Monopoly with Resale

Supplementary Material

Giacomo Calzolari Alessandro Pavan

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1 Restriction to price offers in the resale ultimatum bargaining game

In the model set up, we assume that in the resale ultimatum bargaining game B and T are restricted to simple take-it-or-leave-it price offers. In this appendix, we prove that, not only are price offers sequentially optimal for B and T, but S does not gain from recommending more complex mechanisms.[1] We first characterize the allocations (for the primary and the secondary market) that maximize the monopolist's revenue under minimal sequential rationality constraints for B and T and then show that these allocations can also be sustained in the game where offers in the resale bargaining game are restricted to simple prices.

For simplicity, assume $\lambda_T = 1$ and consider the case where resale is to a third party (the proof for $\lambda_T \in [0, 1]$ and inter-bidder resale follows similar arguments).

Suppose that at $\tau = 2$, T can choose from a topological space of feasible resale mechanisms Π^r. A resale mechanism $\pi = (\mathcal{M}^r, \alpha) \in \Pi^r$ consists of a set of messages $\mathcal{M}^r$ for player B along with a measurable mapping $\alpha : \mathcal{M}^r \to \mathbb{R}\times\Delta(\{0, 1\})$ that assigns to each message $m^r \in \mathcal{M}^r$ a lottery over the decision to trade and an expected payment from T to B. Let Υ denote the set of resale mechanisms that consist of simple take-it-or-leave-it price offers.[2] Without confusion, an element of Υ can be denoted simply by the price t^r. Finally, let Ξ denote the set of direct resale mechanisms $\xi : \Theta_B \mapsto \mathbb{R}\times\Delta(\{0, 1\})$. Since any mechanism in Ξ has the same message space, to save on notation, an element of Ξ will be denoted simply by the mapping ξ.

Now, consider the monopolist. Let Ψ represent a topological space of feasible mechanisms for S. A mechanism $\psi = (\mathcal{M}, \mathcal{R}, \beta) \in \Psi$ consists of a set of messages $\mathcal{M}$ for B, a set of signals/recommendations $\mathcal{R}$ that S can send to T and a measurable mapping $\beta : \mathcal{M} \mapsto \mathbb{R}\times\Delta(\{0, 1\} \times \mathcal{R})$ that assigns to each message $m \in \mathcal{M}$ an expected transfer $t(m) \in \mathbb{R}$ from B to S and a joint lottery $\delta(m) \in \Delta(\{0, 1\} \times \mathcal{R})$ over the decision to trade and the recommendations $\mathcal{R}$. Now, let $\tilde{\Phi}$ denote the set of direct mechanisms $\tilde{\phi} : \Theta_B \mapsto \mathbb{R}\times\Delta(\{0, 1\} \times \tilde{Z})$ in which the recommendations $\tilde{z} = (\xi(\overline{\theta}_T), \xi(\underline{\theta}_T)) \in \tilde{Z} = \Xi^2$ that S sends to T consists in a pair of direct resale mechanisms, respectively for $\overline{\theta}_T$ and $\underline{\theta}_T$. Finally, let Φ denote the set of direct mechanisms $\phi : \Theta_B \mapsto \mathbb{R}\times\Delta(\{0, 1\} \times Z)$ in which S recommends simple take-it-or-leave-it price offers $z = (t^r(\overline{\theta}_T), t^r(\underline{\theta}_T)) \in Z = \Upsilon^2$. In the following, we will denote by $\tilde{\phi} \in \tilde{\Phi}$ an element of $\tilde{\Phi}$ and by $\phi \in \Phi$ an element of Φ.

[1] This does not mean that a stochastic ultimatum bargaining game where B and T are randomly selected to design the resale mechanism is the most favorable resale procedure from the perspective of the initial seller. For example, in the case of inter-bidder resale, if S could choose the resale game, she could simply prohibit any future transaction between B and T and then implement a Myerson optimal auction in the primary market. Alternatively, she could dictate that it is always the resale-seller who makes the offer in the secondary market (as discussed in the paper, sometimes this also allows the monopolist to extract the Myerson revenue – see Zheng (2002)).

[2] Formally, a take-it-or-leave-it price offer is a mechanism $(\mathcal{M}^r, \alpha)$, with $\mathcal{M}^r = \{yes, no\}$, such that, when B chooses the message $m = yes$, the good is transferred to T and B receives a payment t^r, whereas when he chooses $m = no$, he keeps the good and receives no money from T.

Now, let U_S^* represent the highest equilibrium payoff for the monopolist in the restricted game where $\Pi^r = \Upsilon$ and $\Psi = \Phi$, that is in the game where T can only make take-it-or-leave-it price offers and S can only offer direct mechanisms $\phi \in \Phi$, as assumed in the model set up. Similarly, let $\tilde{U}_S^*$ denote the highest equilibrium payoff in an unrestricted game where $\Pi^r \supseteq \Xi \cup \Upsilon$ and $\Psi \supseteq \tilde{\Phi} \cup \Phi$.

Claim A1. *S does not gain from recommending that T and B offer mechanisms in the ultimatum bargaining game more complex than simple price offers*: $\tilde{U}_S^* = U_S^*$.

Proof. We prove the result in three steps. Step 1 shows that in the unrestricted game, $\tilde{U}_S^*$ can be sustained by an equilibrium where S offers a mechanism $\tilde{\phi}^* \in \tilde{\Phi}$ and T follows the monopolist's recommendations. Step 2 characterizes the allocations induced by $\tilde{\phi}^*$. Finally, step 3 shows that these allocations can also be sustained by an equilibrium in the restricted game where $\Pi^r = \Upsilon$ and $\Psi = \Phi$.

Step 1. Given any direct mechanism $\tilde{\phi} \in \tilde{\Phi}$, let $\tilde{Z}(\tilde{\phi})$ denote the set of recommendations $\tilde{z} = (\xi(\overline{\theta}_T), \xi(\underline{\theta}_T))$ in the support of $\tilde{\phi}$ and $\Xi(\tilde{\phi})$ the set of all direct resale mechanisms recommended by $\tilde{\phi}$. Formally, $\Xi(\tilde{\phi}) = \{\xi \in \Xi : \exists\ \tilde{z} = (\xi(\overline{\theta}_T), \xi(\underline{\theta}_T)) \in \tilde{Z}(\tilde{\phi}) \text{ s.t. } \xi = \xi(\overline{\theta}_T) \text{ or } \xi = \xi(\underline{\theta}_T)\}$.

Now, consider a mechanism $\tilde{\phi}^*$ with the following properties:

(i) B finds it optimal to participate and truthfully report his type in $\tilde{\phi}^*$ as well as in any resale mechanism $\xi \in \Xi(\tilde{\phi}^*)$;

(ii) given any $\tilde{z} = (\xi(\overline{\theta}_T), \xi(\underline{\theta}_T)) \in \tilde{Z}(\tilde{\phi}^*)$, the direct mechanism $\xi(\theta_T)$ is optimal for θ_T – any other mechanism $\xi \in \Xi$ that is individually-rational and incentive-compatible for B leads to a lower payoff for θ_T (formally, $\xi(\theta_T)$ is a solution to the program $P_T(r, \theta_T)$ described below with $r = \tilde{z}$).

(iii) $\tilde{\phi}^*$ is optimal for S – any other $\tilde{\phi}$ that strictly dominates $\tilde{\phi}^*$ necessarily violates (i) or (ii).

In the sequel, we prove the following two results. First, for any mechanism $\tilde{\phi}^*$ that satisfies (i)-(iii), there exists *an* equilibrium in the unrestricted game where $\Pi^r \supseteq \Xi \cup \Upsilon$ and $\Psi \supseteq \tilde{\Phi} \cup \Phi$ that supports $\tilde{\phi}^*$. That is, we can construct a (sequentially rational) strategy for B that specifies a complete plan of action for any pair of mechanisms $(\psi, \pi) \in \Psi \times \Pi^r$ – that is, a pair of function $\sigma_B : \Theta_B \times \Psi \to \Delta(\mathcal{M})$ and $\sigma_B : \Theta_B \times \Psi \times \mathcal{M} \times \mathbb{R} \times \{0,1\} \times \mathcal{R} \times \Pi \to \Delta(\mathcal{M}^r)$ – and a (sequentially rational) strategy for T that specifies a reaction to any upstream mechanism ψ – that is, a function $\sigma : \Theta_T \times \Psi \times \mathcal{R} \to \Delta(\Pi^r)$ – such that: (a) S finds it optimal to offer $\tilde{\phi}^*$; (b) T finds it optimal to obey to the recommendations $\tilde{z} \in \tilde{Z}(\tilde{\phi}^*)$; and (c) B finds it optimal to participate and truthfully report his type in $\tilde{\phi}^*$ as well as in any resale mechanism $\xi \in \Xi(\tilde{\phi}^*)$. Second, the monopolist's payoff in the equilibrium supporting $\tilde{\phi}^*$ is higher than in any other equilibrium, i.e. it yields $\tilde{U}_S^*$.[3]

To prove these claims, take any equilibrium of the unrestricted game. Given any upstream

[3] As we show below, $\tilde{\phi}^*$ identifies a profile of allocations – probabilities of trade and transfers for each state (θ_B, θ_T) – that maximize the monopolist's payoff under minimal sequential rationality constraints for B and T.

mechanism $\psi = (\mathcal{M}, \mathcal{R}, \beta) \in \Psi$, let

$$\mathcal{R}(\sigma_B) := \{r \in \mathcal{R} : \exists\ m \in Supp[\sigma_B(\theta_B, \psi)] \text{ s.t. } r \in Supp[\delta(m)] \text{ for some } \theta_B \in \Theta_B\}$$

denote the set of recommendations that, given the buyer's strategy at $\tau = 1$, are sent with positive probability to T. For any recommendation $r \in \mathcal{R}(\sigma_B)$, the reaction $\sigma_T(\theta_T, \psi, r) \in \Delta(\Pi^r)$ is sequentially rational for θ_T if and only if, given the buyer's strategy at $\tau = 2$, it leads to a pair of probability of trade $\{x^r(\overline{\theta}_B), x^r(\underline{\theta}_B)\} \in [0,1]^2$ and a pair of expected transfers $\{t^r(\overline{\theta}_B), t^r(\underline{\theta}_B)\} \in \mathbb{R}^2$ – that solve the following program:[4]

$$\mathcal{P}_T(r, \theta_T) : \begin{cases} \max\limits_{x^r(\cdot), t^r(\cdot)} \sum\limits_{\theta_B} [\theta_T x^r(\theta_B) - t^r(\theta_B)] \Pr(\theta_B | r; \psi) \\ \text{s.t. for any } (\theta_B, \widehat{\theta}_B) \in \Theta_B^2 \\ t^r(\theta_B) - \theta_B x^r(\theta_B) \geq 0 \qquad (IR_B(\theta_B)) \\ t^r(\theta_B) - \theta_B x^r(\theta_B) \geq t^r(\widehat{\theta}_B) - \theta_B x^r(\widehat{\theta}_B) \qquad (IC_B(\theta_B)) \end{cases}$$

where $\Pr(\theta_B | r; \psi)$ is computed using Bayes' rule and the buyer's strategy at $\tau = 1$. Indeed, if the allocations induced by $\sigma_T(\theta_T, \psi, r)$ do not solve $\mathcal{P}_T(r, \theta_T)$, then, θ_T has a profitable deviation that consists in offering a direct mechanism $\xi \in \Xi$ which solves the above program.[5]

We conclude that, given any pair of (sequentially rational) strategies σ_B and σ_T, an upstream mechanism for the monopolist $\psi = (\mathcal{M}, \mathcal{R}, \beta) \in \Psi$ (no matter whether it is on or off the equilibrium path) ultimately leads to a mapping $f : \Theta_B \mapsto \mathbb{R} \times \Delta(\{0,1\} \times \mathcal{R})$ that assigns to each θ_B an expected transfer $t(\theta_B) \in \mathbb{R}$ from B to S and a joint lottery $\delta(\theta_B) \in \Delta(\{0,1\} \times \mathcal{R})$, with the following properties:

(A) type $\theta_B \in \Theta_B$ prefers the outcome $f(\theta_B) = (t(\theta_B), \delta(\theta_B))$ to the outcome $f(\hat{\theta}_B) = (t(\hat{\theta}_B), \delta(\hat{\theta}_B))$ that can be obtained by mimicking the behavior of type $\hat{\theta}_B \neq \theta_B$.

(B) for any $r \in \mathcal{R}(\sigma_B)$, the allocations induced by the reaction of θ_T solve $\mathcal{P}_T(r, \theta_T)$.

Consider the following transformation of $\psi = (\mathcal{M}, \mathcal{R}, \beta)$ into a direct mechanism $\tilde{\phi} \in \tilde{\Phi}$ (using σ_B and σ_T). For any $\tilde{z} = (\xi(\overline{\theta}_T), \xi(\underline{\theta}_T)) \in \tilde{Z} = \Xi^2$, let $\tilde{\mathcal{R}}(\tilde{z}) \subseteq \mathcal{R}(\sigma_B)$ denote the set of recommendations that, given σ_B and σ_T, ultimately lead to the same allocations as those specified in the pair of direct resale mechanisms $(\xi(\overline{\theta}_T), \xi(\underline{\theta}_T))$. Now, take a direct mechanism $\tilde{\phi} : \Theta_B \mapsto \mathbb{R} \times \Delta(\{0,1\} \times \tilde{Z})$ that assigns to each $\theta_B \in \Theta_B$ the same expected transfer as ψ, and a lottery over $\{0,1\} \times \tilde{Z}$ such

[4] Note that, even if T faces an "informed principal" mechanism design problem, since both B and T have quasilinear preferences, private values and finite types, T never gains from hiding her private information to B – see Maskin and Tirole (1990).

[5] To guarantee that, whenever indifferent, B participates and truthfully reveals his type, T may need to increase the transfers $t^r(\overline{\theta}_B)$ and $t^r(\underline{\theta}_B)$, that solve $\mathcal{P}_T(r, \theta_T)$, respectively by ε and δ. However, with quasilinear payoffs, ε and δ can be set arbitrarily close to zero.

that[6]

$$\tilde{\phi}(\tilde{z}|\theta_B) = \sum_{r\in\tilde{\mathcal{R}}(\tilde{z})} \psi(r|\theta_B)$$

where $\psi(r|\theta_B) := \Pr(x = 1, r' = r|\theta_B; \psi)$ and $\tilde{\phi}(\tilde{z}|\theta_B) := \Pr(x = 1, \tilde{z}' = \tilde{z}|\theta_B; \tilde{\phi})$. The mechanism $\tilde{\phi}$ constructed this way maps Θ_B into the same final outcomes – probability of trade and expected payments – as the mechanism ψ. Furthermore, given $\tilde{\phi}$, T has the correct incentives to follow the monopolist's recommendations. To see this, note that when the supports Θ_B and Θ_T overlap, a recommendation $\tilde{z} = (\xi(\overline{\theta}_T), \xi(\underline{\theta}_T)) \in \tilde{Z}(\tilde{\phi})$, is incentive-compatible if and only if $\xi(\underline{\theta}_T)$ is such that

$$x^r(\underline{\theta}_B) = 1,\ x^r(\overline{\theta}_B) = 0,\ t^r(\underline{\theta}_B) = \underline{\theta}_B,\ t^r(\overline{\theta}_B) = 0, \qquad (SR(\tilde{z}, \underline{\theta}_T))$$

and $\xi(\overline{\theta}_T)$ is such that

$$x^r(\underline{\theta}_B) = 1;\ x^r(\overline{\theta}_B) = \begin{cases} 1 \text{ if } \Pr(\overline{\theta}_B|\tilde{z}) > \Delta\theta_B/[\overline{\theta}_T - \underline{\theta}_B] \\ 0 \text{ if } \Pr(\overline{\theta}_B|\tilde{z}) < \Delta\theta_B/[\overline{\theta}_T - \underline{\theta}_B] \\ \text{any } \eta \in [0,1] \text{ if } \Pr(\overline{\theta}_B|\tilde{z}) = \Delta\theta_B/[\overline{\theta}_T - \underline{\theta}_B] \end{cases} \qquad (SR(\tilde{z}, \overline{\theta}_T))$$
$$t^r(\overline{\theta}_B) = x^r(\overline{\theta}_B)\overline{\theta}_B;\ t^r(\underline{\theta}_B) = \underline{\theta}_B + x^r(\overline{\theta}_B)\Delta\theta_B$$

Hence, recommendations may differ only with respect to what S recommends to $\overline{\theta}_T$. Now, take a recommendation $\tilde{z} \in \tilde{Z}(\tilde{\phi})$ such that $\xi(\overline{\theta}_T) = (x^r(\overline{\theta}_B) = x^r(\underline{\theta}_B) = 1; t^r(\overline{\theta}_B) = t^r(\underline{\theta}_B) = \overline{\theta}_B)$. Then, for any $r \in \tilde{\mathcal{R}}(\tilde{z})$,

$$\Pr(\overline{\theta}_B|r;\psi) = \frac{\psi(r|\overline{\theta}_B)\Pr(\overline{\theta}_B)}{\psi(r|\overline{\theta}_B)\Pr(\overline{\theta}_B)+\psi(r|\underline{\theta}_B)\Pr(\underline{\theta}_B)} \geq \Delta\theta_B/[\overline{\theta}_T - \underline{\theta}_B].$$

Since, given $\tilde{\phi}$, T's posterior beliefs when she receives the recommendation $\tilde{z}$ are given by

$$\Pr(\overline{\theta}_B|\tilde{z};\tilde{\phi}) = \frac{\tilde{\phi}(\tilde{z}|\overline{\theta}_B)\Pr(\overline{\theta}_B)}{\tilde{\phi}(\tilde{z}|\overline{\theta}_B)\Pr(\overline{\theta}_B)+\tilde{\phi}(\tilde{z}|\underline{\theta}_B)\Pr(\underline{\theta}_B)} = \frac{\sum_{r\in\tilde{\mathcal{R}}(\tilde{z})}\psi(r|\overline{\theta}_B)\Pr(\overline{\theta}_B)}{\sum_{r\in\tilde{\mathcal{R}}(\tilde{z})}\psi(r|\overline{\theta}_B)\Pr(\overline{\theta}_B)+\sum_{r\in\tilde{\mathcal{R}}(\tilde{z})}\psi(r|\underline{\theta}_B)\Pr(\underline{\theta}_B)},$$

then $\Pr(\overline{\theta}_B|\tilde{z};\tilde{\phi}) \geq \Delta\theta_B/[\overline{\theta}_T - \underline{\theta}_B]$, which implies that $\tilde{z}$ is indeed incentive-compatible. The same result can be established for any $\tilde{z} \in \tilde{Z}(\tilde{\phi})$.

We conclude that for any mechanism ψ, there exists a mechanism $\tilde{\phi}$ satisfying (i) and (ii) that is payoff-equivalent for all players. >From (iii), it is then immediate that in the unrestricted game, there exists an equilibrium sustaining $\tilde{\phi}^*$. Furthermore, the monopolist's payoff in such an equilibrium is necessarily (weakly) higher than in any other equilibrium of the unrestricted game.

Step 2. Now, let $r(\theta_B;\xi) := t^r(\theta_B) - \xi(1|\theta_B)\theta_B$ denote the resale surplus that θ_B obtains when T offers a direct resale mechanism ξ, and $r(\theta_B|\tilde{z}) := p_T r(\theta_B;\xi(\overline{\theta}_T)) + (1-p_T)r(\theta_B;\xi(\underline{\theta}_T))$ the

[6] For simplicity, we assume $\mathcal{R}(\tilde{z})$ is a finite set. If not, then let $\tilde{\phi}(\tilde{z}|\theta_B) = \int_{r\in\mathcal{R}(\tilde{z})} d\delta(r|\theta_B)$, where $\delta(r|\theta_B)$ denotes the probability measure of recommendation r induced by the buyer's strategy at $\tau = 1$.

expected surplus given the recommendation $\tilde{z} = (\xi(\overline{\theta}_T), \xi(\underline{\theta}_T))$. The mechanism $\tilde{\phi}^*$ satisfies (i)-(iii) if and only if it is a solution to the following program

$$\widetilde{\mathcal{P}}_S : \begin{cases} \max_{\tilde{\phi} \in \tilde{\Phi}} \ \mathbb{E}_{\theta_B}\left[t\left(\theta_B\right)\right] \\ \text{s.t.– for any } (\theta_B, \widehat{\theta}_B) \in \Theta_B^2 \text{ –} \\ \quad U(\theta_B) := \sum_{\tilde{z} \in \tilde{Z}} \tilde{\phi}(\tilde{z}|\theta_B)\left\{\theta_B + r(\theta_B|\tilde{z})\right\} - t(\theta_B) \geq 0 \\ \quad U(\theta_B) \geq \sum_{\tilde{z} \in \tilde{Z}} \tilde{\phi}(\tilde{z}|\hat{\theta}_B)\left\{\theta_B + r(\theta_B|\tilde{z})\right\} - t(\hat{\theta}_B) \\ \text{for any } \tilde{z} \in \tilde{Z}(\tilde{\phi}) \text{ and any } \theta_T \in \Theta_T,\ \xi(\theta_T) \text{ satisfies } SR(\tilde{z}, \theta_T)) \\ \tilde{\phi}\left(\tilde{z}|\theta_B\right) \geq 0 \text{ with } \sum_{z \in Z} \tilde{\phi}\left(\tilde{z}|\theta_B\right) \leq 1 \text{ for any } \theta_B \in \Theta_B \qquad (\mathcal{F}) \end{cases}$$

Step 3. Note that S never gains from using a mechanism $\tilde{\phi}$ that recommends a $\xi(\overline{\theta}_T)$ in which $x^r(\overline{\theta}_B) \in (0,1)$. Indeed, for any such mechanism, there exists another mechanism $\tilde{\phi}'$ in which S recommends only $\xi(\overline{\theta}_T)$ such that either $x^r(\overline{\theta}_B) = 0$, or $x^r(\overline{\theta}_B) = 1$, which leads to a higher payoff. It follows that S sends only two possible incentive-compatible recommendations: the first one is for both $\overline{\theta}_T$ and $\underline{\theta}_T$ to trade only with $\underline{\theta}_B$ at a price $t^r = \underline{\theta}_B$; the second is for $\underline{\theta}_T$ to trade only with $\underline{\theta}_B$ at a price $t^r = \underline{\theta}_B$ and for $\overline{\theta}_T$ to trade with both types at a price $t^r = \overline{\theta}_B$. But these are exactly the same resale outcomes that can be implemented recommending simple take-it-or-leave-it price offers. It is then immediate that the solution to $\widetilde{\mathcal{P}}_S$ leads exactly to the same revenue as the solution to $\mathcal{P}_S$ in the main text. We conclude that $\tilde{U}_S^*$ can also be achieved in the game where $\Pi^r = \Upsilon$ and $\Psi = \Phi$. Q.E.D.

2 Implementation of the optimal mechanism of Proposition 1 with price disclosures

Claim A2. *When the direct mechanism of Proposition 1 can not be implemented announcing only the decision to trade, it suffices to disclose the price to implement the optimal informational linkage with the secondary market.*

Proof. The implementations in which S discloses the price but keeps the choice of the contract secret, or discloses the contract with probability less than one, are immediate. In what follows, we prove that S could also fully disclose the choice of the contract by inducing B to play a mixed strategy.

Suppose S offers a menu of two price-lottery pairs. The menu is such that B receives the good with certainty if he pays $t_H = t^*(\overline{\theta}_B)$ and with probability $\delta = [1 - J/K]/[1 - J]$ if he pays $t_L = \delta\left[\underline{\theta}_B + \lambda_B s(\underline{\theta}_B)\right]$, where $t^*(\overline{\theta}_B)$ is the price $\overline{\theta}_B$ pays in the direct mechanism of Proposition 1.

We want to show that it is an equilibrium for the high type to pay t_H and for the low type to randomize over t_H and t_L with probability respectively equal to J and $1-J$. Given this strategy, $\overline{\theta}_T$ offers $t^r = \overline{\theta}_B$ when she observes t_H and $t^r = \underline{\theta}_B$ when she observes t_L, that is t_H and t_L serve the same role as $\bar{z}$ and $\underline{z}$ in the direct mechanism. For the low type to be indifferent between t_H and t_L it must be that

$$\underline{\theta}_B + \lambda_B s(\underline{\theta}_B) + \lambda_T p_T \Delta\theta_B - t_H = \delta\left[\underline{\theta}_B + \lambda_B s(\underline{\theta}_B)\right] - t_L. \tag{1}$$

Since $t_H = t^*(\overline{\theta}_B)$, the left hand side in (1) is also equal to the payoff $\underline{\theta}_B$ obtains by announcing $\theta_B = \overline{\theta}_B$ in the direct mechanism, which is equal to zero since $IC(\underline{\theta}_B)$ and $IR(\underline{\theta}_B)$ bind in the optimal mechanism. As a consequence, $t_L = \delta\left[\underline{\theta}_B + \lambda_B s(\underline{\theta}_B)\right].$

Next, we prove that the high type is also indifferent between t_H and t_L, that is $\overline{\theta}_B - t_H = \delta\left[\overline{\theta}_B + \lambda_B s(\overline{\theta}_B)\right] - t_L$. Using the values of δ and t_L, the previous equality is equivalent to

$$\overline{\theta}_B - t_H - \left[\Delta\theta_B + \lambda_B \Delta s - \lambda_T p_T \Delta\theta_B\right] = 0$$

which holds true since $t_H = t^*(\overline{\theta}_B)$ and in the optimal mechanism both $IR(\underline{\theta}_B)$ and $IC(\overline{\theta}_B)$ are binding, which implies that $0 = \overline{\theta}_B - t_H - \left[\Delta\theta_B + \lambda_B \Delta s - \lambda_T p_T \Delta\theta_B\right].$

Since this mechanism gives B the same payoff and induces the same distribution over x and Z as the optimal direct mechanism, it must also give S the same expected revenue. Q.E.D.

3 Resale to third parties with multiple bidders in the primary market

Claim A3. *A monopolist always benefits from the existence of a secondary market when she is not able to contract with all potential buyers and resale can only be to a third party who does not participate in the primary market.*

Proof. Assume there are $N \geq 2$ potential buyers in the primary market. At the end of the auction, the winner may keep the good for himself or resell it to T in the secondary market, in which case the bargaining game is exactly as in the single-bidder case with λ_i denoting the relative bargaining power of bidder i with respect to T. Continue to assume A1-A4 hold for each bidder and let $\boldsymbol{\theta}_B := (\theta_1, \theta_2, ..., \theta_N) \in \boldsymbol{\Theta}_B := \prod_{i=1}^{N} \Theta_i$ denote a profile of independent private values. Following the same steps as for the single bidder case, one can show that, conditional on bidder i winning the auction, S needs to send only two recommendations: $\underline{z}^i$ must induce $\overline{\theta}_T$ to offer $t^r(\overline{\theta}_T) = \underline{\theta}_i$ and $\bar{z}^i$ to offer $t^r(\overline{\theta}_T) = \overline{\theta}_i$. Let $\phi\left(z^i|\boldsymbol{\theta}_B\right)$ denote the probability the good is assigned to bidder i and a recommendation $z^i \in \{\bar{z}^i, \underline{z}^i\}$ is sent to T when the bidders report $\boldsymbol{\theta}_B$. Also, let

$$V(\theta_i|z^i) := \overline{\theta}_i + \lambda_i s_i(\overline{\theta}_i)$$

$$V_i(\underline{\theta}_i|z^i) := \underline{\theta}_i - \frac{p_i}{1-p_i}\Delta\theta_i + \lambda_i\{s_i(\underline{\theta}_i) - \frac{p_i}{1-p_i}\Delta s_i\} + (1-\lambda_i)\{r_i(\underline{\theta}_i|z^i) - \frac{p_i}{1-p_i}\Delta r_i(z^i)\}$$

denote the resale-augmented virtual valuations of bidder i. Following the same steps as for the single bidder case, we can show that an optimal auction ϕ^* maximizes

$$\mathbb{E}_{\boldsymbol{\theta}_B}[\sum_{i=1}^{N} \sum_{z^i\in\{\bar{z}^i,\underline{z}^i\}} V(\theta_i|z^i)\phi\left(z^i|\boldsymbol{\theta}_B\right)]$$

subject to

$$\begin{aligned}\mathbb{E}_{\boldsymbol{\theta}_{-i}}\left\{\sum_{z^i\in\{\bar{z}^i,\underline{z}^i\}} \phi\left(z^i|\overline{\theta}_i,\boldsymbol{\theta}_{-i}\right)\left[\Delta\theta_i + \lambda_i\Delta s_i + (1-\lambda_i)\Delta r_i(z^i)\right]\right\} \geq \\ \mathbb{E}_{\boldsymbol{\theta}_{-i}}\left\{\sum_{z^i\in\{\bar{z}^i,\underline{z}^i\}} \phi\left(z^i|\underline{\theta}_i,\boldsymbol{\theta}_{-i}\right)\left[\Delta\theta_i + \lambda_i\Delta s_i + (1-\lambda_i)\Delta r_i(z^i)\right]\right\}\end{aligned} \qquad (IC(\underline{\theta}_i))$$

$$\Pr(\overline{\theta}_i|\underline{z}^i) \leq \frac{\Delta\theta_i}{\overline{\theta}_T - \underline{\theta}_i}, \qquad (IC(\underline{z}^i))$$

$$\Pr(\overline{\theta}_i|\bar{z}^i) \geq \frac{\Delta\theta_i}{\overline{\theta}_T - \underline{\theta}_i}, \qquad (IC(\bar{z}^i))$$

for $i \in \{1, 2, ..., N\}$ and $\boldsymbol{\theta}_{-i} := (\theta_1, \theta_2, ..., \theta_{i-1}, \theta_{i+1}, ..., \theta_N)$.

To prove the claim, we compare the expected revenue associated with the solution to the above program with the revenue S could achieve in a Myerson optimal auction without resale. Recall that for any type profile $\boldsymbol{\theta}_B$, a Myerson auction consists in assigning the good to the bidder with the highest virtual valuation, $M(\theta_i)$, provided that $\max_i\{M(\theta_i)\} \geq 0$, and in withholding the good otherwise. The expected revenue of a Myerson optimal auction is thus $\mathbb{E}_{\boldsymbol{\theta}_B}[\max\{0,\ M(\theta_1), ..., M(\theta_N)\}]$, where $M(\overline{\theta}_i) := \overline{\theta}_i$ and $M(\underline{\theta}_i) := \underline{\theta}_i - \frac{p_i}{1-p_i}\Delta\theta_i$, for each $i = 1, ..., N$.

The proof is in two steps. The first step proves that for any $\theta_i \in \Theta_i$ and z^i, the resale-augmented virtual valuations are higher than the corresponding Myerson virtual valuations; that is, $V(\theta_i|z^i) \geq M(\theta_i)$. This follows directly from the fact that $s(\theta_i) \geq 0$, $r_i(z^i) \geq 0$, $\Delta s_i \leq 0$ and $\Delta r_i(z^i) \leq 0$, for any $z^i \in \{\bar{z}^i, \underline{z}^i\}$ and any i.

The second step proves that there exists a recommendation policy that allows to implement Myerson allocation rule with resale. Conditional on i winning the auction, suppose S sends only one recommendation $z^i \in \{\bar{z}^i, \underline{z}^i\}$, independently of whether i announces a low or a high type. The particular recommendation S sends to T depends on the posterior beliefs that are generated by the Myerson allocation rule; that is, S recommends $z^i = \underline{z}^i$ if $\Pr(\overline{\theta}_i|i) \leq \frac{\Delta\theta_i}{\overline{\theta}_T - \underline{\theta}_i}$, and $z^i = \bar{z}^i$ otherwise, where $\Pr(\overline{\theta}_i|i)$ denotes the probability that $\theta_i = \overline{\theta}_i$ given that bidder i wins the auction. Given this policy, which can be trivially implemented disclosing only the identity of the winner, $IC(\underline{z}^i) - IC(\bar{z}^i)$ are clearly satisfied. Furthermore, since Myerson allocation rule is monotonic – i.e. $\mathbb{E}_{\boldsymbol{\theta}_{-i}}\left\{\phi\left(z^i|\overline{\theta}_i,\boldsymbol{\theta}_{-i}\right)\right\} \geq \mathbb{E}_{\boldsymbol{\theta}_{-i}}\left\{\phi\left(z^i|\underline{\theta}_i,\boldsymbol{\theta}_{-i}\right)\right\}$, constraints $IC(\underline{\theta}_i)$ are also satisfied for each i. It follows that Myerson allocation rule remains implementable also in the presence of resale. It is then immediate that the optimal mechanism ϕ^* must satisfy

$$\mathbb{E}_{\boldsymbol{\theta}_B}[\sum_{i=1}^{N} \sum_{z^i\in\{\bar{z}^i,\underline{z}^i\}} V(\theta_i|z^i)\phi^*\left(z^i|\boldsymbol{\theta}_B\right)] \geq \mathbb{E}_{\boldsymbol{\theta}_B}[\max\{0,\ M(\theta_1), ..., M(\theta_N)\}],$$

which proves the result. Q.E.D.

4 Resale to third parties: collusion in the primary market

When S lacks of the commitment not to collude with B, the only credible information that can be disclosed to the secondary market is the decision to trade. Furthermore, the possibility for S to make ϕ public has no strategic effect so that ϕ must be a best response to the strategy T is expected to follow in the secondary market. The optimal mechanism can be designed by looking at the value of the (collusion proof) resale-augmented virtual valuations

$$V(\overline{\theta}_B|\gamma) := \overline{\theta}_B + \lambda_B s(\overline{\theta}_B),$$
$$V(\underline{\theta}_B|\gamma) := \underline{\theta}_B - \tfrac{p_B}{1-p_B}\Delta\theta_B + \lambda_B\left[s(\underline{\theta}_B) - \tfrac{p_B}{1-p_B}\Delta s\right] + \lambda_T\gamma\left[\Delta\theta_B + \tfrac{p_B}{1-p_B}\Delta\theta_B\right],$$

where $\gamma \in [0, p_T]$ is the probability T is expected to offer a high price in the resale game. The seller's optimal (collusion-proof) mechanism then maximizes $U_S := \mathbb{E}_{\theta_B}[V(\theta_B|\gamma)\phi(\theta_B)]$ under the monotonicity condition $\phi(\overline{\theta}_B) \geq \phi(\underline{\theta}_B)$, where $\phi(\theta_B)$ denotes the probability of trade when B reports θ_B.

To see how the informational linkage with the secondary market can be fashioned through a stochastic allocation rule, assume T's prior beliefs are unfavorable ($J < 1$) and $V(\underline{\theta}_B|0) < 0 < V(\underline{\theta}_B|p_T)$. In the unique equilibrium, S sells to $\underline{\theta}_B$ with probability J and to $\overline{\theta}_B$ with certainty. $\overline{\theta}_T$ is then indifferent between offering a high and a low price and randomizes offering $t^r(\overline{\theta}_T) = \overline{\theta}_B$ with probability $\gamma^* \in (0, 1)$ and $t^r(\overline{\theta}_T) = \underline{\theta}_B$ with probability $1 - \gamma^*$, where γ^* solves $V(\underline{\theta}_B|p_T\gamma^*) = 0$ and hence makes S indifferent between selling to the low type and retaining the good.

5 Extended proof of Proposition 2: Optimal auctions with inter-bidder resale

The reduced program is in the Appendix of the paper (proof of Proposition 2). Here we derive a complete characterization of the optimal mechanism in the two polar cases where $\lambda_T = 1$ and $\lambda_B = 1$.

T has all bargaining power (i.e. $\lambda_T = 1$).

In this case, S does not need to disclose any information to B. Therefore, we eliminate z_B from the mechanism ϕ. Furthermore, since $\underline{\theta}_B \leq \underline{\theta}_T \leq \overline{\theta}_B \leq \overline{\theta}_T$, when $h = T$, the only incentive-compatible recommendation for $\overline{\theta}_T$ is to ask $t^r(\overline{\theta}_T) \geq \overline{\theta}_T$ and for $\underline{\theta}_T$ to ask $t^r(\underline{\theta}_T) = \overline{\theta}_B$ if $\Pr(\overline{\theta}_B|\cdot) > 0$ and any $t^r \geq \underline{\theta}_T$ otherwise. Similarly, when $h = B$, the only incentive-compatible recommendation for $\underline{\theta}_T$ is to offer a price $t^r(\underline{\theta}_T) = \underline{\theta}_B$ if $\Pr(\underline{\theta}_B|\cdot) > 0$ and any price $t^r(\underline{\theta}_T) \leq \underline{\theta}_T$ otherwise. Without loss, we will assume that $\underline{\theta}_T$ always asks $t^r(\underline{\theta}_T) = \overline{\theta}_B$ when $h = T$ and offers $t^r(\underline{\theta}_T) = \underline{\theta}_B$ when $h = B$.[7] Since these are the same prices that T offers in the absence of any explicit recommendation, to save on notation, we will drop z_T from the mapping ϕ when $h = T$, or $h = B$ and $\theta_T = \underline{\theta}_T$. When instead $h = B$ and $\theta_T = \overline{\theta}_T$, we will denote by $\bar{z}$ and $\underline{z}$ the recommendations (for $\overline{\theta}_T$) to offer $t^r(\overline{\theta}_T) = \overline{\theta}_B$ and $t^r(\overline{\theta}_T) = \underline{\theta}_B$, respectively. For these recommendations to be incentive compatible, the mechanism ϕ must satisfy

$$\phi(B, \underline{z}|\overline{\theta}_T, \underline{\theta}_B) \geq J\phi(B, \underline{z}|\overline{\theta}_T, \overline{\theta}_B) \qquad \widetilde{IC}(\underline{z}, \overline{\theta}_T)$$
$$\phi(B, \bar{z}|\overline{\theta}_T, \underline{\theta}_B) \leq J\phi(B, \bar{z}|\overline{\theta}_T, \overline{\theta}_B) \qquad \widetilde{IC}(\bar{z}, \overline{\theta}_T)$$

where $J := \frac{p_B(\overline{\theta}_T - \overline{\theta}_B)}{(1-p_B)\Delta\theta_B}$. The maximal revenue for the monopolist can be derived by partitioning the set of direct mechanisms Φ into two classes. The first one, which we denote by Φ_1, is such that $\overline{\theta}_T$ finds it (weakly) optimal to offer a high resale price $t^r(\overline{\theta}_T) = \overline{\theta}_B$ off-equilibrium, after reporting $\hat{\theta}_T = \underline{\theta}_T$. The second is such that $\overline{\theta}_T$ strictly prefers to offer $t^r(\overline{\theta}_T) = \underline{\theta}_B$. For a mechanism ϕ to belong to Φ_1 it must be that

$$\phi(B|\underline{\theta}_T, \underline{\theta}_B) \leq J\phi(B|\underline{\theta}_T, \overline{\theta}_B) \qquad (C_1)$$

Letting $\mathbb{I}_{\phi\in\Phi_1} = 1$ if $\phi \in \Phi_1$ and zero otherwise, and substituting for the values of $s_T(\cdot)$ and $r_B(\cdot)$, the problem for the monopolist reduces to the choice of a mechanism ϕ^* that maximizes[8]

$$\begin{aligned}
U_S = {} & p_T p_B \{\overline{\theta}_T \phi(T|\overline{\theta}_T, \overline{\theta}_B) + \overline{\theta}_T \phi(B, \bar{z}|\overline{\theta}_T, \overline{\theta}_B) + \overline{\theta}_B \phi(B, \underline{z}|\overline{\theta}_T, \overline{\theta}_B)\} \\
& + p_T (1 - p_B) \{\overline{\theta}_T \phi(T|\overline{\theta}_T, \underline{\theta}_B) + \overline{\theta}_T \phi(B, \bar{z}|\overline{\theta}_T, \underline{\theta}_B) + (\overline{\theta}_T - \tfrac{p_B}{1-p_B}\Delta\theta_B)\phi(B, \underline{z}|\overline{\theta}_T, \underline{\theta}_B)\} \\
& + (1 - p_T) p_B \{[\overline{\theta}_B - \tfrac{p_T}{1-p_T}\left(\overline{\theta}_T - \overline{\theta}_B\right)][\phi(T|\underline{\theta}_T, \overline{\theta}_B) + \mathbb{I}_{\phi\in\Phi_1}\phi(B|\underline{\theta}_T, \overline{\theta}_B)] + \overline{\theta}_B[1 - \mathbb{I}_{\phi\in\Phi_1}]\phi(B|\underline{\theta}_T, \overline{\theta}_B)\} \\
& + (1 - p_T)(1 - p_B) \{M(\underline{\theta}_T)\phi(T|\underline{\theta}_T, \underline{\theta}_B) + \mathbb{I}_{\phi\in\Phi_1}[M(\underline{\theta}_T) + (\tfrac{p_T}{1-p_T} - \tfrac{p_B}{1-p_B})\Delta\theta_B]\phi(B|\underline{\theta}_T, \underline{\theta}_B) \\
& + [1 - \mathbb{I}_{\phi\in\Phi_1}](M(\underline{\theta}_T) - \tfrac{p_B}{1-p_B}\Delta\theta_B)\phi(B|\underline{\theta}_T, \underline{\theta}_B)\}
\end{aligned}$$

subject to $\widetilde{IC}(\bar{z}, \overline{\theta}_T)$, $\widetilde{IC}(\underline{z}, \overline{\theta}_T)$ and

$$p_T[\phi\left(B, \underline{z}|\overline{\theta}_T, \overline{\theta}_B\right) - \phi\left(B, \underline{z}|\overline{\theta}_T, \underline{\theta}_B\right)] + (1 - p_T)[\phi\left(B|\underline{\theta}_T, \overline{\theta}_B\right) - \phi\left(B|\underline{\theta}_T, \underline{\theta}_B\right)] \geq 0 \qquad (\widetilde{IC}(\underline{\theta}_B))$$

$$\begin{aligned}
& p_B\left(\overline{\theta}_T - \overline{\theta}_B\right)[\phi(T|\overline{\theta}_T, \overline{\theta}_B) + \phi(B, \bar{z}|\overline{\theta}_T, \overline{\theta}_B) - \phi(T|\underline{\theta}_T, \overline{\theta}_B) - \mathbb{I}_{\phi\in\Phi_1}\phi(B|\underline{\theta}_T, \overline{\theta}_B)] \\
& + (1 - p_B)\Delta\theta_T[\phi(T|\overline{\theta}_T, \underline{\theta}_B) + \phi(B, \underline{z}|\overline{\theta}_T, \underline{\theta}_B) - \phi(T|\underline{\theta}_T, \underline{\theta}_B) - [1 - \mathbb{I}_{\phi\in\Phi_1}]\phi(B|\underline{\theta}_T, \underline{\theta}_B)] \\
& + (1 - p_B)(\Delta\theta_T - \Delta\theta_B)[\phi(B, \bar{z}|\overline{\theta}_T, \underline{\theta}_B) - \mathbb{I}_{\phi\in\Phi_1}\phi(B|\underline{\theta}_T, \underline{\theta}_B)] \geq 0.
\end{aligned}$$
$$(\widetilde{IC}(\underline{\theta}_T))$$

[7] Clearly, S has no incentive to recommend a different price.

[8] Assuming that $\bar{\theta}_T$ offers a high price off-equilibrium when she is indifferent between $t^r = \bar{\theta}_B$ and $t^r = \underline{\theta}_B$ is without loss of generality. Indeed, when $\phi(B|\underline{\theta}_T, \underline{\theta}_B) = J\phi(B|\underline{\theta}_T, \bar{\theta}_B)$, the program for the optimal mechanism is the same no matter whether $\bar{\theta}_T$ offers a low or a high resale price.

Note that the controls $\phi(\cdot|\boldsymbol{\theta})$ associated with the states $\boldsymbol{\theta}=\left(\overline{\theta}_T,\overline{\theta}_B\right)$ and $\boldsymbol{\theta}=\left(\overline{\theta}_T,\underline{\theta}_B\right)$ are linked to the controls associated with the other two states $\boldsymbol{\theta}=\left(\underline{\theta}_T,\overline{\theta}_B\right)$, $\boldsymbol{\theta}=(\underline{\theta}_T,\underline{\theta}_B)$ only through the constraints $\widetilde{IC}(\underline{\theta}_T)$ and $\widetilde{IC}(\underline{\theta}_B)$. In what follows, we disregard $\widetilde{IC}(\underline{\theta}_T)$ since it never binds at the optimum. Also note that it is always optimal to set $\phi^*(T|\boldsymbol{\theta})=1$ for $\boldsymbol{\theta}=(\overline{\theta}_T,\overline{\theta}_B)$ and $\boldsymbol{\theta}=(\overline{\theta}_T,\underline{\theta}_B)$. Indeed, this maximizes U_S and it helps relaxing $\widetilde{IC}(\underline{\theta}_B)$.[9]

To derive the optimal mechanism, it thus suffices to consider the monopolist's payoff in the other two sates $\boldsymbol{\theta}=(\underline{\theta}_T,\overline{\theta}_B)$ and $\boldsymbol{\theta}=(\underline{\theta}_T,\underline{\theta}_B)$.

- Consider first $J\geq 1$.

  1. Suppose $\phi^*\notin\Phi_1$. Then S could reduce $\phi(B|\underline{\theta}_T,\underline{\theta}_B)$ and increase $\phi(T|\underline{\theta}_T,\underline{\theta}_B)$ enhancing her payoff. The optimal mechanism thus necessarily belongs to Φ_1.
  2. When $\overline{\theta}_B-\frac{p_T}{1-p_T}\left(\overline{\theta}_T-\overline{\theta}_B\right)\geq 0$, $\phi^*\left(B|\underline{\theta}_T,\overline{\theta}_B\right)=1$ is clearly optimal. In this case constraints (C_1) and $\widetilde{IC}(\underline{\theta}_B)$ are always satisfied. As for $\boldsymbol{\theta}=(\underline{\theta}_T,\underline{\theta}_B)$, if

$$M(\underline{\theta}_T)+\left(\tfrac{p_T}{1-p_T}-\tfrac{p_B}{1-p_B}\right)\Delta\theta_B\geq\max\left\{0;\ M(\underline{\theta}_T)\right\},$$

  then $\phi^*\left(B|\underline{\theta}_T,\underline{\theta}_B\right)=1$ in which case the revenue is $U_S=(1-p_B)\underline{\theta}_T+p_T\Delta\theta_B+p_B\underline{\theta}_B$. If instead

$$M(\underline{\theta}_T)>\max\left\{0;\ M(\underline{\theta}_T)+\left(\tfrac{p_T}{1-p_T}-\tfrac{p_B}{1-p_B}\right)\Delta\theta_B\right\},$$

  then $\phi^*\left(T|\underline{\theta}_T,\underline{\theta}_B\right)=1$ and the revenue is $(1-p_B)\underline{\theta}_T+p_B\overline{\theta}_B$. Finally, if

$$\max\left\{M(\underline{\theta}_T)+\left(\tfrac{p_T}{1-p_T}-\tfrac{p_B}{1-p_B}\right)\Delta\theta_B;\ M(\underline{\theta}_T)\right\}<0,$$

  then S retains the good when $\boldsymbol{\theta}=(\underline{\theta}_T,\underline{\theta}_B)$ and the revenue is $U_S=p_T\left(1-p_B\right)\overline{\theta}_T+p_B\overline{\theta}_B$.[10]
  Next, assume $\overline{\theta}_B-\frac{p_T}{1-p_T}\left(\overline{\theta}_T-\overline{\theta}_B\right)<0$. In this case $\widetilde{IC}(\underline{\theta}_B)$ necessarily binds, i.e.

$$\phi^*\left(B|\underline{\theta}_T,\overline{\theta}_B\right)=\phi^*\left(B|\underline{\theta}_T,\underline{\theta}_B\right),$$

  and hence (C_1) is always satisfied. Furthermore, since $M(\underline{\theta}_T)\leq\overline{\theta}_B-\frac{p_T}{1-p_T}\left(\overline{\theta}_T-\overline{\theta}_B\right)<0$, S never sells to T when the latter reports a low valuation, i.e. when $\boldsymbol{\theta}=(\underline{\theta}_T,\overline{\theta}_B)$ or $\boldsymbol{\theta}=(\underline{\theta}_T,\underline{\theta}_B)$. At the optimum $\phi^*\left(B|\underline{\theta}_T,\overline{\theta}_B\right)=\phi^*\left(B|\underline{\theta}_T,\underline{\theta}_B\right)=1$ if

$$p_B\left[\overline{\theta}_B-\tfrac{p_T}{1-p_T}\left(\overline{\theta}_T-\overline{\theta}_B\right)\right]+(1-p_B)\left[M(\underline{\theta}_T)+\left(\tfrac{p_T}{1-p_T}-\tfrac{p_B}{1-p_B}\right)\Delta\theta_B\right]\geq 0$$

[9] Note that $\phi\left(T|\boldsymbol{\theta}\right)=1$ is payoff equivalent to $\phi(B,\underline{z}|\boldsymbol{\theta})=1$ for $\boldsymbol{\theta}=(\overline{\theta}_T,\overline{\theta}_B)$, and $\boldsymbol{\theta}=(\overline{\theta}_T,\underline{\theta}_B)$. Nevertheless, selling to T in these two states is more effective in relaxing $\widetilde{IC}(\underline{\theta}_T)$ than selling to B. This also implies that when $\widetilde{IC}(\underline{\theta}_T)$ does not bind, the optimal allocation rule need not be unique.

[10] Again, the solution may not be unique, as $\phi\left(T|\boldsymbol{\theta}\right)=1$ is payoff equivalent to $\phi(B|\boldsymbol{\theta})=1$ for $\boldsymbol{\theta}=(\underline{\theta}_T,\overline{\theta}_B)$. For example, if $M\left(\underline{\theta}_T\right)>\max\left\{0;M(\underline{\theta}_T)+\left(\frac{p_T}{1-p_T}-\frac{p_B}{1-p_B}\right)\Delta\theta_B\right\}$, then $\phi^*\left(T|\underline{\theta}_T,\overline{\theta}_B\right)=1$ is also optimal.

and $\phi^*\left(B|\underline{\theta}_T,\underline{\theta}_B\right) = \phi^*\left(B|\underline{\theta}_T,\underline{\theta}_B\right) = 0$ otherwise. In the first case, the revenue is $U_S = (1-p_B)\underline{\theta}_T + p_T\Delta\theta_B + p_B\underline{\theta}_B$, whereas in the second $U_S = p_T\overline{\theta}_T$.

- Suppose now $J < 1$.

  1. In this case $\widetilde{IC}(\underline{\theta}_B)$ can be neglected as it never binds at the optimum. Furthermore, the optimal mechanism necessarily belongs to Φ_1. The argument is the same as for $J \geq 1$.
  2. Assume now $\overline{\theta}_B - \frac{p_T}{1-p_T}\left(\overline{\theta}_T - \overline{\theta}_B\right) \geq 0$. Then $\phi^*(B|\underline{\theta}_T,\overline{\theta}_B) = 1$. As for $\boldsymbol{\theta} = (\underline{\theta}_T,\underline{\theta}_B)$, if

  $$M(\underline{\theta}_T) + \left(\tfrac{p_T}{1-p_T} - \tfrac{p_B}{1-p_B}\right)\Delta\theta_B \geq \max\left\{0,\ M(\underline{\theta}_T)\right\}$$

  then (C_1) binds and thus $\phi^*\left(B|\underline{\theta}_T,\underline{\theta}_B\right) = J$. If in addition $M(\underline{\theta}_T) \geq 0$, then $\phi^*\left(T|\underline{\theta}_T,\underline{\theta}_B\right) = 1 - J$; otherwise, $\phi^*\left(T|\underline{\theta}_T,\underline{\theta}_B\right) = 0$. In the former case, the expected revenue is $\frac{p_B}{1-p_B}\left[\overline{\theta}_T(p_T - p_B) + (1-p_T)\overline{\theta}_B\right] + \underline{\theta}_T(1-p_B)$, whereas in the latter $(1-p_B)\underline{\theta}_T + p_T\Delta\theta_B + p_B\underline{\theta}_B$. If, on the contrary,

  $$M(\underline{\theta}_T) + \left(\tfrac{p_T}{1-p_T} - \tfrac{p_B}{1-p_B}\right)\Delta\theta_B < \max\left\{0,\ M(\underline{\theta}_T)\right\},$$

  then necessarily $\phi^*\left(B|\underline{\theta}_T,\underline{\theta}_B\right) = 0$. As for $\phi\left(T|\underline{\theta}_T,\underline{\theta}_B\right)$, at the optimum $\phi^*\left(T|\underline{\theta}_T,\underline{\theta}_B\right) = 1$ when $M(\underline{\theta}_T) \geq 0$, whereas $\phi^*\left(T|\underline{\theta}_T,\underline{\theta}_B\right) = 0$ when $M(\underline{\theta}_T) < 0$. The revenue is equal to $(1-p_B)\underline{\theta}_T + p_B\overline{\theta}_B$ in the first case and $(1-p_B)\underline{\theta}_T + p_T\Delta\theta_B + p_B\underline{\theta}_B$ in the second. Next, consider $\overline{\theta}_B - \frac{p_T}{1-p_T}\left(\overline{\theta}_T - \overline{\theta}_B\right) < 0$. At the optimum, (C_1) necessarily binds. It follows that $\phi^*\left(B|\underline{\theta}_T,\overline{\theta}_B\right) = 1$ and $\phi^*\left(B|\underline{\theta}_T,\underline{\theta}_B\right) = J$ if

  $$p_B\left[\overline{\theta}_B - \tfrac{p_T}{1-p_T}\left(\overline{\theta}_T - \overline{\theta}_B\right)\right] + (1-p_B)J\left[M(\underline{\theta}_T) + \left(\tfrac{p_T}{1-p_T} - \tfrac{p_B}{1-p_B}\right)\Delta\theta_B\right] > 0, \qquad (2)$$

  whereas $\phi^*\left(B|\underline{\theta}_T,\overline{\theta}_B\right) = \phi^*\left(B|\underline{\theta}_T,\underline{\theta}_B\right) = 0$ when (2) is reversed . In either case, S never sells to T when the latter reports a low valuation, that is, $\phi^*(T|\boldsymbol{\theta}) = 0$ when $\boldsymbol{\theta} = (\underline{\theta}_T,\overline{\theta}_B)$ and $\boldsymbol{\theta} = (\underline{\theta}_T,\underline{\theta}_B)$. The revenue is

  $$p_T\overline{\theta}_T + p_B(\overline{\theta}_B - p_T\overline{\theta}_T) + (1-p_T)(1-p_B)\left[M(\underline{\theta}_T) + \left(\tfrac{p_T}{1-p_T} - \tfrac{p_B}{1-p_B}\right)\Delta\theta_B\right]J$$

  in the former case, and $p_T\overline{\theta}_T$ in the latter.

B has all bargaining power (i.e. $\lambda_B = 1$).

In this case, S does not need to disclose any information to T. Therefore, we eliminate z_T from the mechanism ϕ. Furthermore, since $\underline{\theta}_B \leq \underline{\theta}_T \leq \overline{\theta}_B \leq \overline{\theta}_T$, when $h = T$, the only incentive-compatible recommendation is for $\underline{\theta}_B$ to offer $t^r(\underline{\theta}_B) \leq \underline{\theta}_B$ and for $\overline{\theta}_B$ to offer $t^r(\overline{\theta}_B) = \underline{\theta}_T$ if $\Pr(\underline{\theta}_T|\cdot) > 0$ and any $t^r \leq \overline{\theta}_B$ otherwise. Similarly, when $h = B$, the only incentive-compatible recommendation for $\overline{\theta}_B$ is to ask a price $t^r(\overline{\theta}_B) = \overline{\theta}_T$ if $\Pr(\overline{\theta}_T|\cdot) > 0$ and any price $t^r \geq \overline{\theta}_B$ otherwise. Without loss of generality, we will assume that $\overline{\theta}_B$ always asks $t^r(\overline{\theta}_B) = \overline{\theta}_T$ when $h = B$ and offers $t^r(\overline{\theta}_B) = \underline{\theta}_T$ when $h = T$. Since these are the same prices that B offers in the absence of any explicit recommendation, to save on notation, we will drop z_B from the mapping ϕ when $h = T$, or $h = B$ and $\theta_B = \overline{\theta}_B$. When instead $h = B$ and $\theta_B = \underline{\theta}_B$, we will denote by $\bar{z}$ and $\underline{z}$ the recommendations (for $\underline{\theta}_B$) to ask $t^r(\underline{\theta}_B) = \overline{\theta}_T$ and $t^r(\underline{\theta}_B) = \underline{\theta}_T$, respectively. For these recommendations to be incentive compatible, the mechanism ϕ must satisfy

$$\begin{aligned} \phi(B, \underline{z}|\underline{\theta}_T, \underline{\theta}_B) &\geq Q\phi(B, \underline{z}|\overline{\theta}_T, \underline{\theta}_B) \qquad \widetilde{IC}(\underline{z}, \underline{\theta}_B) \\ \phi(B, \bar{z}|\underline{\theta}_T, \underline{\theta}_B) &\leq Q\phi(B, \bar{z}|\overline{\theta}_T, \underline{\theta}_B) \qquad \widetilde{IC}(\bar{z}, \underline{\theta}_B) \end{aligned}$$

where $Q := \frac{p_T \Delta\theta_T}{(1-p_T)(\underline{\theta}_T - \underline{\theta}_B)}$. Let Φ_1 denote the set of mechanisms such that $\underline{\theta}_B$ finds it (weakly) optimal to ask a high resale price $t^r(\underline{\theta}_B) = \overline{\theta}_T$ off-equilibrium, after reporting $\hat{\theta}_B = \overline{\theta}_B$. A mechanism $\phi \in \Phi_1$ only if

$$\phi(B|\underline{\theta}_T, \overline{\theta}_B) \leq Q\phi(B|\overline{\theta}_T, \overline{\theta}_B) \qquad (C_1)$$

Letting $\mathbb{I}_{\phi\in\Phi_1} = 1$ if $\phi \in \Phi_1$ and zero otherwise, and substituting for the values of $s_T(\cdot)$ and $r_B(\cdot)$, the program for the monopolist reduces to the choice of a mechanism that maximizes[11]

$$\begin{aligned} U_S = \quad & p_T p_B \left\{\overline{\theta}_T \left[\phi(T|\overline{\theta}_T, \overline{\theta}_B) + \phi(B|\overline{\theta}_T, \overline{\theta}_B)\right]\right\} \\ & + p_T (1-p_B) \left\{\overline{\theta}_T \left[\phi(T|\overline{\theta}_T, \underline{\theta}_B) + \phi(B, \bar{z}|\overline{\theta}_T, \underline{\theta}_B)\right]\right. \\ & \left. + \left[\overline{\theta}_T - \tfrac{p_B}{1-p_B}\Delta\theta_T\right] \phi(B, \underline{z}|\overline{\theta}_T, \underline{\theta}_B)\right\} \\ & + (1-p_T)\, p_B \left\{\left[\overline{\theta}_B - \tfrac{p_T}{1-p_T}\Delta\theta_T\right] \phi(T|\underline{\theta}_T, \overline{\theta}_B) + \overline{\theta}_B \phi(B|\underline{\theta}_T, \overline{\theta}_B)\right\} \\ & + (1-p_T)(1-p_B) \left\{M(\underline{\theta}_B)\phi(B, \bar{z}|\underline{\theta}_T, \underline{\theta}_B)\right. \\ & \left. + \left[M(\underline{\theta}_T) - \tfrac{p_B}{1-p_B}(\overline{\theta}_B - \underline{\theta}_T)\right] \left[\phi(T|\underline{\theta}_T, \underline{\theta}_B) + \phi(B, \underline{z}|\underline{\theta}_T, \underline{\theta}_B)\right]\right\}, \end{aligned}$$

subject to $\widetilde{IC}(\bar{z}, \underline{\theta}_B)$, $\widetilde{IC}(\underline{z}, \underline{\theta}_B)$ and

$$\begin{aligned} & p_B \left[\phi(T|\overline{\theta}_T, \overline{\theta}_B) - \phi(T|\underline{\theta}_T, \overline{\theta}_B)\right] + (1-p_B)\left[\phi(T|\overline{\theta}_T, \underline{\theta}_B) - \phi(T|\underline{\theta}_T, \underline{\theta}_B)\right] \\ & + (1-p_B)\left[\phi(B, \underline{z}|\overline{\theta}_T, \underline{\theta}_B) - \phi(B, \underline{z}|\underline{\theta}_T, \underline{\theta}_B)\right] \geq 0, \end{aligned} \qquad (\widetilde{IC}(\underline{\theta}_T))$$

$$\begin{aligned} & p_T \Delta\theta_T \left[(1-\mathbb{I}_{\phi\in\Phi_1})\phi\left(B|\overline{\theta}_T, \overline{\theta}_B\right) - \phi\left(B, \underline{z}|\overline{\theta}_T, \underline{\theta}_B\right)\right] \\ & + (1-p_T)\Delta\theta_B \left[\mathbb{I}_{\phi\in\Phi_1}\phi\left(B|\underline{\theta}_T, \overline{\theta}_B\right) - \phi\left(B, \bar{z}|\underline{\theta}_T, \underline{\theta}_B\right)\right] \\ & + (1-p_T)\left(\overline{\theta}_B - \underline{\theta}_T\right)\left[\phi\left(T|\underline{\theta}_T, \overline{\theta}_B\right) - \phi\left(T|\underline{\theta}_T, \underline{\theta}_B\right)\right] \\ & + (1-p_T)\left(\overline{\theta}_B - \underline{\theta}_T\right)\left[(1-\mathbb{I}_{\phi\in\Phi_1})\phi\left(B|\underline{\theta}_T, \overline{\theta}_B\right) - \phi\left(B, \underline{z}|\underline{\theta}_T, \underline{\theta}_B\right)\right] \geq 0. \end{aligned} \qquad (\widetilde{IC}(\underline{\theta}_B))$$

[11] Assuming that $\underline{\theta}_B$ asks a high price off-equilibrium when she is indifferent between $t^r = \bar{\theta}_T$ and $t^r = \underline{\theta}_T$ is without loss of generality. Indeed, when $\phi(B|\underline{\theta}_T, \overline{\theta}_B) = Q\phi(B|\overline{\theta}_T, \overline{\theta}_B)$, the program for the optimal mechanism is the same no matter whether $\underline{\theta}_B$ asks a low or a high resale price.

- Assume first $\max\left\{M(\underline{\theta}_T) - \frac{p_B}{1-p_B}(\overline{\theta}_B - \underline{\theta}_T); M(\underline{\theta}_B)\right\} \geq 0.$

  1. If $M(\underline{\theta}_T) - \frac{p_B}{1-p_B}(\overline{\theta}_B - \underline{\theta}_T) \leq M(\underline{\theta}_B)$, then $Q \geq 1$. In this case the mechanism $\phi^*(B|\boldsymbol{\theta}) = 1$ for $\boldsymbol{\theta} = (\underline{\theta}_T, \overline{\theta}_B)$ and $\boldsymbol{\theta} = (\overline{\theta}_T, \overline{\theta}_B)$, and $\phi^*(B, \bar{z}|\boldsymbol{\theta}) = 1$ for $\boldsymbol{\theta} = (\underline{\theta}_T, \underline{\theta}_B)$ and $\boldsymbol{\theta} = (\overline{\theta}_T, \underline{\theta}_B)$ maximizes U_S and satisfies all constraints. For any θ_B, B always asks $t^r(\theta_B) = \overline{\theta}_T$ and thus trade occurs in the secondary market if and only if T has a high valuation. In this case, the final allocation and the expected revenue coincide with that in the Myerson optimal auction if $M(\underline{\theta}_T) \leq M(\underline{\theta}_B)$. If, on the contrary, $M(\underline{\theta}_T) > M(\underline{\theta}_B)$, then in state $\boldsymbol{\theta} = (\underline{\theta}_T, \underline{\theta}_B)$, B retains the good, contrary to what prescribed by the Myerson allocation rule. This in turn induces a loss of expected revenue equal to $(1 - p_T)(1 - p_B)\left[M(\underline{\theta}_T) - M(\underline{\theta}_B)\right]$.
  2. If $M(\underline{\theta}_T) - \frac{p_B}{1-p_B}(\overline{\theta}_B - \underline{\theta}_T) > M(\underline{\theta}_B)$, the following mechanism $\phi^* \notin \Phi_1$ maximizes U_S and satisfies all constraints

$$\phi^*(T|\overline{\theta}_T, \underline{\theta}_B) = \phi^*(T|\underline{\theta}_T, \underline{\theta}_B) = \phi^*(T|\overline{\theta}_T, \overline{\theta}_B) = \phi^*(B|\underline{\theta}_T, \overline{\theta}_B) = 1.$$

  Trade does not occur in the secondary market, the final allocation is exactly as in Myerson, but the expected revenue is just $p_T p_B \overline{\theta}_T + (1 - p_T p_B)\,\underline{\theta}_T$ instead of

$$\mathbb{E}_{\boldsymbol{\theta}}\left[\max\left\{0,\ M(\theta_T),\ M(\theta_B)\right\}\right] = p_B[p_T\overline{\theta}_T + (1 - p_T)\overline{\theta}_B] + (1 - p_B)\underline{\theta}_T.$$

- Assume now $\max\left\{M(\underline{\theta}_T) - \frac{p_B}{1-p_B}(\overline{\theta}_B - \underline{\theta}_T); M(\underline{\theta}_B)\right\} < 0$. In this case, S finds it optimal to retain the good when $\boldsymbol{\theta} = (\underline{\theta}_T, \underline{\theta}_B)$. As for the other states, the following mechanism maximizes U_S and satisfies all constraints

$$\phi^*(T|\overline{\theta}_T, \overline{\theta}_B) = \phi^*(T|\overline{\theta}_T, \underline{\theta}_B) = \phi^*(B|\underline{\theta}_T, \overline{\theta}_B) = 1.$$

  The monopolist's expected revenue is $p_T\overline{\theta}_T + (1 - p_T)\,p_B\overline{\theta}_B$, trade occurs in the primary market if and only if at least one of the two bidders has a high valuation, and no offers are made in the resale game. If $M(\underline{\theta}_T) \leq 0$, the expected revenue is the same as in the Myerson auction. On the contrary, if $M(\underline{\theta}_T) > 0 > M(\underline{\theta}_B)$, S incurs a loss equal to $(1 - p_T)\,(1 - p_B)\,M(\underline{\theta}_T)$.

We conclude that when $\lambda_B = 1$, the impossibility to prohibit resale results in a loss of expected revenue for the monopolist if and only if $M(\underline{\theta}_T) > \max\left\{0, M(\underline{\theta}_B)\right\}$. ■

References

[1] Maskin, E. and J. Tirole "The Principal-Agent Relationship with an Informed Principal: The Case of Private Values." *Econometrica* 1990 Vol. 58, pp. 379–409.