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Tilting the supply schedule to enhance competition in uniform-price auctions

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Abstract

Uniform-price auctions of a divisible good in fixed supply admit underpricing equilibria, where bidders submit high inframarginal bids to prevent competition on prices. The seller can obstruct this behavior by tilting her supply schedule and making the amount of divisible good on offer change endogenously with its (uniform) price. Precommitting to an increasing supply curve is a strategic instrument to reward aggressive bidding and enhance expected revenue. A fixed supply may not be optimal even when accounting for the cost to the seller of issuing a quantity different from her target supply.

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1. Introduction

In the last few years, uniform-price auctions have become a popular mechanism to allocate divisible goods. For instance, since September 1998, the U.S. Department of Treasury has switched from a traditional discriminatory format to the uniform-price auction to issue all its securities.[1] Similarly, uniform-price auctions are now commonly used to run on-line initial public offerings (IPOs) of unseasoned shares (open IPOs), as well as in electricity markets and in markets for emission permits.

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[1] The decision to extend the uniform price format to all Treasury securities was taken at the completion of a period of 9 years in which this format had been limited to 2 years, 5 years and inflation-adjusted bonds.

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In a uniform-price auction, bidders compete by simultaneously submitting their demand schedules for the divisible good on offer. The seller compares the aggregate demand with her aggregate supply and computes a clearing (stop-out) price. Demand above the stop-out price is awarded in full, while marginal demand at the stop-out price is prorated. Since all buyers pay the same price, the uniform-price auction is analogous to a Walrasian market, with the only difference that demand schedules are submitted strategically; see Nyborg (2002).

This difference makes uniform-price auctions susceptible to substantial underpricing, because bidders can submit high inframarginal demands that prevent competition on prices and support equilibria where the stop-out price is lower than its Walrasian equivalent. The possibility of underpricing equilibria was first proven in Wilson (1979), Maxwell (1983) and Back and Zender (1993). This result has been shown robust to different model specifications by Ausubel and Cramton (1998), Biais and Faugeron-Crouzet (2002), Engelbrecht-Wiggans and Kahn (1998), Noussair (1994) and Wang and Zender (2002).

A common assumption across these papers is that the supply of the auctioned good is fixed in advance. This seemingly innocuous assumption implies a strategic asymmetry between the bidders and the seller: The former can use their demand schedules to inhibit price competition, but the latter cannot alter her supply schedule to enhance it. It is plausible to expect that the introduction of an adjustable supply should prevent at least some underpricing equilibria. Intuitively, while the steepness of the competitors' demand curves has a price effect which increases the marginal cost of a higher bid, an increasing supply function induces a quantity effect that raises its marginal revenue. Making the quantity effect greater than the price effect inhibits coordination on low prices.

Only a few papers have studied the equilibria of a uniform-price auction with a variable supply. Back and Zender (2001) shows that, if the seller reserves the right to decrease her supply *after* receiving the bids, underpricing – while still possible – is severely curtailed. McAdams (2001) derives a similar result and then shows that underpricing is eliminated if the seller reserves the right to increase or arbitrarily adjust her supply. Lengwiler (1999) assumes that the seller produces the good at a constant marginal cost which is private information to her and studies how the right to restrict supply affects the bidders' demand schedules.

These papers share the assumption that the supply is adjustable *after* the seller has observed the bid schedules. However, there are situations where it may be necessary to precommit and declare the supply schedule *before* observing the bid schedules. For instance, declaring the supply schedule ex ante increases transparency in IPOs of unseasoned shares and thus should reduce the winner's curse. In electricity markets [2] near peak capacity, there may simply be no time to allow for ex post adjustments.

This paper studies the existence of underpricing equilibria when the seller precommits to an increasing supply schedule, as suggested in Pavan (1996). We find that underpricing is still possible, although to a lesser extent than in the case of an ex post decreasable supply. Committing ex ante to an increasing supply attaches a positive

[2] In these markets, the auctioned good is the right to service the exogenous demand for electricity.

quantity effect to price competition. This effect more than compensates the flexibility lost by giving up ex post reductions. On the other hand, note that precommitment entails the risk of losing control on the quantity sold. Therefore, we show also that a fixed supply is in general suboptimal even if the seller faces increasing costs for selling a quantity diverging from her supply target. The expected gain from reducing underpricing may offset the expected loss from selling a quantity potentially different from the target.

A variable supply is not the only means for the seller to obstruct underpricing in uniform-price auctions. Kremer (2001) and Nyborg (2002) suggest adopting different rationing rules. McAdams (2001) proposes to offer discounts to marginal bidders. Some fine-grained institutional details also hamper underpricing: Nyborg (2002) considers allowing only a finite number of bids, or imposing a tick size for price or quantities; Back and Zender (1993) considers the uncertainty about supply induced by the presence of noncompetitive bidders.

Some of these factors may go towards explaining why, in spite of their theoretical ubiquity, the degree in which underpricing equilibria occur is still controversial. The empirical literature has concentrated mostly on the question whether more revenue is raised by a discriminatory or by a uniform-price auction; see Binmore and Swierzbinski (2001) for a critical review. However, the experimental evidence reported in Goswami et al. (1996) shows that bidders manage to coordinate on underpricing, at least in environments where nonbinding preplay communication is possible. Evidence of underpricing is reported by Tenorio (1997) for foreign currency auctions in Zambia, by Kandel et al. (1999) for IPO auctions in Israel, and by Bjonnes (2001) for Treasury auctions in Norway. Keloharju et al. (2002) confirms the underpricing in Treasury auctions in Finland, but argues that it is not due to strategic manipulation.

The rest of the paper is organized as follows. Section 2 describes the model, which is a straightforward variation on the setup in Back and Zender (1993). Section 3 characterizes a large class of symmetric equilibria under fixed supply, which contains as special cases all the symmetric underpricing equilibria studied in the literature. Section 4 studies the effects of an increasing supply schedule and generalizes the equilibria of Section 3 to the case of an increasing and concave supply schedule. Section 5 analyzes the symmetric underpricing equilibria under a linear supply schedule. Section 6 studies the seller's ex ante choice of a linear supply schedule that maximizes her expected profit and provides an example with an explicit derivation. Finally, Section 7 rounds up the paper with a few comments. All proofs are in Appendix A.

2. The model

A single (female) seller wishes to auction a homogenous and perfectly divisible good using a uniform-price format. She can offer a fixed supply Q or, more generally, she can post a (weakly) increasing[3] and right-continuous (aggregate) supply schedule $S(p)$. She can also set a reserve price $p_L \geqslant 0$, under which no sale occurs.

[3] In the following, this and similar qualifiers always hold in the weak sense, unless otherwise noted.

There are $n \geqslant 2$ (male) risk-neutral bidders. The per unit value of the good to each bidder is v. This value is commonly known[4] to the bidders (or, equivalently, is the expected value of a commonly known distribution), while the seller knows only that v is distributed over some nonempty interval $[v_{\mathrm{L}}, v_{\mathrm{H}}]$, with c.d.f. $F(v)$. Each bidder i competes by simultaneously submitting a decreasing and left-continuous demand schedule $d_i(p)$, representing his cumulative demand for the good at a price not greater than p. The resulting aggregate demand schedule $D(p) = \sum_{i=1}^{n} d_i(p)$ is also decreasing and left-continuous.

Following Back and Zender (1993), we define the stop-out price by

$$P = \sup\{p \geqslant p_{\mathrm{L}} \mid D(p) \geqslant S(p)\}$$

when the set $\{p \geqslant p_{\mathrm{L}} \mid D(p) \geqslant S(p)\}$ is not empty, and otherwise we let $P = p_{\mathrm{L}}$. When possible, this definition ensures that the stop-out P clears the market. Moreover, if there are multiple clearing prices, it selects the highest one; if there is no clearing price because of a discontinuity, it selects the price at the discontinuity point; if there is not sufficient demand at p_{L}, it forces the stop-out price to be p_{L} and the good is not auctioned in full.

The rest of the allocation rule is as follows. If P clears the market, each bidder i is awarded a quantity $\hat{d}_i(P) = d_i(P)$. Otherwise, there is an excess demand[5] $E(P) = D(P) - S(P) > 0$, which is rationed pro rata at the margin. Let $\Delta d_i(P) = d_i(P) - \lim_{p \downarrow P} d_i(p)$ and $\Delta D(P) = \sum_{i=1}^{n} \Delta d_i(P)$. Then bidder i receives

$$\hat{d}_i(P) = d_i(P) - \frac{\Delta d_i(P)}{\Delta D(P)} E(P).$$

3. Underpricing equilibria under fixed supply

Throughout this section, we assume that the divisible good is in fixed supply at a level Q. Except for her early choice of the uniform-price format and the reserve price p_{L}, the seller plays no strategic role and we restrict attention to the (sub)game among the n bidders engaged in the auction. The payoff to bidder i is $\pi_i = (v - P)\hat{d}_i(P)$, where P now depends only on the bidders' choice of their demand schedules. If $v < p_{\mathrm{L}}$, participating in the auction is not profitable. We focus on the case where $v \geqslant p_{\mathrm{L}}$. Since bid schedules at prices $p < p_{\mathrm{L}}$ are immaterial, we omit them for simplicity.

The natural benchmark case is the competitive equilibrium, where the market clears at the Walrasian price $P = v$. The next proposition establishes the existence of a wide class of symmetric underpricing equilibria in pure strategies. For any price p^* between p_{L} and v, bidders inhibit price competition by submitting steep demand curves and split

[4] A natural generalization of this assumption is to endow bidders with proprietary information stemming from private signals. This is carried out in Wilson (1979) and in Back and Zender (1993), but – like us – they study equilibria which do not depend on the signals received; therefore, the generalization would be inconsequential. Signal-dependent symmetric equilibria are studied in Wang and Zender (2002).

[5] Without loss of generality, our continuity assumptions rule out the possibility of a strictly positive excess supply.

symmetrically the fixed supply of the divisible good. This behavior is self-enforcing because players' inframarginal bids (made costless in equilibrium by the uniform price format) ensure that each bidders' marginal cost is higher than p^*. This rules out any incentive to raise the price in order to acquire higher quantities.

Proposition 1. *Assume $v \geqslant p_L$ and a fixed supply $S(p) = Q$ for all $p \geqslant p_L$. For any price p^* in $[p_L, v]$, there exists a symmetric Nash equilibrium in pure strategies such that the stop-out price is $P = p^*$. The equilibrium demand schedule for bidder i is*

$$d_i^*(p) = \begin{cases} 0 & \text{if } p > v, \\ y(p) & \text{if } p^* < p \leqslant v, \\ z(p) & \text{if } p_L \leqslant p \leqslant p^*, \end{cases} \tag{1}$$

where $y(p)$ is any positive, decreasing and twice differentiable function on $(p^*, v]$ such that

$$\lim_{p \downarrow p^*} y(p) = \frac{Q}{n}, \tag{2}$$

$$(v - p^*)\left[-\lim_{p \downarrow p^*} y'(p)\right] \leqslant \frac{Q}{n(n-1)} \tag{3}$$

and

$$2y'(p) \leqslant (v - p)y''(p) \quad \text{for all } p \text{ in } (p^*, v) \tag{4}$$

and $z(p)$ is any positive, decreasing, convex and continuous function on $[p_L, p^*]$ such that

$$z(p^*) \geqslant \max\left[\frac{Q}{n}, \frac{Q}{n-1} + (v - p^*) \cdot z'_-(p^*)\right]. \tag{5}$$

There are a few good reasons for our exhibiting this class. First, the equilibria of Proposition 1 encompass the notable cases of symmetric underpricing equilibria when v is commonly known, as listed in Nyborg (2002). These include the linear equilibria of Wilson (1979), and both the linear equilibria of Theorem 1 and the nonlinear equilibria of Theorem 4 in Back and Zender (1993). Notwithstanding this, we note that the assumptions of twice differentiability of $y(p)$ and convexity of $z(p)$ can be relaxed and thus there exist other symmetric equilibria outside of this class.

Second, when the supply schedule is increasing and concave, the equilibria in this class can be generalized to provide a natural mapping between the cases of fixed and increasing (concave) supply. This is carried out in Section 4. Third, in Section 5 (Proposition 4) we derive the choice of $y(p)$ and $z(p)$ that is most conducive to underpricing for an arbitrary increasing supply. We prove that the resulting profile of demand schedules supports the largest set of symmetric equilibrium prices (both within and without the class of equilibria in Proposition 1). Therefore, this class contains the symmetric equilibrium which is most conducive to underpricing under any increasing supply.

Proposition 1 shows that, once the seller has announced a fixed supply Q and set a reserve price p_L, any stop-out price between p_L and the expected value v can be sustained in equilibrium. Since the interval $[p_L, v]$ exhausts the set of feasible and individually rational stop-out prices, Proposition 1 is akin to a "folk theorem" for uniform-price auctions in fixed supply. In particular, for $p_L \leqslant v_L$ bidders can always induce prices below the minimum resale value and thus earn positive profits with certainty. For those Treasury auctions which set the reserve price to zero, this makes the possibility of an underpricing equilibrium far from rare.

Proposition 1 allows for a continuum of equilibrium prices. However, under a fixed supply, all bidders prefer a lower price. Thus, the most reasonable prediction is that the stop-out price should be $P = p_L$, which is the only Pareto efficient outcome for the bidders. This idea is formally captured by applying the coalition-proofness refinement proposed in Bernheim et al. (1987). Since $P = p_L$ is the worst outcome from the seller's viewpoint, its prominence makes it important for her to enhance price competition. The next section suggests a possible route.

4. Underpricing equilibria under increasing supply

Under a fixed supply Q of the divisible good, Proposition 1 shows that the n bidders can sustain an underpricing equilibrium at a stop-out price p^* in $[p_L, v)$ and split symmetrically the quantity Q by posting the profile of demand schedules $\{d_j^*(p)\}_{j=1}^n$ as in Proposition 1.

Suppose from now on that the seller commits ex ante to an increasing supply schedule $S(p)$. Given p^*, assume that $S(p^*) = Q$ so that coordination on p^* is still feasible. The next proposition establishes that, if $S(p)$ is sufficiently elastic at p^*, the profile $\{d_j^*(p)\}_{j=1}^n$ is no longer self-enforcing.

Let $\delta(p) = -[pd'_+(p)]/[d(p)]$ be the (right-hand) price elasticity of the demand schedules $\{d_j^*(p)\}_{j=1}^n$ at a price p in $[p_L, v)$ and set $\delta(p^*) = \lim_{p \downarrow p^*} \delta(p)$. Similarly, let $\sigma(p)$ and $\sigma(p^*)$ be the corresponding price elasticities for the supply schedule $S(p)$.

Proposition 2. *Assume $v > p_L$ and an increasing, absolutely continuous supply schedule $S(p)$. Given p^* in $[p_L, v)$ and the demand schedules $\{d_j^*(p)\}_{j=1}^n$, compute the positive quantity*

$$\alpha(p^*, v, n) = \frac{1}{n}\left[\frac{p^*}{(v - p^*)} - (n-1)\delta(p^*)\right]. \tag{6}$$

If $\sigma(p^*) > \alpha(p^*, v, n)$, then coordination on p^* by submitting the profile of demand schedules $\{d_j^*(p)\}_{j=1}^n$ is no longer an equilibrium for the n bidders.

Intuitively, when the elasticity of the supply schedule is sufficiently high, the negative price effect on bidder i's profits due to the increase in his purchase price is more than compensated by the positive quantity effect induced by the increase in the quantity he wins. This makes i's payoff (locally) increasing to the right of p^* and induces him to bid more aggressively, raising the stop-out price above p^*. Note that $\alpha(p^*, v, n)$ is decreasing in the number of bidders. As n increases, the price effect sustaining p^* has less bite and the quantity effect necessary to countervail it can be achieved with a lower elasticity of the supply schedule.

It is important to clarify the scope of Proposition 2. We do not assume that the seller knows v. She cannot compute α and she cannot directly use it to undermine an underpricing equilibrium. Proposition 2 shows only that bidders' coordination on p^* through the profile of demand schedules $\{d_j^*(p)\}_{j=1}^n$ is not enforceable when the bidders' common value v is such that

$$\frac{v-p^*}{p^*} > [n\sigma(p^*)+(n-1)\delta(p^*)]^{-1}. \tag{7}$$

However, as it stands, Proposition 2 suggests that the seller may have an incentive to strategically precommit to an elastic supply schedule. In Section 6 we show that this is indeed correct.

We point out the analogy with the oligopoly game discussed in Klemperer and Meyer (1989). Here, the role of the firms is played by the intermediaries, usually primary dealers, who buy in the auction at a uniform price p^* and resale at a common price v. Similarly, the role of the demand function is played by the supply curve adopted by the seller; in particular, note that the expression on the left-hand side of (7) is the analog of the Lerner's index. When the stop-out price is p^*, bidder i's profit is $\pi_i(p^*)=(v-p^*)\hat{d}_i(p^*)$. A necessary condition for p^* to be an equilibrium price is that the (right-hand) derivative $\partial_p^+\pi_i(p^*)\leqslant 0$ for all i. For $\hat{d}_i(p^*)=S(p^*)-\lim_{p\downarrow p^*}\sum_{k\neq i} d_k(p)$, this is equivalent to $(v-p^*)/p^*\leqslant\gamma^{-1}(p^*)$, where $\gamma(p)$ is the (right-hand) price elasticity of the residual supply for bidder i. In the case of symmetric equilibria, $\gamma^{-1}(p^*)$ reduces to the expression on the right-hand side of (7).

Proposition 2 does not rule out the possibility of underpricing equilibria under an increasing supply schedule. The next proposition is an existence result that characterizes a large class of underpricing equilibria. It is a natural generalization of Proposition 1, which is precisely recovered for a fixed supply $S(p)=Q$ for any $p\geqslant p_{\mathrm{L}}$.

Proposition 3. *Assume $v\geqslant p_{\mathrm{L}}$ and an increasing, concave, continuous supply $S(p)$ for all $p\geqslant p_{\mathrm{L}}$. Given a price p^* in $[p_{\mathrm{L}},v]$, suppose that there exists a positive, decreasing and twice differentiable function on $(p^*,v]$ such that*

$$\lim_{p\downarrow p^*} y(p)=\frac{S(p^*)}{n}, \tag{8}$$

$$(v-p^*)\left[\frac{S'_+(P^*)}{n-1}-\lim_{p\downarrow p^*} y'(p)\right]\leqslant\frac{S(p^*)}{n(n-1)} \tag{9}$$

and

$$2y'(p)\leqslant(v-p)y''(p)\quad \textit{for all } p \textit{ in } (p^*,v) \tag{10}$$

and a positive, decreasing, convex and continuous function $z(p)$ on $[p_{\mathrm{L}},p^*]$ such that

$$z(p^*)\geqslant\max\left\{\frac{S(p^*)}{n},\frac{S(p^*)}{n-1}+(v-p^*)\cdot\left[z'_-(p^*)-\frac{S'_+(p^*)}{n-1}\right]\right\}. \tag{11}$$

Then there exists a symmetric Nash equilibrium in pure strategies such that the stop-out price is $P = p^*$. The equilibrium demand schedule for bidder i is

$$d_i^*(p) = \begin{cases} 0 & \text{if } p > v, \\ y(p) & \text{if } p^* < p \leqslant v, \\ z(p) & \text{if } p_{\mathrm{L}} \leqslant p \leqslant p^*. \end{cases} \tag{12}$$

Note that, consistently with Proposition 2, Condition (9) implies that, if for a given v the slope $S'_+(p^*)$ of the supply function is sufficiently high, these profiles of demand schedules no longer support an underpricing equilibrium at $p = p^*$.

5. Underpricing equilibria under linear supply

Proposition 2 suggests how the seller might be able to induce more aggressive bidding by posting an increasing supply schedule. In essence, what she has to accomplish is making the quantity effect sufficiently high to compensate for the highest possible price effect that bidders' strategies can achieve. On the other hand, bidders submit their demand schedules only after the supply curve has been announced. Thus, it seems reasonable to assume that they can try to contrast the quantity effect induced by an increasing supply and sustain low prices by resorting to steeper demand schedules.

The next proposition characterizes the set of all prices that can be supported as a symmetric equilibrium in a uniform-price auction when the supply increases endogenously with its price. It turns out that adopting perfectly inelastic demand schedules (with a flat at the equilibrium price) is the best way for bidders to sustain low stop-out prices when they face an increasing supply curve.

The intuition is the following. When the supply increases with its (uniform) price, there is an incentive to bid more aggressively and win a higher amount of the good. To sustain a low price, bidders need to compensate this positive quantity effect by reducing the residual supply available to their competitors. This is most effectively done by submitting perfectly inelastic demand schedules.

Proposition 4. *Assume $v \geqslant p_{\mathrm{L}}$ and an increasing, continuous supply schedule $S(p)$. Let T be the set of all stop-out prices that can be supported by a symmetric equilibrium where players submit decreasing demand schedules. Consider the set T^0 of all stop-out prices p^* than are supported by the following profile of (symmetric) demand schedules:*

$$d_i^*(p) = \begin{cases} 0 & \text{if } p > v, \\ \dfrac{S(p^*)}{n} & \text{if } p^* < p \leqslant v, \\ \dfrac{S(p^*)}{n-1} & \text{if } p_{\mathrm{L}} \leqslant p \leqslant p^*, \end{cases} \tag{13}$$

for $i = 1, \ldots, n$. Then $T = T^0$.

The equilibria of Proposition 4 are again a special case of Proposition 1. However, the power of the result lies elsewhere: For any supply schedule announced by the seller, the inelastic schedules described in (13) represent the best chance for the bidders to sustain an underpricing equilibrium at price p^*. Thus, there is no loss of generality in restrict attention to this profile of bids. We apply this result to the analysis of the strategic choice of a supply schedule by the seller.

We postpone the analysis of the full game to the next section and consider here the second stage, in which bidders compete *after* the seller has announced her choice of the supply schedule. For tractability, we make the assumption that the seller has posted a reserve price $p_{\mathrm{L}} \geqslant 0$ and an increasing (piecewise) linear supply function

$$S(p) = \begin{cases} r + s(p - p_{\mathrm{L}}) & \text{if } p \geqslant p_{\mathrm{L}}, \\ 0 & \text{otherwise,} \end{cases} \tag{14}$$

with $r, s \geqslant 0$. The triple $\{p_{\mathrm{L}}, r, s\}$ defines the linear supply mechanism chosen by the seller. The special case of a fixed supply corresponds to a choice of $s = 0$.

Given the supply mechanism $\{p_{\mathrm{L}}, r, s\}$, for $v < p_{\mathrm{L}}$ there is no trade. If $v \geqslant p_{\mathrm{L}}$, we know from Proposition 4 that without loss of generality we can assume that bidders post the demand schedules given in (13). Hence, substituting $\delta(p^*) = 0$ in (7), we find that bidders can coordinate only on prices such that

$$\frac{v - p^*}{p^*} \leqslant [n\sigma(p^*)]^{-1}. \tag{15}$$

Substituting for

$$\sigma(p^*) = \frac{s p^*}{r + s(p^* - p_{\mathrm{L}})},$$

the set of possible equilibrium stop-out prices turns out to be the interval $[p_c, v]$, where

$$p_c = \max\left\{ p_{\mathrm{L}}, \frac{nv + p_{\mathrm{L}}}{n+1} - \frac{r}{(n+1)s} \right\}. \tag{16}$$

The lower bound on underpricing is p_c, which is properly defined for $s > 0$. (For $s = 0$, the supply is fixed and thus $p_c = p_{\mathrm{L}}$.) This bound is increasing in the number of bidders. Therefore, when the supply is strictly increasing, attracting bidders works to the advantage of the seller. Moreover, as the number n of bidders increases, p_c tends to v and thus the stop-out price must converge to the competitive benchmark. With a strictly increasing supply and an infinite number of players, the seller could extract all the surplus from the bidders. Contrast this with the case of a fixed supply in Proposition 1, where the number of bidders does not affect the set of underpricing equilibria.

Using p_c, we can compare the extent of the possible underpricing when the seller commits ex ante to an increasing supply schedule or reserves the right to decrease ex post a fixed supply. Assuming $p_{\mathrm{L}} = 0$ for simplicity, Back and Zender (2001) shows that the set of possible equilibrium stop-out prices is the interval $[p_z, v]$, with

$$p_z = \frac{n-1}{n} v.$$

Since $p_c \geqslant p_z$ for $r/s \leqslant v/n$, neither procedure is a priori more effective in restricting the risk of underpricing. However, since the support of v is the interval $[v_L, v_H]$, by choosing r and s with $r/s < v_L/n$ the seller can ensure that her ex ante commitment is strictly less conducive to underpricing for any v. Tilting the supply schedule ex ante provides more flexibility than the right to shift it backwards ex post.

5.1. The case of supply uncertainty

When $p_c > p_L$, the use of an increasing supply schedule genuinely reduces the scope for underpricing with respect to the mere introduction of a reserve price p_L. This occurs for $v > p_L + (r/ns)$, when the negative price effect on bidder i's profits due to the increase in the purchase price cannot be made sufficiently strong – even assuming perfectly inelastic demand schedules – to overturn the positive quantity effect due to the increasing supply. Roughly speaking, then, an increasing supply really makes a difference only when $v > p_L + (r/ns)$. It is a natural question to ask whether this would remain true under different specifications of the bidding environment.

We consider the important case when there is supply uncertainty or, more generally, when the bidders have private information [6] about the exact amount of the good on offer. The leading example is the case of Treasury auctions, where noncompetitive bidders are allowed to submit demands to be filled at the stop-out price before the (remaining) quantity is awarded to the competitive bidders. When the amount of noncompetitive demand is not known to the competitive bidders, they face supply uncertainty. Another example arises in electricity markets, when additional power may unexpectedly become available.

Under supply uncertainty, Back and Zender (1993) has derived a class of symmetric equilibria [7] which on average leads to strictly less underpricing than in the standard case. Under a mild assumption on the support of the noncompetitive demand, Nyborg (2002) has proved that these are essentially the only symmetric equilibria robust to supply uncertainty. Therefore, the presence of this form of uncertainty among the bidders reduces (but does not necessarily rule out) the extent of the expected underpricing.

The next proposition derives the analog of these symmetric equilibria in the case of an increasing linear supply schedule as given in (14). The analysis confirms that supply uncertainty reduces but does not eliminate the expected underpricing with respect to the case without uncertainty. Moreover, the linearity of the supply schedule makes it possible to separate the effects of an increasing supply schedule and of supply uncertainty: as in the standard case, the introduction of an increasing supply genuinely

[6] In equilibrium, each bidder submits a demand schedule which is optimal for all realizations of the uncertainty over the supply. Therefore, it is not necessary that bidders agree on the probability distribution, provided that the support is the same. We assume for simplicity that the distribution is unique.

[7] These equilibria are a special case of Proposition 1. They can be read off Proposition 5 by substituting $S(p) = Q$ and $s = 0$.

makes a difference exactly when $v > p_L + (r/ns)$. And when it makes a difference, it eliminates the (symmetric) underpricing equilibria with the lowest stop-out prices.

To model supply uncertainty, we follow Back and Zender (1993) and add to the basic model the assumption that the supply available to the competitive bidders at price p is $\max\{S(p) - \eta, 0\}$, where $\eta \geqslant 0$ is the random reduction due to noncompetitive demand. We assume that the support of η is the interval $[0, \lim_{p\uparrow+\infty} S(p)]$ and that each competitive bidder is allowed to demand as much as he wishes. As a function of $S(p) - \eta$, the stop-out price is now a random variable. We say that underpricing occurs if the realized stop-out price is strictly lower than v for some value[8] of η.

Proposition 5. *Assume $v \geqslant p_L$ and an increasing linear supply schedule $S(p)$ as in (14).*

If $n \geqslant 3$, the only equilibria of Proposition 3 which can survive supply uncertainty have

$$y(p) = \alpha \left(\frac{v - p}{v - p_L} \right)^{1/(n-1)} - s\,\frac{(v - p)}{n - 2} \tag{17}$$

for $p^* < p \leqslant v$, where α is a positive constant such that $y(p^*) = [S(p^*)/n]$.

If $n = 2$, the only equilibria of Proposition 3 which can survive supply uncertainty have

$$y(p) = \alpha \left(\frac{v - p}{v - p_L} \right) - s(v - p) \ln \left(\frac{v - p}{v - p_L} \right) \tag{18}$$

for $p^* < p \leqslant v$, where α is any positive constant such that $y(p^*) = [S(p^*)/2]$.

The remaining specification of the equilibrium demand schedule $d_i^*(p)$ is identical to (12) in Proposition 3. For $s = 0$, we obtain the equilibria under fixed supply given in Back and Zender (1993). For $p^* \uparrow v$, letting $\alpha \uparrow +\infty$ recovers the competitive benchmark, where $P = v$ and no underpricing occurs. Note that for a given $p^* \in [p_L, v)$, when the function $y(p)$ in (17) or (18) is not decreasing over the interval $(p^*, v]$, none of the underpricing equilibria of Proposition 3 sustaining p^* as a stop-out price survives supply uncertainty.

The realized stop-out price is defined implicitly by the equation

$$nd^*(p) = \max\{S(p) - \eta, 0\} \tag{19}$$

as a function of η. When the noncompetitive demand is $\eta = 0$, the stop-out price is p^*. If we parameterize the equilibria of Proposition 5 by p^* in $[p_L, v)$, we can view the stop-out price $P(\eta; p^*)$ as a function of η and the parameter p^*. Since $y(p)$ in the corresponding equilibrium is uniformly higher for higher values of p^*, it follows that $p_1^* < p_2^*$ implies $P(\eta; p_1^*) \leqslant P(\eta; p_2^*)$ for all η. Roughly speaking, the equilibrium associated with a lower p^* generates a higher level of underpricing. In particular, the most severe underpricing occurs when $p^* = p_L$ and $y(p_L) = r/n$.

[8] This is equivalent to the milder requirement that underpricing occurs with positive probability because the equilibrium demand schedules are decreasing and convex.

Note that (19) implies also that the stop-out price $P(\eta; p^*)$ is increasing in η for any p^*. Hence, $P(0; p^*)$ provides an immediate bound for the underpricing which can occur at the equilibrium associated with p^*. Moreover, since the support of η is a nondegenerate interval, the expected underpricing is strictly less than implied by $P(0; p^*)$ confirming Back and Zender's (1993) result. The exact computation of the expected underpricing depends on the distribution of η and is in general long and tedious: Even in the simpler case of a fixed supply, Back and Zender's (1993) works out only the special case $n=2$. However, the point that an increasing supply genuinely makes a difference exactly when $v > p_L + (r/ns)$ can be proved directly.

Suppose $n = 2$. (The argument for $n \geqslant 3$ is analogous.) For any p^* in $[p_L, v)$, substituting for α in (18) implies that the only possible (symmetric) equilibrium demand schedule over $(p^*, v]$ is

$$y(p) = \left(\frac{v-p}{v-p^*}\right)\frac{S(p^*)}{2} - s(v-p)\ln\left(\frac{v-p}{v-p^*}\right).$$

But $y(p)$ can be a piece of the equilibrium demand schedule only if it is decreasing or, equivalently, if its derivative is strictly negative. Computing the derivative, this occurs if

$$s\left[1+\ln\left(\frac{v-p}{v-p^*}\right)\right] < \frac{S(p^*)}{2(v-p^*)}.$$

The left-hand side is bounded above by s because $p^* < p \leqslant v$. The right-hand side is bounded below by $(r/2)[1/(v-p_L)]$ because $S(p^*) \geqslant r$ and $p^* \geqslant p_L$. Therefore, an underpricing equilibrium can always occur if $v \leqslant p_L + (r/2s)$.

Finally, note that the profile of strategies in Proposition 4 is no longer an equilibrium. Intuitively, the difficulty is that supply uncertainty requires that bidders' demand curves must be ex post optimal for the stop-out price associated with any realization of η, while the schedules of Proposition 4 are optimal only at p^*. This situation is the analog of Klemperer and Meyer's (1989) analysis of supply function equilibria in oligopoly, where uncertainty about the market demand narrows down the set of symmetric equilibria.

6. The choice of a supply schedule

In this section we let the seller explicitly use her supply schedule as a strategic variable. We consider a two-stage game where the seller first publicly commits to an increasing linear supply curve and then bidders compete simultaneously on demand schedules within a uniform-price auction.

We assume that the seller's payoff π^s is the difference between the revenue she collects when selling a quantity Q at a uniform price of P and a cost function $C(Q)$; that is, $\pi^s = P \cdot Q - C(Q)$. In the standard case where the supply is fixed and $C(Q) = 0$, maximizing π^s is consistent with avoiding underpricing equilibria. More generally, this formulation takes into account also the costs of auctioning different quantities. Besides the obvious costs of running the auction, this may incorporate institutional considerations about the effects of issuing debt in the case of Treasury auctions or of diluting control in the case of IPO auctions or of risking a blackout in the electricity market.

For tractability, we make the following assumptions. The seller's cost function for selling a quantity Q is $C(Q)=\alpha+\beta(Q-\bar{Q})^2$, with $\alpha \geqslant 0$ and $\beta > 0$; here, $\bar{Q}$ represents her target quantity. Thus, $\bar{Q}$ could be the number of securities that the Treasury would ideally like to auction, the target quantity of shares to be issued through an IPO or the current demand for power in an electricity market. The seller sets a linear supply schedule by choosing a triple $\{p_{\mathrm{L}}, r, s\}$ of positive reals. There is no supply uncertainty.

We know from Section 5 that bidders can coordinate on any stop-out price in $[p_c, v]$ and therefore the second stage allows for multiple equilibria. However, bidder i's payoff in the symmetric equilibrium with stop-out price p is

$$\pi_i(p) = (v-p)\hat{d}_i(p) = (v-p)\left[\frac{r+s(p-p_{\mathrm{L}})}{n}\right]. \tag{20}$$

This quadratic function attains its maximum at $\hat{p}=(1/2)[v+p_{\mathrm{L}}-(r/s)] < p_c$. Assuming that the bidders select the only Pareto efficient (and coalition-proof) equilibrium, we refine the set of possible stop-out prices arising in the second-stage to the singleton p_c in (16).

Although a stop-out price lower than v cannot be ruled out, increasing the elasticity of the supply schedule enhances price competition among bidders and may lead to a higher equilibrium price. On the other hand, making the supply schedule more elastic contrasts with the objective of maintaining control on the total quantity auctioned, which is better served by a fixed supply. There is an obvious trade-off between price competition and quantity control. The next proposition states that the best trade-off is not struck at either extreme.

Proposition 6. *Suppose that the seller believes v to be uniformly distributed on* $[0,1]$. *The optimal linear supply mechanism exists and it is strictly increasing for $p \geqslant p_{\mathrm{L}}$, with $0 < s^* < +\infty$.*

Note that, if the simple assumption of a linear supply mechanism suffices to rule out the optimality of a constant supply, this is true a fortiori for more general supply mechanisms. Therefore, the restriction to linear supply mechanisms does not detract from the result. A second advantage of this restriction is that one can explicitly solve the game and derive the optimal supply mechanism, making comparative statics possible.

For an example, the next proposition characterizes the optimal linear supply mechanism with two bidders and a target quantity $\bar{Q}=0$.

Proposition 7. *Suppose that the seller believes v to be uniformly distributed on* $[0,1]$. *Assume $n=2$ and a cost function $C(Q)=a+bQ^2$ with $a \geqslant 0$ and $b > 0$. The optimal linear supply mechanism has $p_{\mathrm{L}}=\frac{2}{5}$, $r=0$ and $s=5/(4b)$ and the supply schedule is*

$$S(p) = \max\left\{\frac{5}{4b}\,p - \frac{1}{2b}, 0\right\}.$$

We briefly comment on this example. Under the optimal linear supply mechanism, the seller's profit is $E(\pi^s) = 1/(20b) - a$. The cost function has two parameters: a may be interpreted as the fixed (and unavoidable) cost of running the auction; instead, b is positively related to the marginal cost of expanding the quantity issued. Not surprisingly, the seller's profit increases as either parameter decreases. Note that the reserve price is set below the bidders' expected value (that is, $E(v) = \frac{1}{2}$): When the supply is not constrained to be fixed, the seller gets higher profits by reducing the risk of not selling the good while inhibiting bidders' coordination on low prices.

Finally, the slope of the supply schedule decreases as b increases: As the cost of expanding Q increases, the supply becomes more inelastic. In particular, a fixed supply would become optimal for $b \to +\infty$. On the other hand, as $b \to 0$, the optimal linear supply mechanism would fix the reserve price at $p_{\mathrm{L}} = \frac{2}{5}$ and let the supply be perfectly elastic. This would make infinite both the quantity auctioned and the seller's profit. However, these extreme cases must be taken with a grain of salt because they depend heavily on the implicit assumption that the valuation of the buyers does not depend on the quantity issued. We find it more plausible to assume that our model holds only over an intermediate range of parameters.

7. Concluding remarks

We close the paper with some remarks on the implications of our analysis for the two prominent examples of Treasury auctions and initial public offerings.

The market for Treasury securities is by far the most relevant example of a widespread use of uniform-price auctions for divisible goods. This paper suggests that, for uniform-price auctions, the practice to combine a fixed supply with a reserve price below market values can be suboptimal for the Treasury. The adoption of an elastic supply with an appropriate reserve price may allow the Treasury to enhance price competition and raise higher expected revenues. Moreover, the use of an elastic supply would enable the Treasury to exploit a positive correlation between the supply of securities in the primary market and the stop-out price. Successful auctions with high stop-out prices (and low yields) would be associated to higher issuances, while unsuccessful auctions would turn out in a lower issuance. Hence, the introduction of uniform-price auctions with an elastic supply would go toward a reduction of the overall cost of public debt.

There is an obvious trade-off between controlling the interest rates and the supply of securities to the market. Therefore, while selecting an increasing supply schedule, the Treasury must compromise between different objectives. A fixed supply may be adequate when the prime objective of the Treasury is the amount of debt rather than its unit cost. Instead, a perfectly elastic supply may be appropriate when the cost of issuing debt in a variable supply is small compared to the benefit of controlling the interest rate in the primary market. This latter choice, used for example in Italy up to 1962, lets the monetary authorities know in advance the cost of issuing new debt.

However, it also implies that the Treasury loses control on the amount of securities[9] supplied to the market.

Our analysis suggests that an optimal way to share the control on yields and quantities between the market and the monetary authorities is to post an increasing (but not perfectly elastic) supply schedule and a reserve price close to the expected resale value in the secondary market. The frequently observed choice of a zero reserve price adopted in several primary markets seems inadequate. Moreover, it may work to the advantage of the Treasury to encourage participation: Increasing the number of dealers in the primary market leads to stronger price competition and reduces the need for an elastic supply schedule, leading to a tighter control on the liquidity of the market.

Another prominent example of using uniform-price auctions for divisible goods concerns the IPOs of unseasoned shares; see Sherman (2001) for a review of the different methods used. Although book building has been the dominant mechanism for most of the 1990s, in the last few years an increasing number of companies have decided to issue stocks on-line using Internet-based uniform-price auctions; see for example the site http://www.openipo.com. Also, Israel frequently conducts its IPOs using uniform-price auctions.

The participants in an IPO are usually a seller (the company which goes public), a financial intermediary and the investors. As in Biais and Faugeron-Crouzet (2002), assume that the seller seeks to maximize the proceeds from the IPO and the financial intermediary acts in the seller's best interest. Then a uniform-price IPO auction fits the simple model of Section 2 and our analysis implies that a fixed supply may lead to possibly large underpricing. This theoretical result is confirmed in Biais and Faugeron-Crouzet (2002). Likewise, by analyzing 27 IPO uniform-price auctions held in Israel between 1993 and 1996, Kandel et al. (1999) has found significant underpricing, with demand schedules that have a flat around the IPO price and are very similar to those described in Proposition 1.

Our analysis shows that a company which goes public has a simple way to reduce the possibility of large underpricing and raise more money. Instead of announcing a fixed supply, the company should make the supply of shares a function of the stop-out price. This would encourage investors to bid more aggressively in the hope of being awarded a higher amount of shares. As suggested in Proposition 6, this mechanism is robust to the possibility the company faces a cost in selling a number of shares different from a target supply.

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[9] A partial solution is offered by the possibility of reopening the auction; see Scalia (1997).

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Appendix A

Proof of Proposition 1. The proof is in three steps. First, we compute the residual supply for bidder i when his competitors follow the equilibrium strategies. Second, we check that, when the stop-out price is p^*, his best reply is to post a demand schedule such that $\lim_{p\downarrow p^*} d_i(p) = Q/n$ and thus, in particular, the schedule described in (1). Third, we establish that p^* is his preferred stop-out price.

Consider first bidder i's residual supply. Suppose that his competitors follow the equilibrium strategies. Their aggregate demand is $D_{-i}(p) = \sum_{k\neq i} d_k^*(p)$ and therefore the residual supply curve for i is

$$x_i(p) = Q - D_{-i}(p) = \begin{cases} Q & \text{if } \ p > v, \\ Q - (n-1)y(p) & \text{if } \ p^* < p \leqslant v, \\ Q - (n-1)z(p) & \text{if } \ p_{\mathrm{L}} \leqslant p \leqslant p^*. \end{cases}$$

Consider now bidder i's best reply when the stop-out price is p^*. Player i can win any quantity Q^* between $\max\{Q-(n-1)z(p^*), 0\}$ and Q/n by submitting an appropriate demand schedule $d_i(p)$ such that $\lim_{p\downarrow p^*} d_i(p) = Q^*$. Since bidder i's payoff at the stop-out price p^* is $\pi_i = (v - p^*)\hat{d}_i(p^*)$, the best he can do is maximizing his assigned quantity $\hat{d}_i(p^*)$ by going for $Q^* = Q/n$.

Finally, we check that the preferred stop-out price is indeed p^*. To avoid negative pay-offs, P cannot be greater than v. Furthermore, $P \geqslant \hat{p}$ where $\hat{p} = \sup\{p \geqslant p_{\mathrm{L}} : x_i(p) \leqslant 0\}$ if $\{p \geqslant p_{\mathrm{L}} : x_i(p) \leqslant 0\}$ is not empty, and $\hat{p} = p_{\mathrm{L}}$, otherwise: at prices below $\hat{p}$, the fixed supply of the divisible good is entirely demanded by his competitors and therefore bidder i cannot drive the stop-out price below $\hat{p}$. Thus, it suffices to show that, as a function of the stop-out price $P \in [\hat{p}, v]$, π_i achieves its maximum at p^*. We check this separately over the two intervals $[\hat{p}, p^*]$ and $[p^*, v]$. Since $y(p)$ and $z(p)$ are continuous, bidder i has no incentive to raise the aggregate demand above Q at any $p \neq p^*$ because only his demand would be rationed; hence, we can assume that no rationing takes place at any $p \neq p^*$ and let $\pi_i(p) = (v - p)x_i(p)$ for any $p \in [\hat{p}, p^*) \cup (p^*, v]$.

The function $(v - p)x_i(p)$ is continuous and concave over $[\hat{p}, p^*]$ and is increasing over this interval if its (left-hand) derivative $(n-1)z(p) - Q - (n-1)(v-p)z'_-(p) \geqslant 0$, which follows from (5). Furthermore, as $z(p^*) \geqslant Q/n$, $\pi_i(p^*) \geqslant (v - p^*)x_i(p^*)$, and hence $\pi_i(p^*) \geqslant \pi_i(p)$ for any $p \in [\hat{p}, p^*]$. Consider next the interval $[p^*, v]$. Similarly to the above, $(v-p)x_i(p)$ is continuous over $(p^*, v]$ and since $\pi_i(p^*) = (v - p^*)(Q/n)$, from (2) π_i is right-continuous (and hence continuous) at $p = p^*$ as well. Thus, π_i achieves its maximum at p^* if its (right-hand) derivative $\partial_p^+ \pi_i(p^*) \leqslant 0$ and its second derivative $\partial^2 \pi_i(p)/\partial p^2 \leqslant 0$ for all p in (p^*, v). These two inequalities follow, respectively, from (3) and (4). $\square$

Proof of Proposition 2. Note that $\alpha(p^*, v, n) > 0$ by (3). It is obvious that the profile $\{d_j^*(p)\}_{j=1}^n$ described in (1) can still achieve the stop-out price p^* and a symmetric split of Q, giving each bidder a profit $\pi_i = (v - p^*)(Q/n)$. We claim that, for $\sigma(p^*) > \alpha(p^*, v, n)$, $\{d_j^*(p)\}_{j=1}^n$ is no longer a Nash equilibrium because any bidder i strictly prefers to deviate. More precisely, we show that i's profits are increasing in a right neighborhood of p^* and thus he prefers to bid more aggressively, raising the stop-out price above p^*.

Suppose that i's competitors follow their part of the strategy profile $\{d_j^*(p)\}_{j=1}^n$. Substituting $S(p)$ for Q in the proof of Proposition 1, the residual supply curve for bidder i in the interval $(p^*, v]$ is

$$x_i(p) = S(p) - (n-1)y(p).$$

Moreover (see Proof of Proposition 1), bidder i's profit function $\pi_i(p)$ is right-continuous at $p = p^*$ with $\pi_i(p) = (v - p)x_i(p)$ for any $p \in (p^*, v]$. Therefore, it suffices to show that its (right-hand) derivative is strictly positive in a (right) neighborhood of p^*. Since $S(p)$ is increasing with $S(p^*) = Q$ and $y(p)$ is decreasing with $\lim_{p \downarrow p^*} y(p) = (Q/n)$, for $p > p^*$ we have

$$\begin{aligned}\partial_p^+ \pi_i(p) &= (v - p)[S'_+(p) - (n-1)y'_+(p)] - S(p) + (n-1)y(p) \\ &\geqslant (v - p)[S'_+(p) - (n-1)y'_+(p)] - (Q/n).\end{aligned}$$

For $\varepsilon > 0$ sufficiently small, (6) and continuity imply that the last expression is strictly positive in $(p^*, p^* + \varepsilon)$, which establishes the claim. □

Proof of Proposition 3. The proof is very similar to the one of Proposition 1. The residual supply curve for bidder i is now

$$x_i(p) = \begin{cases} S(p) & \text{if } p > v, \\ S(p) - (n-1)y(p) & \text{if } p^* < p \leqslant v, \\ S(p) - (n-1)z(p) & \text{if } p_{\mathrm{L}} \leqslant p \leqslant p^*. \end{cases}$$

Consider bidder i's best reply when the stop-out price is p^*. He can win any quantity Q^* between $\max\{S(p^*) - (n-1)z(p^*), 0\}$ and $S(p^*)/n$ by submitting an appropriate demand schedule $d_i(p)$ with $\lim_{p \downarrow p^*} d_i(p) = Q^*$. Since bidder i's payoff at the stop-out price p^* is $\pi_i = (v - p^*)\hat{d}_i(p^*)$, the best he can do is maximizing his assigned quantity $\hat{d}_i(p^*)$ by going for $Q^* = S(p^*)/n$.

To check that the preferred stop-out price is p^*, it suffices to show that π_i achieves its maximum at p^*. Since $y(p)$ and $z(p)$ are continuous, bidder i has no incentive to raise the aggregate demand above $S(p)$ at any $p \neq p^*$ because only his demand would be rationed; hence, $\pi_i(p) = (v - p)x_i(p)$ for any $p \in [\hat{p}, p^*) \cup (p^*, v]$, where $\hat{p} = \sup\{p \geqslant p_{\mathrm{L}}: x_i(p) \leqslant 0\}$ if $\{p \geqslant p_{\mathrm{L}}: x_i(p) \leqslant 0\}$ is nonempty, and $\hat{p} = p_{\mathrm{L}}$ otherwise.

Over $[\hat{p}, p^*]$, $(v - p)x_i(p)$ is continuous and concave. Therefore, it is increasing over this interval if its (left-hand) derivative is positive at $p = p^*$. This follows from (11). Furthermore, as $z(p^*) \geqslant S(p^*)/n, \pi_i(p^*) \geqslant (v - p^*)x_i(p^*) \geqslant \pi_i(p)$ for any $p \in [\hat{p}, p^*]$. Consider next the interval $[p^*, v]$. As above, $(v - p)x_i(p)$ is continuous over $(p^*, v]$ and since $\pi_i(p^*) = (v - p^*)(S(p^*)/n)$, from (8) π_i is right-continuous (and hence continuous) at $p=p^*$ as well. Thus, π_i achieves its maximum at p^* if its (right-hand) derivative $\partial_p^+\pi_i(p^*) \leqslant 0$ and its second derivative $\partial^2\pi_i(p)/\partial p^2 \leqslant 0$ for all p in (p^*, v). The first inequality follows immediately from (9). The second inequality follows from (10), because $S(p)$ is increasing and concave. □

Proof of Proposition 4. It is obvious that $T^0 \subseteq T$. We show that $T \subseteq T^0$. Suppose that p^* in T is supported by a symmetric equilibrium where each bidder i posts the same demand schedule $d_i(p)=d(p)$. By symmetry, each bidder wins a quantity $S(p^*)/n$ of the divisible good[10] at the price p^*.

By the definition of stop-out price, $\lim_{p\downarrow p^*}[nd(p) - S(p)] \leqslant 0$. As $d(p)$ is decreasing and $S(p)$ is continuous, this implies $d(p) \leqslant S(p^*)/n$ for $p > p^*$. Therefore, the residual supply to bidder i for $p > p^*$ under the equilibrium profile $\{d_i(\cdot)\}_{i=1}^n$ is greater than under the profile in (13).

Hence, if bidder i does not find profitable to increase the stop-out price above p^* when his competitors post $d(p)$, this must remain true when they submit the demand schedules in (13). Furthermore, bidder i cannot make the stop-out price go below p^* because for $p < p^*$ the supply is entirely demanded by his competitors. It follows that the profile of demands in (13) supports p^* as a stop-out price. □

Proof of Proposition 5. Since noncompetitive demand makes the stop-out price a random variable, bidder i now wants to maximize his expected payoff $E\pi_i = E[(v-p)\hat{d}_i(p)]$. As in the proofs of Propositions 1 and 3, this problem can be reduced to the choice of the optimal stop-out price (given others' equilibrium strategies).

Bidder i maximizes $E\pi_i$ by making sure that his demand schedule is optimal for almost all realizations of η. Given the support of η and the equilibrium strategies in Proposition 5, bidder i has no incentive to make the stop-out price go below p^*. This can be avoided by submitting $d^*(p)$, in which case the stop-out price can span at most the interval $[p^*, v]$. The function $y(p)$ is then part of an equilibrium demand schedule only if it satisfies the first-order condition

$$(v-p)[S'_+(p) - (n-1)y'_+(p)] = y(p)$$

for almost all p in $[p^*, v)$. By the assumed monotonicity of $y(p)$, this differential equation uniquely identifies $d^*(p)$ over this interval. For $S'(p)=s$, the solutions of this linear differential equation for $n \geqslant 3$ and $n=2$ are, respectively, (17) and (18). The boundary condition follows from (8). □

Proof of Proposition 6. We need to show that there exists an optimal linear supply mechanism and that it has $s>0$. For convenience, rewrite the cost function $C(Q)=\alpha+\beta(Q-\bar{Q})^2$ as $C(Q)=(\alpha+\beta\bar{Q}^2)-(2\beta\bar{Q})Q+\beta Q^2$. Renaming the parameters, we study the (seemingly) simple cost function $C(Q)=a-cQ+bQ^2$ for $a \geqslant 0, b>0$ and $c \geqslant 0$. This cost function has a minimum in $Q^*=c/(2b)$, where its value is $C(Q^*)=4ab-c^2$; we assume $4ab-c^2 \geqslant 0$ and make sure that the cost is never negative.

[10] By pro-rate rationing, in a symmetric equilibrium each bidder is assigned a quantity $\hat{d}_i(p^*)=S(p^*)/n$ even if there are flats at the stop-out price that make aggregate demand exceed supply.

Consider the first stage of the game. Given the supply mechanism, the second stage leads to a stop-out price

$$P(v;\, p_{\mathrm{L}}, r, s, n) = \max\left\{ p_{\mathrm{L}}, \frac{nv + p_{\mathrm{L}}}{n+1} - \frac{r}{(n+1)s} \right\}$$

that determines the total quantity auctioned $\hat{D}(P)$. The optimal linear supply mechanism for the uniform-price auction is obtained by selecting the triple $p_{\mathrm{L}} \geqslant 0$, $r \geqslant 0$, and $s \geqslant 0$ that maximizes the expected profit for the seller, which is given by

$$E(\pi^s) = \int_0^1 \pi^s[P(v;\cdot), \hat{D}(P(v;\cdot)]\, \mathrm{d}v.$$

Depending on the parameters, we can distinguish three cases: (i) if $v < p_{\mathrm{L}}$, there is no sale; (ii) if $p_{\mathrm{L}} \leqslant v \leqslant p_{\mathrm{L}} + (r/ns)$, the stop-out price is p_{L} and the quantity auctioned is $\hat{D}(p_{\mathrm{L}}) = r$; and (iii) if $p_{\mathrm{L}} + [r/(ns)] < v \leqslant 1$, the stop-out price is $P(v) = (nsv - r + sp_{\mathrm{L}})/[(n+1)s]$ and the quantity auctioned is $\hat{D}(P(v)) = [n/(n+1)](sv + r - sp_{\mathrm{L}})$.

Therefore, the expectation can be written as the sum of three integrals over the (possibly empty) supports $v < p_{\mathrm{L}}$, $p_{\mathrm{L}} \leqslant v \leqslant p_{\mathrm{L}} + [r/(ns)]$ and $p_{\mathrm{L}} + [r/(ns)] < v \leqslant 1$. Writing p instead of p_{L} and $f(p,r,s)$ instead of $E(\pi^s)$ for convenience, we have

$$\begin{aligned} f(p,r,s) = {} & \int_0^p [-C(0)]\, \mathrm{d}v + \int_p^{p+r/ns} [pr - C(r)]\, \mathrm{d}v \\ & + \int_{p+\frac{r}{ns}}^1 [P(v) \cdot \hat{D}[P(v)] - C(\hat{D}[P(v)])]\, \mathrm{d}v, \end{aligned} \tag{21}$$

where integrals over empty supports are meant to be null. There are three possible cases: (A1) if $p > 1$, only the support of the first integral is not null; (A2) if $p \leqslant 1 \leqslant p + (r/ns)$, only the support of the first two integrals are not null; and (A3) if $p + [r/(ns)] < 1$, all the three supports are not null.

The rest of the proof is in four steps. The first step shows that $s = 0$ can be part of an optimal linear supply only if cases A1 or A2 hold. The second step determines the best triple under case A2 and checks that it generates a strictly higher profit than any triple under case A1; therefore, the optimal triple does not occur in case A1. The third step exhibits a triple for case A3 which generates an even higher profit; therefore, the optimal triple does not occur in case A2 either. It follows that the optimal triple (if it exists) must occur in case A3. The fourth step establishes existence and concludes the proof.

Step 1: For $r = 0$ and $s = 0$ the supply is zero, which is obviously not optimal. If $r > 0$ and $s = 0$, we are in A2. Hence, $s = 0$ only if we are in A1 or A2.

Step 2: We begin by noting that A1 occurs for $p > 1$, so the second and third integral in (21) have an empty support. Since the reserve price is set too high and no one ever buys, the seller's profit is just $E(\pi^s) = -C(0) = -a$.

Consider now A2, which occurs for $p < 1 \leqslant p + (r/ns)$: Only the third integral has an empty support. Computing the other two integrals, we find that the seller's expected profit is

$$f(p,r,s) = -a + \int_p^1 [p \cdot r + (cr - br^2)]\,\mathrm{d}v$$

$$= -a + (1-p)[(p+c)r - br^2]. \tag{22}$$

This expression is a second-degree polynomial in p and r which does not explicitly depend on s (provided that $p + (r/2s) \geqslant 1$). Thus, we ignore s momentarily. We begin from the boundary of the admissible region. Conditional on $p = 0$, the optimal choice is $r = c/(2b)$ which gives $f(0, c/(2b), s) = -a + c^2/(4b) \geqslant -a$. Any triple with $p = 1$ or $r = 0$ can be discarded, because it fails the test of making $f(p,r,s)$ greater than $-a$.

Moving to the interior of the admissible region, the only stationary point which survives this test has $p = (2-c)/3$ and $r = (1+c)/3b$. As $p \geqslant 0$, this is interesting only for $c \leqslant 2$. Substituting in (22), we obtain

$$f\left(\frac{2-c}{3}, \frac{1+c}{3b}, s\right) = -a + \frac{(1+c)^3}{27b} \geqslant -a + \frac{c^2}{4b} = f\left(0, \frac{c}{2b}, s\right), \tag{23}$$

with equality holding at $c=2$. Hence, the highest profit in A2 is attained for $p=(2-c)/3$ and $r = (1+c)/3b$ if $c \leqslant 2$ and for $p = 0$ and $r = c/(2b)$ if $c > 2$.

Reintroducing s in the picture, recall that this class requires $p + (r/ns) \geqslant 1$. This places an upper bound on the admissible values of s. More precisely, the optimal supply mechanisms are: (a) $p_L = (2-c)/3$, $r = (1+c)/3b$ and $s \leqslant 1/(nb)$ if $c \leqslant 2$; and (b) $p = 0$, $r = c/(2b)$ and $s \leqslant c/(2nb)$ if $c > 2$. They include as special cases the possibility of setting $s = 0$. Finally, (23) shows that the seller's profit is higher than in A1.

Step 3: Consider now A3, under which no integral has an empty support a priori. The seller's expected profit is

$$f(p,r,s) = -a + \frac{r}{ns}[(p+c)r - br^2]$$
$$+ \int_{p+r/ns}^1 \left[\frac{n[nsv - (r-sp)][sv + (r-sp)]}{(n+1)^2 s} + \left(c\,\frac{nsv + n(r-sp)}{n+1} - b\left[\frac{nsv + n(r-sp)}{n+1}\right]^2 \right) \right] \mathrm{d}v.$$

Carrying out the computation [11] and rearranging, this gives

$$f(p,r,s,) = -a + \frac{1}{6n(n+1)}[(2bn^3s^2 + 3n^2s + n^3s)p^3$$
$$- (6bn^3rs + 9n^2r - 3cn^2s - 3cn^3s + 6n^2s + 3n^3r + 6bn^3s^2)p^2$$

[11] All computations from here on have been carried out using Maple V (Release 5).

$$+\frac{1}{s}(6bn^3s^3 - 6cn^3rs - 6cn^2s^2 - 3n^3s^2 + 6bn^3r^2s + 12bn^3rs^2$$
$$+12n^2rs + 3nr^2 - 6cn^3s^2 - 6cn^2rs + 3r^2 + 3n^2s^2 + 6n^2r^2)p$$
$$+\frac{1}{s^2}(3nr^3 - 3n^2rs^2 - 6bn^3s^3r + 2n^3s^3 - 6bn^3r^2s^2 + 3cn^2s^3$$
$$+3n^3rs^2 - 6n^2r^2s - 6bnr^3s + 6cn^3rs^2 + 6cn^2rs^2 + 3cnr^2s - 4br^3s$$
$$+r^3 - 2bn^3s^4 + 3cn^3s^3 + 3cr^2s)]. \tag{24}$$

Finding the maximizer of this expression is a hard task, but fortunately our argument requires only to exhibit a triple (p,r,s) satisfying A3 and achieving a higher profit than A2. Choosing the triple

$$\hat{p} = 0, \quad \hat{r} = 0, \quad \hat{s} = \frac{2n + 3c(n+1)}{4bn}$$

in A3, we obtain

$$f(\hat{p},\hat{r},\hat{s}) = -a + \frac{2n + 3c(n+1)}{48b(n+1)},$$

which we compare against the values in (23). For $c \leqslant 2$ (and $n \geqslant 2$), it is easily checked that

$$-a + \frac{2n + 3c(n+1)}{48b(n+1)} \geqslant -a + \frac{(1+c)^3}{27b},$$

similarly,

$$-a + \frac{2n + 3c(n+1)}{48b(n+1)} \geqslant -a + \frac{c^2}{4b}$$

for $c > 2$. Hence, $(\hat{p},\hat{r},\hat{s})$ – which we do not claim is optimal – does better than any triple satisfying A1 or A2. Note in particular that $\hat{s} > 0$.

Step 4: It remains to be shown that there actually exists an optimal triple in A3. While $f(p,r,s)$ is a continuous function, the region defining A3 is not compact because s is unbounded from below; thus the maximization problem may not have a solution at all. Consider what may happen for $s \to +\infty$. Since A3 imposes $p + r/(sn) \leqslant 1$, either (i) $r = o(s)$ and $r \to 0$; or (ii) $r = O(s)$ and the ratio (r/s) stay bounded between 0 and $(1-p)n$.

If $r = o(s)$,

$$f(p,r,s) \sim -\frac{1}{3n(n+1)}[bn^3(1-p)^3]s^2$$

as $s \to +\infty$. As $p < 1$, this guarantees that eventually $f(p,r,s) < -a$. Therefore, by a standard argument, the set of triples over which the maximum of $f(p,r,s)$ should be searched can be trimmed and made compact without loss of generality. Then Weierstrass' Theorem ensures that a maximum exists.

If $r = O(s)$,

$$f(p,r,s) \sim -\frac{1}{3n(n+1)} \times \left[bn^3(1-p)^3 s^2 + 3bn^3(1-p)[r^2 + (1-p)rs] + \frac{b(3nr^3 + 2r^3)}{s}\right]$$

as $s \to +\infty$. Then $p < 1$ guarantees again that eventually $f(p,r,s) < -a$, and the argument given above applies as well.

We conclude that the triple p^*, r^*, s^* exists and belongs to A3, which in turn implies that $s^* > 0$. □

Proof of Proposition 7. We know from Proposition 6 that the optimal linear mechanism belongs to A3. Replacing $n = 2$ and $c = 0$ into (24) we have

$$\begin{aligned} E(\pi^s) = -a + \frac{1}{108s^2}(&33pr^2s - 16br^3s + 16s^3 + 12rs^2 - 12ps^3 - 24r^2s \\ &+ 48prs^2 - 24p^2s^3 - 48br^2s^2 + 48pbr^2s^2 - 48brs^3 \\ &+ 96bprs^3 - 48bp^2rs^3 - 16bs^4 + 48bps^4 - 48bp^2s^4 \\ &+ 16bp^3s^4 + 20p^3s^3 - 60p^2rs^2 + 7r^3). \end{aligned}$$

To increase readability, apply an increasing linear transformation and let $g(p,r,s) = 108[E(\pi^s) + a]$. Then g and $E(\pi^s)$ have the same maximizers. Collecting terms with respect to p, note that g is a cubic (in p) and that the coefficient of its leading term is positive

$$\begin{aligned} g(p,r,s) = {} & (16bs^2 + 20s)p^3 - (48brs + 24s + 60r + 48bs^2)p^2 \\ & + \left(\frac{96brs^3 - 12s^3 + 48br^2s^2 + 48rs^2 + 33r^2s + 48bs^4}{s^2}\right)p \\ & - \left(\frac{16bs^4 + 16br^3s - 16s^3 - 12rs^2 + 48br^2s^2 + 24r^2s - 7r^3 + 48brs^3}{s^2}\right). \end{aligned}$$

We ignore momentarily r and s. Differentiating with respect to p, we find

$$\begin{aligned} \frac{\partial g}{\partial p} = {} & (48bs^2 + 60s)p^2 - (96brs + 48s + 120r + 96bs^2)p \\ & + \left(\frac{96brs^3 - 12s^3 + 48br^2s^2 + 48rs^2 + 33r^2s + 48bs^4}{s^2}\right). \end{aligned}$$

Hence, the two stationary points of g with respect to p are

$$p_1 = \frac{8brs + 4s + 8bs^2 + 10r - 3\sqrt{4br^2s + 4s^2 + 5r^2}}{8bs^2 + 10s}$$

and

$$p_2 = \frac{8brs + 4s + 8bs^2 + 10r + 3\sqrt{4br^2s + 4s^2 + 5r^2}}{8bs^2 + 10s}.$$

The cubic g has a local maximum in $p = p_1$ and a local minimum in $p = p_2$. Since $p_2 \geqslant 1$ for any pair (r,s) in the admissible set, this implies that the maximizers of g can only occur at $p=0$ or $p=p_1$ depending on whether $p_1 < 0$ or $p_1 \geqslant 0$, respectively. We check separately the two subcases.

First, suppose $p = 0$. Then the function

$$g(0,r,s) = -\left(\frac{16bs^4+16br^3s-16s^3-12rs^2+48br^2s^2+24r^2s-7r^3+48brs^3}{s^2}\right)$$

has only two stationary points

$$\left(r = \frac{-3-2\sqrt{2}}{b}, s = \frac{3+\sqrt{2}}{2b}\right) \quad \text{and} \quad \left(r = \frac{-3+2\sqrt{2}}{b}, s = \frac{3-\sqrt{2}}{2b}\right).$$

Since both have $r < 0$, they fall outside of the admissible set and the maximizer (if it exists) can only be a corner solution with $r = 0$. Substituting $r = 0$ and maximizing $g(0,0,s)$, we find $s = 1/(2b)$. However, $p = 0$, $r = 0$, and $s = 1/(2b)$ imply

$$p_1 = \frac{8brs + 4s + 8bs^2 + 10r - 3\sqrt{4br^2s + 4s^2 + 5r^2}}{8bs^2 + 10s} = \frac{1}{7} > 0,$$

contradicting the initial assumption of $p_1 < 0$.

Now, suppose $p = p_1$. Then the function

$$g(p_1,r,s) = \frac{27(+8s^3 - 28br^2s^2 - 16b^2r^2s^3 - 10r^2s + \sqrt{(4br^2s + 4s^2 + 5r^2)^3})}{s^2(4bs+5)^2}$$

has only three stationary points:

$$\left(r = 0, s = \frac{5}{4b}\right), \quad \left(r = \frac{1}{3b}, s = \frac{1}{2b}\right) \quad \text{and} \quad \left(r = -\frac{1}{3b}, s = \frac{1}{2b}\right).$$

We rule out the third one because it is not admissible and the second one because it yields $(\partial^2 g/\partial r^2) > 0$. The first candidate, instead, passes the second order conditions. Hence, noting that $p_1 = \frac{2}{5}$ for $r = 0$ and $s = 5/(4b)$, we conclude that the maximizer is at $p = \frac{2}{5}$, $r = 0$ and $s = 5/(4b)$. This is the optimal linear supply mechanism we were after. □

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