{Q}$, where $\mathcal{Q} \equiv \bigcup_{\hat{J}\in\{1,...,J-1\}}[\underline{\theta}(\hat{J}; M), \bar{\theta}(\hat{J}; M)]$, with $\mathcal{Q} \neq \varnothing$ and $\mathcal{Q} \subset (-M, +M)$.

It is easy to show that multiplicity pertains also for $\alpha = -(J-1)$, which completes the proof of the second part of the proposition.

Step 2. We now show that $-(J-1) < \alpha < 1$ is, not only necessary, but also sufficient for uniqueness.

Take any strategy profile and pick any arbitrary player i. Then, for any given θ, there are exactly three possibilities:

$$(C1) \qquad -M \leq (1-\alpha)\theta + \alpha K_{-i} \leq +M$$
$$(C2) \qquad (1-\alpha)\theta + \alpha K_{-i} > M$$
$$(C3) \qquad (1-\alpha)\theta + \alpha K_{-i} < -M$$

In the following, we show that (C1) can be satisfied by an equilibrium strategy profile for some θ and some i if and only if $\theta \in [-M, +M]$ and $k_j = \theta$ for all j. Similarly, (C2) can be satisfied for some θ and some i if and only if $\theta > M$ and $k_j = M$ for all j. Finally, (C3) can be satisfied for some θ and some i if and only if $\theta < -M$ and $k_j = -M$ for all j. Together these properties imply that the unique equilibrium strategy profile is the symmetric one given by

$$k_i = \begin{cases} M & \text{when } \theta > M \\ \theta & \text{for } \theta \in [-M, +M] \\ -M & \text{for } \theta < -M. \end{cases} \tag{3}$$

Consider first (C1). If a strategy profile satisfies (C1) for some i and some θ and this profile is an equilibrium, then, by (1), it must be that $k_i = (1-\alpha)\,\theta + \alpha K_{-i}$ for the particular θ and the particular player i. Noting that $K_{-i} = \frac{1}{J-1}\sum_{l=1}^{J} k_l - \frac{1}{J-1}k_i$, we have that

$$k_i = \frac{1}{1+\frac{\alpha}{J-1}}\,(1-\alpha)\,\theta + \frac{\frac{\alpha}{J-1}}{1+\frac{\alpha}{J-1}}\sum_{l=1}^{J} k_l, \tag{4}$$

in which case condition (C1) can be restated as

$$-M \;\leq\; \frac{1}{1+\frac{\alpha}{J-1}}\,(1-\alpha)\,\theta + \frac{\frac{\alpha}{J-1}}{1+\frac{\alpha}{J-1}}\sum_{l=1}^{J} k_l \;\leq\; M. \tag{5}$$

We now argue that (5) ensures $-M \leq (1-\alpha)\,\theta + \alpha K_{-j} \leq M$ also for any $j \neq i$. Suppose, towards a contradiction, that $(1-\alpha)\,\theta + \alpha K_{-j} > M$ [resp., $(1-\alpha)\,\theta + \alpha K_{-j} < -M$] for some j, in which case k_j must equal M [resp., $-M$]. But then combining $(1-\alpha)\,\theta + \alpha K_{-j} > M = k_j$ [resp., $(1-\alpha)\,\theta + \alpha K_{-j} < -M = k_j$] with $K_{-j} = \frac{1}{J-1}\sum_{l=1}^{J} k_l - \frac{1}{J-1}k_j$ and $1/\left[1+\alpha/(J-1)\right] > 0$ (where the latter is true because $-(J-1) < \alpha$), gives

$$\frac{1}{1+\frac{\alpha}{J-1}}\,(1-\alpha)\,\theta + \frac{\frac{\alpha}{J-1}}{1+\frac{\alpha}{J-1}}\sum_{l=1}^{J} k_l > M \quad [\text{resp.}, < -M],$$

which contradicts (5).

That $-M \leq (1-\alpha)\,\theta + \alpha K_{-j} \leq M$ in turn implies, by (1), that (4) must hold also for $j \neq i$. But since the right-hand-side of (4) is identical for all j, it must be that k_j is the same for all j. Letting $k_1 = \cdots = k_J = k$ in (4) for some k, we then have that $k = \theta$, in which case (5) reduces to $\theta \in [-M, +M]$. We conclude that, whenever (C1) is satisfied in equilibrium for some θ and some i, it is necessarily the case that $\theta \in [-M, +M]$ and $k_j = \theta$ for all j.

Next, consider case (C2). If a strategy profile satisfies (C2) for some i and some θ and this profile is an equilibrium, then, by (1), it must be that $k_i = M < (1-\alpha)\,\theta + \alpha K_{-i}$. From the same steps as in case (C1), the latter implies that

$$M \ < \ \frac{1}{1+\frac{\alpha}{J-1}}\left(1-\alpha\right)\theta+\frac{\frac{\alpha}{J-1}}{1+\frac{\alpha}{J-1}}\sum_{l=1}^{J}k_l. \tag{6}$$

We now argue that (6) implies $k_j = M$ also for any $j \neq i$. Suppose, towards a contradiction, that there is some $j \neq i$ for which $k_j < M$. Then, by (1), it must be that $(1-\alpha)\theta + \alpha K_{-j} \leq k_j$, which together with $k_j < M$ implies

$$\frac{1}{1+\frac{\alpha}{J-1}}\left(1-\alpha\right)\theta+\frac{\frac{\alpha}{J-1}}{1+\frac{\alpha}{J-1}}\sum_{l=1}^{J}k_l \leq k_j < M,$$

contradicting (6). Hence, it must be that $k_j = M$ also for j. But then, from (1), $k_j = M$ for all j, along with $\alpha < 1$, implies $\theta > M$. We conclude that, whenever condition (C2) holds in equilibrium for some θ and some i, it must be that $\theta > M$ and $k_j = M$ for all j.

An argument symmetric to that for case (C2) implies that, whenever condition (C3) is satisfied in equilibrium for some θ and some i, then necessarily $\theta < -M$ and $k_j = -M$ for all j.

Combining the above three cases, and noting that they exhaust all possibilities and correspond to non-overlapping intervals of θ, we conclude that the strategy profile in which all players follow the strategy in (3) is indeed the unique equilibrium whenever $-(J-1) < \alpha < 1$. *QED*

When $\alpha > 1$, multiplicity emerges no matter the number of players J and the bound M. Moreover, the region of θ for which multiplicity pertains is independent of J and is increasing in M, covering the entire real line as $M \to \infty$. When instead, $\alpha < 1$, uniqueness is ensured for any M once J is large enough. In particular, any $J \geq 2$ when $\alpha \in (-1, 1)$, and any $J \geq 1-\alpha$ when $\alpha \leq -1$ suffices for uniqueness no matter M. Therefore, if we view our model as the limit in which both J and M go to infinity (whatever the order), then the restriction $\alpha < 1$ is "essentially" necessary and sufficient for uniqueness.

To be precise, our model does not impose any bound on k and assumes a continuum of players. If one takes this literally, then the best response, which reduces to $k = (1-\alpha)\theta + \alpha K$ for all (θ, K), admits a unique fixed point in $\mathbb{R}$ if and only if $\alpha \neq 1$. However, when $\alpha > 1$ this uniqueness is an artifact of the fact that there are no bounds for k. Indeed, if we were to introduce any bound $M < \infty$, then multiple equilibria would emerge whenever $\alpha > 1$ no matter how large M is. Furthermore, the multiplicity would be more pervasive the larger M : for any $\theta \in (-M, +M)$, there would exist three equilibria, the interior equilibrium in which $k = \theta$, along with the two extreme equilibria in which $k = -M$ and $k = +M$. The interval of θ for which multiplicity emerges would thus cover to the entire real line as $M \to \infty$. At the same time, the two extreme equilibria diverge to $-\infty$ and $+\infty$, which explains why these two extreme equilibria disappear in our model in which there are no bounds.

In the paper, we impose no bounds on k so as to make the analysis tractable under incomplete information, but we continue to restrict $\alpha < 1$ so as to rule out the possibility that uniqueness is an artifact of the unboundedness assumption.

Remarks. The preceding analysis did not address the role of α for the stability of the equilibrium (in the sense of iterated best responses). It is easy to check that the equilibrium $\kappa(\theta)$ is stable when $\alpha \in (-1, 1)$, but it is unstable when either $\alpha < -1$ or $\alpha > 1$.[2] Clearly, none of the results in the paper is affected if one restricts attention to $\alpha \in (-1, 1)$ so as to ensure stability.

Also, the preceding analysis did not address whether $\alpha < 1$ is either necessary or sufficient for equilibrium uniqueness under *incomplete* information. Once we restrict k in $[-M, +M]$ for some $M < \infty$, it is easy to construct examples where $\alpha > 1$ leads to multiplicity also under incomplete information (at least for sufficiently precise public information).[3] It is also possible to show that $\alpha \in (-1, 1)$ suffices for uniqueness, following the same argument as in Morris and Shin (2002). What is not obvious is whether uniqueness survives with incomplete information also when $\alpha < -1$; we have no reason to expect otherwise, but we have not proved it.

A2. Comparative Statics

The aforementioned discussion focused on the role of the restriction $\alpha < 1$ for the determinacy of equilibria. We now discuss why the same restriction rules out comparative statics that would seem to be paradoxical for applied purposes.

To see this, consider the investment example of Section 6.2 in the paper: the payoff of an agent is $U(k, K, \sigma, \theta) = Ak - k^2/2$, where $A \equiv (1-a)\theta + aK$. Here k is interpreted as investment, A as the return to investment, $k^2/2$ as the cost to investment; the slope of the best response is $\alpha = a$.

When $\alpha < 1$, the return A, the best response $BR(K;\theta)$, and the equilibrium $\kappa(\theta)$ are all increasing in θ. When instead $\alpha > 1$, the return to investment is decreasing in θ and, by implication,

[2] To see this, consider the case with a continuum of players and unbounded actions, and let $BR^1(K;\theta) \equiv (1-\alpha)\kappa(\theta) + \alpha K$ and $BR^t(K;\theta) \equiv BR^1\left(BR^{t-1}(K;\theta);\theta\right)$ for $t \geq 2$. The following are then true: when $\alpha \in (-1, 1)$, as $t \to \infty$, $BR^t(K;\theta) \to \kappa(\theta)$ for any $K \neq \kappa(\theta)$; when instead $\alpha < 1$ or $\alpha > 1$, as $t \to \infty$, $BR^t(K;\theta) \to \infty$ for any $K > \kappa(\theta)$ and $BR^t(K;\theta) \to -\infty$ for any $K < \kappa(\theta)$.

[3] We considered a related example in Section 2 of Angeletos and Pavan (2004). For tractability, that example assumed an upward discontinuity in the best response, which can be interpreted as the limit for $\alpha \to \infty$.

the best response $BR(K;\theta)$ is also decreasing in θ for any given K; nevertheless, the equilibrium $\kappa(\theta) = \theta$ is increasing in θ. That is, the equilibrium investment moves in the *opposite* direction than the return to investment. Why? Because the equilibrium is sustained by the belief that the reduction in A caused by the increase in θ comes with a more-than-offsetting increase in others' investment, so that at the end the total effect is a increase in A, which in turn makes everybody invest more, thus making the belief self-fulfilling.

Clearly, this issue is closely related to multiplicity. To see this, think of a variant in which the best-response function $BR(K;\theta)$ is continuous, differentiable, and strictly monotonic in both K and θ, and satisfies, for any θ, $\lim_{K\to-\infty}[BR(K;\theta) - K] > 0$ and $\lim_{K\to\infty}[BR(K;\theta) - K] < 0$. Let $\alpha(K;\theta) \equiv \partial BR(K;\theta)/\partial K$ be the slope of the best response, which now may be a function of K and/or θ. Then, whenever there is an unstable equilibrium (i.e., a fixed point to the best response with $\alpha > 1$), this equilibrium is necessarily surrounded by two stable equilibria (i.e., fixed points with $\alpha < 1$). While the former equilibrium has the opposite comparative statics with respect to θ than the best response, the latter two equilibria have the same comparative statics. Although this property need not be disturbing per se, it becomes unappealing once translated in specific applications, which explains why it is standard practice in applied models with complementarities to ignore unstable fixed points and focus on the stable ones.

Because such paradoxical comparative statics emerge only when $\alpha > 1$, this result provides a complementary reason for restricting the analysis to economies in which $\alpha < 1$.[4]

A3. Summary

The combination of the aforementioned considerations led us to restrict attention to economies in which $\alpha < 1$. This is not to say that the case $\alpha > 1$ is uninteresting—it is only to say that our analysis is not appropriate for this case. First, one must recognize that $\alpha > 1$ is endemic to multiple equilibria, in which case the information structure may matter, not only for the local comparative statics of any given equilibrium, but also for the determinacy of equilibria. Second, even if one is happy to select one particular equilibrium and focus on its local comparative statics, doing so with the linear equilibrium identified in the paper may be problematic, for this equilibrium has the opposite comparative statics than individual best responses, which seems unappealing for many applied purposes.

[4] The aforementioned discussion focused on the case in which θ is commonly known; similar arguments apply to the case of incomplete information.