Matching Auctions

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Motivation

- Mediated matching central to "sharing economy"

- Most matching markets intrinsically **dynamic — re-matching**
  - shocks to profitability of existing matching allocations
  - gradual resolution of uncertainty about attractiveness
  - preference for variety

- Re-matching, while pervasive, largely ignored by matching theory
This paper

- Dynamic matching
  - mediated (many-to-many) interactions
  - evolving private information
  - payments
  - capacity constraints

- Applications
  - scientific outsourcing (Science Exchange)
  - lobbying
  - sponsored search
  - internet display advertising
  - lending (Prospect, LendingClub)
  - B2B
  - health-care (MEDIGO)
  - organized events (meetings.com)

- Matching auctions

- Dynamics under profit vs welfare maximization
Plan

- Model
  - Matching auctions
  - Truthful bidding
  - Profit maximization
  - Distortions
- Endogenous processes
- Conclusions
Model

- Profit-maximizing platform mediates interactions between 2 sides, $A, B$
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- Agents: $N_A = \{1, \ldots, n_A\}$ and $N_B = \{1, \ldots, n_B\}$, $n_A, n_B \in \mathbb{N}$
Model

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- Agents: $N_A = \{1, \ldots, n_A\}$ and $N_B = \{1, \ldots, n_B\}$, $n_A, n_B \in \mathbb{N}$

- Period-$t$ match between agents $(i, j) \in N_A \times N_B$ yields gross payoffs
  \[ v_{ijt}^A = \theta_i^A \cdot \epsilon_{ijt}^A \quad \text{and} \quad v_{ijt}^B = \theta_j^B \cdot \epsilon_{ijt}^B \]

  \[ \theta_i^k: \text{"vertical" type} \]

  \[ \epsilon_{ijt}^k: \text{"horizontal" type (time-varying match-specific)} \]
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  $\theta_i^k$: "vertical" type

  $\varepsilon_{ijt}^k$: "horizontal" type (time-varying match-specific)

- Agent $i$'s period-$t$ (flow) type ($i \in N_A$):
  \[ v_{it}^A = (v_{i1t}^A, v_{i2t}^A, ..., v_{in_B t}^A) \]
Model

- Profit-maximizing platform mediates interactions between 2 sides, $A, B$

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\nu_{ijt}^A = \theta_i^A \cdot \varepsilon_{ijt}^A \quad \text{and} \quad \nu_{ijt}^B = \theta_j^B \cdot \varepsilon_{ijt}^B
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\[
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\]

- Agent $i$’s payoff $(i \in N_A)$:

\[
U_i^A = \sum_{t=0}^{\infty} \delta^t \sum_{j \in N_B} \nu_{ijt}^A \cdot x_{ijt} - \sum_{t=0}^{\infty} \delta^t p_{it}^A
\]

with $x_{ijt} = 1$ if $(i, j)$-match active, $x_{ijt} = 0$ otherwise.
Model

- Platform’s profits:

\[
\sum_{t=0}^{\infty} \delta^t \left( \sum_{i \in N_A} p^A_{it} + \sum_{j \in N_B} p^B_{jt} - \sum_{i \in N_A} \sum_{j \in N_B} c_{ijt} \cdot x_{ijt} \right)
\]
Model

- In each period $t \geq 1$, each agent $l \in N^k$ from each side $k = A, B$ can be matched to at most $m^k_l$ agents from side $-k$.
  - **one-to-one matching**: $m^k_l = 1$ all $l = 1, ..., n^k$, $k = A, B$
  - **many-to-many matching with no binding capacity constraints**: $m^k_l \geq n^{-k}$, all $l = 1, ..., n^k$, $k = A, B$
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In each period $t \geq 1$, platform can match up to $M$ pairs of agents

- space, time, services constraint

- platform can delete previously formed matches and create new ones.

Total number of existing matches cannot exceed $M$ in all periods.
Each $\theta_i^k$ drawn independently from (abs cont.) $F_i^k$ over $\Theta_i^k = [\theta_i^k, \bar{\theta}_i^k]$
Model

- Each $\theta_i^k$ drawn independently from (abs cont.) $F_i^k$ over $\Theta_i^k = [\theta_i^k, \bar{\theta}_i^k]$.

- Period-$t$ horizontal type $\epsilon_{ijt}^k$ drawn from cdf $G_{ijt}^k(\epsilon_{ijt}^k | \epsilon_{ijt-1}^k)$. 
Model

- Each $\theta^k_i$ drawn independently from (abs cont.) $F^k_i$ over $\Theta^k_i = [\theta^k_i, \bar{\theta}^k_i]$

- Period-$t$ horizontal type $\varepsilon^k_{ijt}$ drawn from cdf $G^k_{ijt}(\varepsilon^k_{ijt} | \varepsilon^k_{ijt-1})$

- Agents observe $\theta^k_i$ prior to joining, but learn $(\varepsilon^k_{ijt})$ over time
Plan

• Model

• Matching auctions

• Truthful bidding

• Profit maximization

• Distortions

• Endogenous processes

• Conclusions
Matching auctions

- At $t = 0$ (i.e., upon joining the platform), each agent $i \in N_k$ purchases membership status $\theta_i^k \in \Theta_i^k$ at price $p_i^k(\theta)$
  - higher status $\rightarrow$ more favorable treatment in subsequent auctions

Full transparency - bids, payments, membership, matches all public.
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- At any $t \geq 1$: 

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- At any \( t \geq 1 \):
  - agents bid \( b^k_{lt} \equiv (b^k_{ljt})_{j \in N_{-k}} \), one for each partner from side \( -k \)
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    \[
    S_{ijt} \equiv \beta^A_i(\theta^A_i) \cdot b_{ijt}^A + \beta^B_j(\theta^B_j) \cdot b_{ijt}^B - c_{ijt}
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  - matches maximizing sum of scores s.t. individual and aggregate capacity constraints implemented
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  - unmatched agents pay nothing

Full transparency - bids, payments, membership, matches all public.
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- Full transparency - bids, payments, membership, matches all public.
Payments (PST + BV)

- Fixing weights $\beta$, *weighted surplus*:

$$w_t \equiv \sum_{i \in N_A} \sum_{j \in N_B} S_{ijt} \cdot \chi_{ijt}$$

- $w_{t}^{-i,A} =$ weighted surplus in absence of agent $i \in N_A$ (same as $W_t$, but with $S_{ij}=0$, all $j \in N_B$).

- Period-$t$ payments, $t \geq 1$:

$$\psi_{it}^A = \sum_{j \in N_B} b_{ijt}^A \cdot \chi_{ijt} - \frac{w_t - w_{t}^{-i,A}}{\beta_i^A (\theta_i^A)}$$
(Horizontal) match quality under rule $\chi$:

$$D_f^A(\theta) \equiv \mathbb{E}^{\lambda|\chi|\theta} \left[ \sum_{t=1}^{\infty} \delta^t \sum_{j \in N_B} \epsilon_{ijt}^A \chi_{ijt} \right]$$

Period-0 membership fees:

$$\psi_{i0}^A = \theta_i^A D_i^A(\theta) - \int_{\theta_i^A}^{\theta_i} D_i^A(\theta_{-i}, y) dy - \mathbb{E}^{\lambda|\chi|\theta_0} \left[ \sum_{t=1}^{\infty} \delta^t \psi_{it}^A \right] - L_i^A$$
Payments

- Payments similar to GSPA for sponsored search but adjusted for
  
  - dynamic externalities
  - costs of information rents (captured by $\beta$)
  - matches need not maximize true surplus
Plan

- Model
- Matching auctions
- **Truthful bidding**
- Profit maximization
- Distortions
- Endogenous processes
- Conclusions
Definition

Strategy profile $\sigma = (\sigma^k)_{k=A,B}^{i\in\Omega^k}$ is truthful if each agent
- selects membership status corresponding to true vertical type
- at each $t \geq 1$, bids given by $b^k_{ijt} = v^k_{ijt} = \theta^k_i \cdot \varepsilon^k_{ijt}$, all $(i,j) \in \Omega^A \times \Omega^B$, $k = A, B$, irrespective of membership status selected at $t = 0$ and of past bids.

Truthful equilibrium is an equilibrium in which strategy profile is truthful.
Truthful bidding

Theorem

Any matching auction in which $L^k_i$ large enough admits an equilibrium in which all agents participate in each period and follow truthful strategies. Furthermore, such truthful equilibria are periodic ex-post (agents’ strategies are sequentially rational, regardless of beliefs about other agents’ past and current types).
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Profit maximization

Theorem

Let

\[ \beta_{l,k}^{l,P}(\theta_{l,k}) \equiv 1 - \frac{1 - F_{l}^{k}(\theta_{l}^{k})}{f_{l}^{k}(\theta_{l}^{k})\theta_{l}^{k}}, \quad \text{all } l \in N^{k}, \ k = A, B. \tag{1} \]

Suppose \( D_{l}^{k}(\theta^{-l,k}, \theta_{l,k}^{k}; \beta^{P}) \geq 0, \quad \text{all } l \in N^{k}, \ k = A, B, \) and all \( \theta^{-l,k}. \)

Matching auctions with weights \( \beta^{P} \) and payments s.t. \( L_{l}^{k} = 0, \) all \( l \in N^{k}, \)

\( k = A, B, \) maximize platform's profits across all possible mechanisms.
Plan

- Model
- Dynamic matching auctions
- Truthful bidding
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Welfare maximization

Theorem

Let $\beta^k_{l} W_l (\theta^k_l) = 1$, all $\theta^k_l$, $l \in N^k$, $k = A, B$.

(i) Matching auctions with weights $\beta^W$ and payments with $L^k_l$ large enough, all $l \in N^k$, $k = A, B$, maximize ex-ante welfare over all possible mechanisms.

(ii) Suppose $D^k_l (\theta^{l-1,k}, \theta^k_l; \beta^W) \geq 0$, all $l \in N^k$, $k = A, B$, and all $\theta^k_l$.

Matching auctions with payment s.t. $L^k_l = 0$, all $l \in N^k$, $k = A, B$, admit ex-post periodic equilibria in which agents participate and follow truthful strategies at all histories. Furthermore, such auctions maximize the platform's profits over all mechanisms implementing welfare-maximizing matches and inducing the agents to join platform in period zero.
Distortions

**Theorem**

Assume horizontal types ε non-negative

(1) If none of capacity constraints binds

\[ \chi_{ijt}^P = 1 \implies \chi_{ijt}^W = 1 \]

(2) If only platform’s capacity constraint potentially binding

\[ \sum_{(i,j) \in N^A \times N^B} \chi_{ijt}^W \geq \sum_{(i,j) \in N^A \times N^B} \chi_{ijt}^P \]

(3) If some of individual capacity constraints potentially binding,

\[ \sum_{(i,j) \in N^A \times N^B} \chi_{ijt}^P > 0 \implies \sum_{(i,j) \in N^A \times N^B} \chi_{ijt}^W > 0. \]

(*) Above conclusions can be reversed with negative horizontal types (upward distortions)
Plan

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- Dynamic matching auctions
- Truthful bidding
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Endogenous Processes

- **Endogenous processes:**
  - when \( x_{ijt-1} = 0 \), \( \varepsilon_{ijt}^k = \varepsilon_{ijt-1}^k \) a.s.
  - when \( x_{ijt-1} = 1 \), kernel \( G_{ijt} \) depends on \( \sum_{s=1}^{t-1} x_{ijt} \)
  - costs \( c_{ijt} \) may also depend on \( \sum_{s=1}^{t-1} x_{ijt} \)
  - example 1: experimentation in Gaussian world \( (\varepsilon_{ijt}^k = \mathbb{E}[\omega_{ij}^k(z_{ijt}^k)_s]) \)
  - example 2: preference for variety
Endogenous Processes

- **Endogenous processes:**

  - when $x_{ijt-1} = 0$, $\varepsilon_{ijt} = \varepsilon_{ijt-1} \ a.s.$

  - when $x_{ijt-1} = 1$, kernel $G_{ijt}$ depends on $\sum_{s=1}^{t-1} x_{ijs}$

  - costs $c_{ijt}$ may also depend on $\sum_{s=1}^{t-1} x_{ijs}$

  - example 1: experimentation in Gaussian world ($\varepsilon_{ijt}^{k} = \mathbb{E}[\omega_{ij}\mid(z_{ijs}^{k})_{s}]$)

  - example 2: preference for variety

- $\varepsilon$ drawn independently across agents and from $\theta$, given $x$
Index scores

- Suppose that either $M_t = 1$ all $t$, or all capacity constraints are non-binding
Suppose that either \( M_t = 1 \) all \( t \), or all capacity constraints are non-binding.

Auctions similar to those above but where at each \( t \) agents adjust membership status to \( \theta^k_{lt} \in \Theta^k_l \) and scores given by following indexes

\[
S_{ijt} \equiv \sup_{\tau} \frac{\mathbb{E}_{\lambda_{ij}|\theta_0, \theta_t, b_t, x^{t-1}} \left[ \sum_{s=t}^{\tau} \delta^{s-t} \left( \beta^A_i (\theta^A_{i0}) \cdot b^A_{ijt} + \beta^B_j (\theta^B_{j0}) \cdot b^B_{ijt} - c_{ij}^A (x^{s-1}) \right) \right]}{\mathbb{E}_{\lambda_{ij}|\theta_0, \theta_t, b_t, x^{t-1}} \left[ \sum_{s=t}^{\tau} \delta^{s-t} \right]}
\]

where

\( \tau \): stopping time

\( \lambda_{ij}|\theta_0, \theta_t, b_t, x^{t-1} \): process over bids under truthful bidding, when

\( \varepsilon^k_{ijt} = \frac{b^k_{ijt}}{\theta^k_{it}} \)
Index scores

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- Auctions similar to those above but where at each \( t \) agents adjust membership status to \( \theta^k_{lt} \in \Theta^k_l \) and scores given by following indexes:

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S_{ijt} = \sup_{\tau} \frac{\mathbb{E}_{\lambda_{ij} | \theta_0, \theta_t, b_t, x^{t-1}} \left[ \sum_{s=t}^{\tau} \delta^{s-t} \left( \beta^A_i (\theta_{i0}) \cdot b^A_{ijt} + \beta^B_j (\theta_{j0}) \cdot b^B_{ijt} - c_{ij}(x^{s-1}) \right) \right]}{\mathbb{E}_{\lambda_{ij} | \theta_0, \theta_t, b_t, x^{t-1}} \left[ \sum_{s=t}^{\tau} \delta^{s-t} \right]}
\]

where

\( \tau \): stopping time

\( \lambda_{ij} | \theta_0, \theta_t, b_t, x^{t-1} \): process over bids under truthful bidding, when

\( \varepsilon^k_{ijt} = \frac{b^k_{ijt}}{\theta^k_{it}} \)

- Same qualitative conclusions as for exogenous processes.
Conclusions

- Mediated (dynamic) matching
  - agents learn about attractiveness of partners over time
  - shocks to profitability of matching allocations

- Matching auctions
  - similar in spirit to GSPA for sponsored search BUT
    (i) richer externalities
    (i) costs of info rents

- Ongoing work:
  - searching for arms/partners
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Thank You!