Supplement to

A Rational Theory of "Irrational Exuberance"

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Abstract

In this document we prove three results discussed in the main text without proof. Section A1 considers the variant in which entrepreneurs receive multiple private signals with correlated errors. Section A2 considers the variant in which the entrepreneurs' cost of investment is subject to idiosyncratic shocks. Section A3 considers the variant in which aggregate investment is observed with noise.

A1. Multiple private signals with correlated errors

Consider the following variant of the baseline model in which each entrepreneur receives two private signals

$$x_i = \theta + \varepsilon_1 + \xi_i \quad \text{and} \quad y_i = \theta + \varepsilon_2 + \eta_i,$$

where ε_1 and ε_2 are independent common noises, Normally distributed with variances $1/\pi_1$ and $1/\pi_2$, whereas ξ_i and η_i are private noises, Normally distributed with variances $1/\pi_\xi$ and $1/\pi_\eta$, independent and identically distributed across agents, and independent of any other random variable in the economy.

Let $\pi_x \equiv \pi_1\pi_\xi/(\pi_1+\pi_\xi)$, $\pi_y \equiv \pi_2\pi_\eta/(\pi_2+\pi_\eta)$, $\pi \equiv \pi_\theta+\pi_x+\pi_y$, $\delta_0 \equiv \pi_\theta/\pi$, $\delta_1 \equiv \pi_x/\pi$, and $\delta_2 \equiv \pi_y/\pi$. The posterior about θ of an entrepreneur with signals x and y is Normal with mean

$$\mathbb{E}[\theta|x,y] = \delta_0\mu + \delta_1 x + \delta_2 y$$

and variance $1/\pi$. His posterior about the common noise components satisfies

$$\mathbb{E}\left[\varepsilon_1|x,y\right]=\rho_1\left[x-\mathbb{E}\left[\theta|x,y\right]\right] \quad \text{and} \quad \mathbb{E}\left[\varepsilon_2|x,y\right]=\rho_2\left[y-\mathbb{E}\left[\theta|x,y\right]\right]$$

where

$$\rho_1 \equiv \tfrac{\pi_\xi}{\pi_\xi+\pi_1}=\frac{\pi_x}{\pi_1} \quad \text{and} \quad \rho_2 \equiv \tfrac{\pi_\eta}{\pi_\eta+\pi_2}=\frac{\pi_y}{\pi_2}$$

are the correlations of the errors across the entrepreneurs, i.e. $\rho_1 = Corr(\varepsilon_1+\xi_i,\varepsilon_1+\xi_j)$ and $\rho_2 = Corr(\varepsilon_2+\eta_i,\varepsilon_2+\eta_j)$,for all i,j with $i\neq j$.

In any linear equilibrium, there exist coefficients $(\beta_0,\beta_1,\beta_2)$ such that individual investment satisfies

$$k\left(x,y\right)=\beta_0+\beta_1 x+\beta_2 y, \tag{1}$$

in which case aggregate investment is given by

$$K\left(\theta,\varepsilon_1,\varepsilon_2\right)=\beta_0+\left(\beta_1+\beta_2\right)\theta+\beta_1\varepsilon_1+\beta_2\varepsilon_2.$$

It follows that, from the perspective of the uninformed traders, observing K is equivalent to observing a Gaussian signal

$$z\equiv\frac{K-\beta_0}{\beta_1+\beta_2}=\theta+\frac{\beta_1}{\beta_1+\beta_2}\varepsilon_1+\frac{\beta_2}{\beta_1+\beta_2}\varepsilon_2$$

with precision

$$\pi_z=\left[\left(\frac{\beta_1}{\beta_1+\beta_2}\right)^2\pi_1^{-1}+\left(\frac{\beta_2}{\beta_1+\beta_2}\right)^2\pi_2^{-1}\right]^{-1}.$$

It is then immediate that the equilibrium price satisfies

$$p=\gamma_0+\gamma_1 K,$$

with $\gamma_1>0$ if and only if $\beta_1+\beta_2>0$. By implication, $\alpha\equiv\lambda\gamma_1>0$ if and only if $\beta_1+\beta_2>0$, as in the baseline model. It follows that, an entrepreneur's best response satisfies

$$k\left(x,y\right)=\left(1-\alpha\right)\mathbb{E}\left[\kappa(\theta)|x,y\right]+\alpha\mathbb{E}\left[K|x,y\right] \tag{2}$$

where $\kappa(\theta)\equiv\kappa_0+\kappa_1\theta$, with

$$\kappa_0\equiv\frac{\lambda\gamma_0}{1-\alpha} \text{ and } \kappa_1\equiv\frac{1-\lambda}{1-\alpha}.$$

Substituting

$$\mathbb{E}\left[K|x,y\right] = \beta_0 + \left[\beta_1\left(1-\rho_1\right) + \beta_2\left(1-\rho_2\right)\right]\mathbb{E}\left[\theta|x,y\right] + \beta_1\rho_1 x + \beta_2\rho_2 y$$

into (2) we have that

$$\begin{aligned} k\left(x,y\right) &= \left(1-\alpha\right)\mathbb{E}\left[\kappa_0 + \kappa_1\theta|x,y\right] + \alpha\mathbb{E}\left[K|x,y\right] \\ &= \left(1-\alpha\right)\kappa_0 + \left(1-\alpha\right)\kappa_1\mathbb{E}\left[\theta|x,y\right] \\ &\quad + \alpha\beta_0 + \alpha\left[\beta_1\left(1-\rho_1\right)+\beta_2\left(1-\rho_2\right)\right]\mathbb{E}\left[\theta|x,y\right] + \alpha\beta_1\rho_1 x + \alpha\beta_2\rho_2 y \end{aligned}$$

For the above to coincide with (1) for all x and y, the coefficients $(\beta_0, \beta_1, \beta_2)$ must solve the following system:

$$\left(\beta_0 - \kappa_0\right)\left(1-\alpha\right) = \left\{\left(1-\alpha\right)\kappa_1 + \alpha\left[\beta_1\left(1-\rho_1\right)+\beta_2\left(1-\rho_2\right)\right]\right\}\delta_0\mu \tag{3}$$

$$\beta_1\left(1-\alpha\rho_1\right) = \left\{\left(1-\alpha\right)\kappa_1 + \alpha\left[\beta_1\left(1-\rho_1\right)+\beta_2\left(1-\rho_2\right)\right]\right\}\delta_1 \tag{4}$$

$$\beta_2\left(1-\alpha\rho_2\right) = \left\{\left(1-\alpha\right)\kappa_1 + \alpha\left[\beta_1\left(1-\rho_1\right)+\beta_2\left(1-\rho_2\right)\right]\right\}\delta_2 \tag{5}$$

It is then immediate that

$$\frac{\beta_2}{\beta_1} = \frac{\delta_2}{\delta_1}\frac{1-\alpha\rho_1}{1-\alpha\rho_2}. \tag{6}$$

Hence, as in the benchmark model, a positive complementarity amplifies the relative impact of the signal with the higher correlation of errors across agents (in other words, the piece of information with the higher "commonality" across agents).

In the remainder of this section, instead of solving for the equilibrium degree of complementarity α, we illustrate directly the effect of α on the ratio of non-fundamental to fundamental volatility. This permits us to illustrate the robustness of our key positive and normative predictions to the more general information structures considered here. We thus proceed as follows.

First, we solve (3)-(5) for β_1 and β_2, which gives

$$\beta_1 = \kappa_1\frac{\left(1-\alpha\right)\left(1-\alpha\rho_2\right)\pi_x}{\left(1-\alpha\rho_1\right)\left(1-\alpha\rho_2\right)\pi_\theta + \left(1-\alpha\right)\left(1-\alpha\rho_2\right)\pi_x + \left(1-\alpha\right)\left(1-\alpha\rho_1\right)\pi_y} \tag{7}$$

$$\beta_2 = \kappa_1\frac{\left(1-\alpha\right)\left(1-\alpha\rho_1\right)\pi_y}{\left(1-\alpha\rho_1\right)\left(1-\alpha\rho_2\right)\pi_\theta + \left(1-\alpha\right)\left(1-\alpha\rho_2\right)\pi_x + \left(1-\alpha\right)\left(1-\alpha\rho_1\right)\pi_y} \tag{8}$$

Next, we define fundamental and non-fundamental volatility as follows:

$$FV \equiv \mathbb{E}\left[Var\left(K|\varepsilon_1,\varepsilon_2\right)\right] = \left(\beta_1+\beta_2\right)^2 Var\left(\theta\right) = \left(\beta_1+\beta_2\right)^2\pi_\theta^{-1} \tag{9}$$

$$NFV \equiv \mathbb{E}\left[Var\left(K|\theta\right)\right] = \left(\beta_1\right)^2 Var\left(\varepsilon_1\right) + \left(\beta_2\right)^2 Var\left(\varepsilon_2\right) = \left(\beta_1\right)^2\rho_1/\pi_x + \left(\beta_2\right)^2\rho_2/\pi_y. \tag{10}$$

The former measures the variation in K that is due to variation in θ, whereas the latter measures the variation in K that is due to variation in the common errors.

Substituting the equilibrium values of β_1 and β_2 into (9) and (10), we then get that

$$\begin{aligned} FV &= \frac{\kappa_1^2 (1-\alpha)^2 \left[(1-\alpha\rho_2)\,\pi_x + (1-\alpha\rho_1)\,\pi_y\right]^2}{(denom)^2\,\pi_\theta}, \\ NFV &= \frac{\kappa_1^2 (1-\alpha)^2 \left[(1-\alpha\rho_2)^2 \rho_1\pi_x + (1-\alpha\rho_1)^2 \rho_2\pi_y\right]}{(denom)^2}. \end{aligned}$$

where $denom \equiv (1-\alpha\rho_1)(1-\alpha\rho_2)\,\pi_\theta + (1-\alpha)(1-\alpha\rho_2)\,\pi_x + (1-\alpha)(1-\alpha\rho_1)\pi_y$. It follows that the ratio of non-fundamental to fundamental volatility is given by

$$R \equiv \frac{NFV}{FV} = \frac{(1-\alpha\rho_2)^2 \rho_1\pi_x + (1-\alpha\rho_1)^2 \rho_2\pi_y}{\left[(1-\alpha\rho_2)\,\pi_x + (1-\alpha\rho_1)\,\pi_y\right]^2}\pi_\theta,$$

and hence that

$$\frac{\partial R}{\partial \alpha} = \frac{2\pi_x\pi_y\pi_\theta\,(\rho_1-\rho_2)^2}{\left[(1-\alpha\rho_2)\,\pi_x + (1-\alpha\rho_1)\,\pi_y\right]^3}. \tag{11}$$

Hence in any equilibrium in which $\alpha < \min\{\rho_1^{-1}, \rho_2^{-1}\}$, necessarily $\partial R/\partial\alpha \geq 0$ (with strict inequality except in the degenerate case in which $\rho_1 = \rho_2$). The result actually holds true more generally for any equilibrium in which $\beta_1, \beta_2 > 0$. To see this, suppose, towards a contradiction, that $\beta_1, \beta_2 > 0$ and that $\partial R/\partial\alpha < 0$, or equivalently that $(1-\alpha\rho_2)\,\pi_x + (1-\alpha\rho_1)\,\pi_y < 0$, which requires that $\alpha > 1$. Because $\beta_1, \beta_2 > 0$, from (7) and (8) we then have that $sign(1-\alpha\rho_1) = sign\,(1-\alpha\rho_2)$. But then necessarily

$$(1-\alpha\rho_1)(1-\alpha\rho_2)\,\pi_\theta + (1-\alpha)[(1-\alpha\rho_2)\,\pi_x + (1-\alpha\rho_1)\pi_y] > 0.$$

Because $\kappa_1(1-\alpha) = 1-\lambda > 0$, from (7) and (8), we then have that $(1-\alpha\rho_1) > 0$ and $(1-\alpha\rho_2) > 0$, in which case necessarily $\partial R/\partial\alpha > 0$, a contradiction. Hence, in any equilibrium in which investment responds positively to both sources of information, an increase in the complementarity perceived by the entrepreneurs necessarily leads to an increase in the non-fundamental volatility of aggregate investment relative to its fundamental volatility.

We conclude that the main positive prediction of the benchmark model (Corollary 1) extends to these more general information structures. Furthermore, because the efficient strategy is always $k_i = \mathbb{E}_i[\theta]$ for all i, the main normative prediction (Corollary 2) also extends.

A2. Heterogeneity in the entrepreneurs' cost of investment

Next, consider a variant of the benchmark model in which each entrepreneur receives only a private signal $x_i = \theta + \xi_i$ about the fundamental θ. The cost of investment for entrepreneur i is now $k_i^2/2 - w_i k_i$, where w_i is a random variable reflecting a shock to the marginal cost of investment. Assume that $w_i = \omega + \zeta_i$, where the common shock ω is Normally distributed with mean 0 and variance σ_ω^2, while the idiosyncratic shocks ζ_i are independent of ω, Normally distributed with mean 0 and variance σ_ζ^2 and independent across entrepreneurs. Each entrepreneur observes w_i, but not ω.

This environment differs from the benchmark model in two ways. First, there is no piece of information that is commonly known by the entrepreneurs but not by the traders. Second, the problem is no longer one of pure common values: the shocks w_i introduce a private-value component. At the same time, the role of the aggregate shock ω is similar to the role of the expectational shock ε in the benchmark model: it is an unobserved random variable that is uncorrelated with θ and that moves aggregate investment.

In this environment, the price continues to satisfy $p = \mathbb{E}[\theta|K]$, but now an entrepreneur's investment depends on his private signal x_i and on his private cost w_i. Following steps similar to those in Proposition 10 in the main text, we can establish the following result.

Proposition. **(i)** *In any equilibrium in which high investment is good news for θ, there exist a scalar $\alpha > 0$ and a function $\kappa(\theta, w)$ such that*

$$k(x,w) = \mathbb{E}\left[\ (1-\alpha)\,\kappa(\theta,w) + \alpha K(\theta,\omega)\ \mid x, w\ \right]. \tag{12}$$

(ii) *λ small enough suffices for the equilibrium to be unique and for investment to increase with both θ and ω.*

(iii) *The efficient investment satisfies*

$$k(x,w) = \mathbb{E}\left[\ \theta + w \mid x, w\ \right]. \tag{13}$$

(iv) *In any equilibrium in which investment increases with both θ and ω, investment underreacts to θ and overreacts to ω.*

Proof. *Part (i).* The best-response condition is now given by

$$k(x,w) = w + \mathbb{E}\left[(1-\lambda)\,\theta + \lambda p(\theta,\omega)\,|x,w\right]. \tag{14}$$

It follows that, in any equilibrium in which $p(\theta,\omega)$ is linear in (θ,ω), there are coefficients $(\beta_0, \beta_1, \beta_2)$ such that $k(x,w) = \beta_0 + \beta_1 x + \beta_2 w$ and therefore $K(\theta,\omega) = \beta_0 + \beta_1\theta + \beta_2\omega$. The equilibrium price then satisfies $p(\theta,\omega)=\gamma_0+\gamma_1 K(\theta,\omega)$, where

$$\gamma_0 \equiv \frac{\pi_\theta}{\pi_\theta+\pi_z}\mu - \frac{\pi_z}{\pi_\theta+\pi_z}\frac{\beta_0}{\beta_1} \qquad \text{and} \qquad \gamma_1 \equiv \frac{\pi_z}{\pi_\theta+\pi_z}\frac{1}{\beta_1}$$

with $\pi_z \equiv (\beta_1/\beta_2)^2 \sigma_\omega^{-2}$. Hence, $\gamma_1 > 0$ if and only if $\beta_1 > 0$. The result then follows by letting $\alpha = \lambda\gamma_1$ and $\kappa(\theta,w) = \frac{w+(1-\lambda)\theta+\lambda\gamma_0}{1-\lambda\gamma_1}$, where $\alpha=\lambda\gamma_1$ is the degree of complementarity in investment decisions, while $\kappa(\theta,w)$ is the equilibrium investment in the game among the entrepreneurs, when they all receive a shock w, θ is common knowledge, and the price function is $p=\gamma_0+\gamma_1 K$.

Part (ii). This follows from the same steps as in the benchmark model.

Part (iii). The result is immediate.

Part (iv). By (14), in any linear equilibrium,

$$\begin{aligned}
k(x,w) &= w+\mathbb{E}\left[(1-\lambda)\theta+\lambda(\gamma_0+\gamma_1 K(\theta,\omega))|x,w\right] \\
&= w+\mathbb{E}\left[(1-\lambda)\theta+\lambda\left(\gamma_0+\gamma_1\left(\beta_0+\beta_1\theta+\beta_2\omega\right)\right)|x,w\right] \\
&= w+\mathbb{E}\left[(1-\lambda)\theta+\lambda\gamma_0+\lambda\gamma_1\beta_0+\lambda\gamma_1\beta_1\theta\right]+\mathbb{E}\left[\lambda\gamma_1\beta_2\omega|x,w\right].
\end{aligned}$$

Since $\mathbb{E}[\theta|x,w]=\mathbb{E}[\theta|x]=\frac{\pi_\theta}{\pi_\theta+\pi_x}\mu+\frac{\pi_x}{\pi_\theta+\pi_x}x$ and $\mathbb{E}[\omega|x,w]=\mathbb{E}[\omega|w]=\eta w$, where $\eta\equiv\sigma_w^{-2}/(\sigma_w^{-2}+\sigma_\omega^{-2})>0$, we have that

$$k(x,w)=f(x)+(1+\lambda\gamma_1\beta_2\eta)\,w$$

where $f(x)\equiv\mathbb{E}\left[(1-\lambda)\theta+\lambda\gamma_0+\lambda\gamma_1\beta_0+\lambda\gamma_1\beta_1\theta|x\right]$. It follows that β_2 must equal $1+\lambda\gamma_1\beta_2\eta$, or equivalently $\beta_2=1/(1-\lambda\gamma_1\eta)$. In any equilibrium in which $\beta_1>0$ and $\beta_2>0$, we have that $\gamma_1>0$ and $\lambda\gamma_1\eta<1$. But then $\beta_2>1$. Furthermore, β_1 must equal $[(1-\lambda)+\lambda\gamma_1\beta_1]\frac{\pi_x}{\pi_\theta+\pi_x}$ or equivalently

$$\beta_1=\frac{\pi_x}{\pi_\theta+\pi_x}\left(\frac{(1-\lambda)}{1-\lambda\gamma_1\frac{\pi_x}{\pi_\theta+\pi_x}}\right)<\frac{\pi_x}{\pi_\theta+\pi_x}$$

because $\gamma_1\equiv\left[\beta_1\sigma_\omega^{-2}\right]/\left[\beta_2^2\pi_\theta+\beta_1^2\sigma_\omega^{-2}\right]<1$.

In contrast, in the efficient allocation, $k(x,w)=w+\mathbb{E}[\theta|x]$, which gives the result. ■

As in the benchmark model, whenever high investment is "good news" for θ, the equilibrium price increases with aggregate investment, inducing complementarity in investment decisions. In fact, we can now guarantee that this is the case in *every* equilibrium. Once again, this complementarity is unwarranted from a social perspective and it now amplifies the response of investment to the common cost shock ω.

A3. Optimal release of information

Finally, consider a variant in which traders observe aggregate investment with noise, as discussed in Section 5.3 in the main text. That is, assume traders observe $\tilde{K} = K + \eta$, where η is aggregate measurement error, independent of all other shocks, with mean zero and variance $1/\pi_\eta$. The following result was claimed in footnote 18 without proof.

Claim. *Any* (β_1, β_2) *that can be sustained with a stabilization policy* $\tau(p) = \tau_1 p$, *with* $\tau_1 \in [0, +\infty)$, *can also be sustained with a policy that control the quality of information* $\pi_\eta \in [0, +\infty)$ *about* K.

Proof. First, consider the game in which the policy maker stabilizes prices with a tax $\tau(p) = \tau_1 p$. An investment strategy in which the sensitivity to private and common information is (β_1, β_2) can be sustained as an equilibrium if and only if there exists a $\tau_1 \in [0, +\infty)$ such that

$$\begin{aligned} \beta_1 &= \frac{(1-\lambda)\delta_1}{1-\lambda\hat{\alpha}(\tau_1;\beta_1,\beta_2)\delta_1} \\ \beta_2 &= \frac{[1-\lambda+\lambda\hat{\alpha}(\tau_1;\beta_1,\beta_2)\beta_1]\delta_2}{1-\lambda\hat{\alpha}(\tau_1;\beta_1,\beta_2)} \end{aligned} \tag{15}$$

where

$$\hat{\alpha}(\tau_1;\beta_1,\beta_2) \equiv \frac{\gamma_1(\beta_1,\beta_2)}{1+\tau_1}$$

and

$$\gamma_1(\beta_1,\beta_2) \equiv \frac{\pi_z}{\pi_\theta+\pi_z}\frac{1}{\beta_1+\beta_2}$$

with

$$\pi_z = \pi_z(\beta_1,\beta_2) \equiv \left(\frac{\beta_1+\beta_2}{\beta_2}\right)^2 \pi_y.$$

To see this, note that when the entrepreneurs follow a linear strategy $k(x,y) = \beta_0 + \beta_1 x + \beta_2 y$ the price is equal to

$$p = \frac{1}{1+\tau_1}[\gamma_0 + \gamma_1 K]$$

where $\gamma_0 = \gamma_0(\beta_0,\beta_1,\beta_2) \equiv \frac{\pi_\theta}{\pi_\theta+\pi_z}\mu - \frac{\pi_z}{\pi_\theta+\pi_z}\frac{\beta_0}{\beta_1+\beta_2}$, $\gamma_1 = \gamma_1(\beta_1,\beta_2)$ and $K = \beta_0 + \beta_1\theta + \beta_2 y$. The best response for each entrepreneur then consists in following the strategy $k(x,y) = \tilde{\beta}_0 + \tilde{\beta}_1 x + \tilde{\beta}_2 y$ given by

$$\begin{aligned} \tilde{\beta}_0 &= \frac{\lambda\gamma_0}{1+\tau_1} + (1-\lambda)\delta_0\mu + \lambda\hat{\alpha}[\beta_0 + \beta_1\delta_0\mu] \\ \tilde{\beta}_1 &= (1-\lambda)\delta_1 + \lambda\hat{\alpha}\beta_1\delta_1 \\ \tilde{\beta}_2 &= (1-\lambda)\delta_2 + \lambda\hat{\alpha}[\beta_1\delta_2 + \beta_2] \end{aligned}$$

For this strategy to coincide with $k\left(x,y\right)=\beta_0+\beta_1 x+\beta_2 y$, (β_1,β_2) must satisfy (15) and β_0 must solve

$$\beta_0=\frac{\lambda\gamma_0(\beta_0,\beta_1,\beta_2)}{1+\tau_1}+\left(1-\lambda\right)\delta_0\mu+\lambda\hat{\alpha}(\beta_1,\beta_2;\pi_1)\left[\beta_0+\beta_1\delta_0\mu\right] \tag{16}$$

Because (16) is linear in β_0, a solution to (16) always exists. We conclude that (15) are both necessary and sufficient for the existence of an equilibrium in which the sensitivity to private and common information is (β_1,β_2).

Next, consider the game in which the policy maker controls the quality of information π_η about K. A (linear) strategy in which the sensitivity to private and common information is (β_1,β_2) can be sustained as an equilibrium if and only if there exists a $\pi_\eta^{-1}\in[0,+\infty)$ such that

$$\begin{aligned}\beta_1 &= \frac{(1-\lambda)\delta_1}{1-\lambda\gamma_1(\pi_\eta^{-1};\beta_1,\beta_2)\delta_1} \\ \beta_2 &= \frac{\left[1-\lambda+\lambda\gamma_1(\pi_\eta^{-1};\beta_1,\beta_2)\beta_1\right]\delta_2}{1-\lambda\gamma_1(\pi_\eta^{-1};\beta_1,\beta_2)}\end{aligned} \tag{17}$$

where

$$\gamma_1(\pi_\eta^{-1};\beta_1,\beta_2)\equiv\frac{\pi_z}{\pi_\theta+\pi_z}\frac{1}{\beta_1+\beta_2}$$

and

$$\pi_z=\pi_z(\pi_\eta^{-1};\beta_1,\beta_2)\equiv\left(\left(\frac{\beta_2}{\beta_1+\beta_2}\right)^2\pi_y^{-1}+\left(\frac{1}{\beta_1+\beta_2}\right)^2\pi_\eta^{-1}\right)^{-1}. \tag{18}$$

To see this, note that if $k(x,y)=\beta_0+\beta_1 x+\beta_2 y$ can be sustained in equilibrium, then the price must be equal to $p=\gamma_0+\gamma_1[K+\eta]$, where $K=\beta_0+\beta_1\theta+\beta_2 y$, $\gamma_1=\gamma_1(\beta_1,\beta_2;\pi_\eta)$ and

$$\gamma_0=\gamma_0(\beta_0,\beta_1,\beta_2;\pi_\eta)\equiv\frac{\pi_\theta}{\pi_\theta+\pi_z}\mu-\frac{\pi_z}{\pi_\theta+\pi_z}\frac{\beta_0}{\beta_1+\beta_2}$$

with π_z as in (18). Replacing K and p into

$$k\left(x,y\right)=\mathbb{E}\left[\ (1-\lambda)\,\theta+\lambda p\mid x,y\ \right]. \tag{19}$$

we have that the entrepreneurs' best response consists in following the strategy $k\left(x,y\right)=\tilde{\beta}_0+\tilde{\beta}_1 x+\tilde{\beta}_2 y$ given by

$$\begin{aligned}\tilde{\beta}_0 &= \left(1-\lambda\right)\delta_0\mu+\lambda\gamma_0+\lambda\gamma_1\left[\beta_0+\beta_1\delta_0\mu\right] \\ \tilde{\beta}_1 &= \left(1-\lambda\right)\delta_1+\lambda\gamma_1\beta_1\delta_1 \\ \tilde{\beta}_2 &= \left(1-\lambda\right)\delta_2+\lambda\gamma_1\left[\beta_1\delta_2+\beta_2\right]\end{aligned}$$

For this strategy to coincide with $k(x,y) = \beta_0 + \beta_1 x + \beta_2 y$ it must be that (β_1, β_2) satisfy (17) and that β_0 solves

$$\beta_0 = \lambda \frac{\gamma_0(\beta_0, \beta_1, \beta_2; \pi_\eta)}{1 - \lambda\gamma_1(\beta_1, \beta_2; \pi_\eta)} + \frac{(1-\lambda)\,\delta_0\mu}{[1 - \lambda\gamma_1(\beta_1, \beta_2; \pi_\eta)]} + \lambda \frac{\gamma_1(\beta_1, \beta_2; \pi_\eta)}{1 - \lambda\gamma_1(\beta_1, \beta_2; \pi_\eta)} \beta_1 \delta_0 \mu \tag{20}$$

Because (20) is linear in β_0, a solution to (20) always exists. We conclude that (17) are both necessary and sufficient for the existence of an equilibrium in which the sensitivity to private and common information is (β_1, β_2).

The result then follows from the fact that, given any (β_1, β_2), the functions $\hat{\alpha}(\cdot; \beta_1, \beta_2)$ and $\gamma_1(\cdot; \beta_1, \beta_2)$ are continuous over $[0, +\infty)$ with the same range. ■