Matching Auctions
Supplementary Material

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This document contains additional results for the manuscript "Matching Auctions." Section S.1 contains an example of an environment with an intermediate capacity constraint (that is, $1 < M < n_A \cdot n_B$) in which the separability condition is violated and an index policy fails to be optimal, illustrating the role of this condition. Section S.2 contains an example of an environment with endogenous processes, nonnegative match values, and an intermediate capacity constraint, for which the aggregate level of interactions is weakly higher in each period under profit maximization than under welfare maximization, and strictly higher in some periods. Finally, Section S.3 contains an example demonstrating how such upward distortions may arise when agents dislike certain interactions.

All numbered items (i.e., sections, definitions, results, and equations) in this document contain the prefix S. Any numbered reference without a prefix refers to an item in the main text. Please refer to the main text for notation and definitions.

S.1 Example - suboptimality of index policies

The following example shows how in an environment with intermediate capacity levels, the index rule $\chi^{I;\beta}$ may be suboptimal when separability fails.

Suppose $N_A = \{1, 2, 3\}$, $N_B = \{1\}$, and $M = 2$. Assume that the myopic score $S_{ijt}^{m;\beta}$ of each match (i, j) evolves according to the following process, where n denotes here the number of past interactions:

	$n = 0$	$n = 1$	$n = 2$	$n \geq 3$
$S_{11}^{m;\beta}$	3	3	3	-1
$S_{21}^{m;\beta}$	4	2	2	-1
$S_{31}^{m;\beta}$	5	1	1	-1

Since each $S_{ijt}^{m;\beta}(n)$ decreases with n, the associated index rule (as defined in Definition 3 in the main text) is myopic, and hence implements the matches $(2, 1)$ and $(3, 1)$ in period 1, the matches

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$(1,1)$ and $(2,1)$ in period 2 and again in period 3, the matches $(1,1)$ and $(3,1)$ in period 4, the match $(3,1)$ in period 5, and no match from period 6 onwards.[1] Consider now an alternative rule, implementing the matches $(1,1)$ and $(3,1)$ in period 1, the matches $(1,1)$ and $(2,1)$ in period 2 and again in period 3, the matches $(2,1)$ and $(3,1)$ in period 4, the match $(3,1)$ in period 5, and no match thereafter. The difference in the corresponding profits is $1-2\delta+\delta^3$, which is negative for $\delta > \frac{\sqrt{5}-1}{2}$. Thus, for all $\delta > \frac{\sqrt{5}-1}{2}$, an index rule does not maximize weighted surplus. Finally, note that, in this example, the separability condition in Definition 5 in the main text is violated for $\delta > \frac{\sqrt{5}-1}{2}$.

S.2 Example - upward distortions with nonnegative values

The following example demonstrates that under endogenous processes, with intermediate capacity levels, profit-maximization may involve an inefficiently high aggregate number of interactions in some periods, and even intertemporally.

Suppose $N_A = \{1,2,3\}$, $N_B = \{1\}$ and $M = 2$. Consider the following deterministic processes governing the evolution of the myopic scores, respectively under welfare maximization (left panel), and under profit maximization (right panel).

	$n=0$	$n=1$	$n=2$	$n\geq 3$		$n=0$	$n=1$	$n=2$	$n\geq 3$
$S_{1,1}^{m;\beta^W}$	7	-1	-1	-1	$S_{1,1}^{m;\hat{\beta}}$	3	-1	-1	-1
$S_{2,1}^{m;\beta^W}$	4	2	2	2	$S_{2,1}^{m;\hat{\beta}}$	4	2	2	2
$S_{3,1}^{m;\beta^W}$	5	1	-1	-1	$S_{3,1}^{m;\hat{\beta}}$	5	1	-1	-1

These processes can be generated by the truthful-strategies equilibria of Theorems 2 and 3, in the following environment. The vertical types are given by $\Theta_2^A = \Theta_3^A = \Theta_1^B = \{1\}$, whereas $\Theta_1^A = [2,3]$, with F_1^A uniform over $[2,3]$. The platform's costs are such that $c_{11t} = c_{31t} = 1$, $c_{21t} = 0$. Lastly, the horizontal types are such that $\varepsilon_{i1t}^B = 0$, all $i = 1,2,3$ all $t \geq 1$. For the side-A agents, instead, the horizontal types evolve deterministically over time according to the table below (with n indicating the number of previous interactions with agent 1 from side B). The processes above then correspond to those for the realized vertical type $\theta_i^A = 2$.

	$n=0$	$n=1$	$n=2$	$n\geq 3$
ε_{11}^A	4	0	0	0
ε_{21}^A	4	2	2	2
ε_{31}^A	6	2	0	0

[1] More generally, suppose processes are endogenous and match quality deteriorates over time, in the sense that, for all $t \geq 1$, $(i,j) \in N_A \times N_B$, $k = A, B$, $x^{t-1} \in X^{t-1}$, whenever $x_{ijt} = 1$, then $\varepsilon_{ijt+1}^k \leq \varepsilon_{ijt}^k$ a.s. and $c_{ijt+1} \geq c_{ijt}$. In such environment, irrespective of the weights β, for any pair $(i,j) \in N_A \times N_B$, any $t \geq 1$, $S_{ijt}^{I;\beta} = S_{ijt}^{m;\beta}$. This can be seen by noting that the optimal stopping time satisfies the property $\tau_{ijt} = \inf\{s > t \mid S_{ijs}^{I;\beta} \leq S_{ijt}^{I;\beta}\}$. That is, it is the first time at which the process of $S_{ij}^{m;\beta}$ reaches a state in which $S_{ij}^{I;\beta}$ drops (weakly) below its period-t value.

As in the example in Section S.1, since the myopic scores decline over time, the indexes coincide with the myopic scores, and hence both the profit-maximizing and the welfare-maximizing auctions simply match the two pairs of agents with the highest myopic score. Also note that, for sufficiently low δ, the environment in this example is separable both under $S^{I;\hat{\beta}}$ and $S^{I;\beta^W}$. The welfare-maximizing auction matches in period 1 the pairs $(1,1)$ and $(3,1)$, in period 2 the pairs $(2,1)$ and $(3,1)$, and in any subsequent period only the pair $(2,1)$. The profit-maximizing auction matches in period 1 the pairs $(2,1)$ and $(3,1)$, in period 2 the pairs $(1,1)$ and $(2,1)$, in period 3 the pairs $(2,1)$ and $(3,1)$, and from period 4 onwards only the pair $(2,1)$. Thus, at any point in time, the total number of matches under profit maximization is weakly higher than under welfare maximization (strictly higher in period 3).

S.3 Example - upward distortions under negative values

Consider the following environment where processes are exogenous, $N_A = N_B = \{1\}$, $M = 1$, and $c_{11t} = 0$, all $t \geq 1$. The vertical types are given by $\Theta_1^B = \{1\}$ and $\Theta_1^A = [1+\varsigma, 2+\varsigma]$, $\varsigma > 0$, with F_1^A uniform over Θ_1^A. At each period $t \geq 1$, regardless of past realizations, $\varepsilon_{11t}^B = 1$, whereas ε_{11t}^A is drawn uniformly from $\{-3, +3\}$. Suppose the realized vertical type of agent 1 from side A is equal to $1+\varsigma$, in which case the weights used under profit maximization to scale the two agents' bids are given by $\hat{\beta}_1^A(\theta_1^A) = \varsigma$ and $\hat{\beta}_1^B(\theta_1^B) = 1$. Furthermore, consider a realized sequence $(\varepsilon_{11t}^A)_{t=1}^{\infty}$ of horizontal types for agent 1 from side A such that $\varepsilon_{11t}^A = -3$, all $t \geq 1$. Then, for sufficiently small ς and δ, the pair is matched in each period under profit maximization, despite matching being inefficient.