Matching Auctions
Supplementary Material

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This document contains an extensive proof of Theorem 5 in the main text. The proof in the main text contains the key steps but omits those that follow from arguments similar to those establishing Theorems 2 and 3 in the main text. The complete proof in the present supplement contains all steps. To facilitate the reading, we also reproduce here the claims that the theorem makes.

A Proof of Theorem 5 in the main text

Theorem 5 in the main text states the following:

Theorem 5. *Suppose that either $M = 1$, or none of the capacity constraints binds (i.e., $M \geq n^A \cdot n^B$, and $m_l^m \geq n^{-k}$, all $l \in N^k$, $k = A, B$). Then conclusions analogous to those in Theorems 1-3 apply to the matching auctions where the scores are given by the indexes $S_{ijt}^{I;\beta}$ and the payments are given by Conditions (6) and (15) in the main text. Furthermore, the same conclusions as in parts (1) and (2) of Theorem 4 hold (with $M = 1$, for part 2).*

The proof is in 4 parts. Part 1 establishes that conclusions analogous to those in Theorem 1 hold. Part 2 establishes that conclusions analogous to those in Theorem 2 hold. Part 3 establishes that conclusions analogous to those in Theorem 3 hold. Finally, Part 4 establishes that the same conclusions as in parts (1) and (2) of Theorem 4 hold (with $M = 1$, for part 2).

Part 1. We want to show that the following claims are true*: "Any matching auction in which (a) the scores are given by the indexes $S_{ijt}^{I;\beta}$ defined in the main text, with arbitrary weights β, (b) the period-t payments, $t > 0$, are the ones defined by Condition (15) in the main text, and (c) the period-*0 *payments are the one defined in Condition (6) in the main text, with L_l^k large enough, all $l \in N^k$, $k = A, B$, admits an equilibrium in which all agents participate in each period and follow truthful strategies (i.e., in each period, they select the membership status corresponding to their true*

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vertical type and then bid truthfully their myopic values for all matches). Furthermore, such truthful equilibria are periodic ex-post (that is, the agents' strategies are sequentially rational, regardless of the agents' beliefs about other agents' past and current types)."

Proof of Part 1. The proof parallels the one for Theorem 1 in the main text. Step 1 shows that, when all agents follow truthful strategies, the matches implemented in equilibrium maximize continuation weighted surplus, starting from any period-t history, any $t \geq 1$. It then uses this property to establish that, when the period-t payments are the ones defined in Condition (15) in the main text, any $t \geq 1$, then participating in the auctions and following truthful strategies constitutes a periodic ex-post continuation equilibrium, after any period-t history, any $t \geq 1$. Step 2 completes the proof by showing that the matches sustained under truthful strategies satisfy the same dynamic monotonicity conditions as in Lemma 2 in the main text. It then shows that such monotonicities imply that, when the period-0 membership fees are the ones defined by Condition (6) in the main text, with L_l^k large enough, all $l \in N^k$, $k = A, B$, any agent who expects all other agents to participate and selecting the membership status equal to their true vertical type finds it optimal
to do the same, regardless of the agent's beliefs about the other agents' vertical types.

Step 1. We first establish that, when either $M = 1$, or none of the capacity constraints binds (i.e., $M \geq n^A \cdot n^B$, and $m_l^m \geq n^{-k}$, all $l \in N^k$, $k = A, B$), the rule $\chi^{I;\beta}$ that selects in each period the matches for which the index scores $S_{ijt}^{I;\beta}$ are the highest, among those for which the scores are non-negative, maximizes continuation weighted surplus. That is, irrespective of the particular history that led to the selection of the period-0 membership statuses θ_0 and of the past matches x^{t-1}, in the continuation game that starts with period $t \geq 1$, when the true vertical type profile is θ_t, the true profile of horizontal types is ε_t —with ε_t obtained from θ_t and b_t by letting, for each $(i, j) \in N^A \times N^B$ and $k = A, B$,

$$\varepsilon_{ijt}^k = \begin{cases} b_{ijt}^k / \theta_{it}^k & \text{if } b_{ijt}^k / \theta_{it}^k \in \mathcal{E}_{ijt}^k \\ \arg \min\limits_{\hat{\varepsilon}_{ijt}^k \in \mathcal{E}_{ijt}^k} \left\{ |b_{ijt}^k / \theta_{it}^k - \hat{\varepsilon}_{ijt}^k| \right\} & \text{otherwise} \end{cases} \tag{A.1}$$

—and all agents follow truthful strategies from period t onwards, then the matches under $\chi^{I;\beta}$ maximize continuation weighted surplus

$$W_t \equiv \mathbb{E}^{\lambda[\chi]|\theta_t, b_t, x^{t-1}} \left[\sum_{s=t}^{\infty} \delta^{s-t} \sum_{i \in N^A} \sum_{j \in N^B} \left(\beta_i^A(\theta_{i0}^A) v_{ijs}^A + \beta_j^B(\theta_{j0}^B) v_{ijs}^B - c_{ijs} \right) \cdot \chi_{ijs} \right]$$

over the entire set $\mathcal{X}$ of feasible matching rules χ. To see this, note that the problem of maximizing W_t can be viewed as a multi-armed bandit problem, with each arm corresponding to a potential match, and with the flow period-t reward of activating each arm (i, j) given by the myopic score

$$S_{ijt}^{m;\beta} \equiv \beta_i^A(\theta_{i0}^A) v_{ijt}^A + \beta_j^B(\theta_{j0}^B) v_{ijt}^B - c_{ijt}.$$

When $M = 1$, that $\chi^{I;\beta}$ maximizes W_t at all histories follows from known results (see, e.g., Whittle (1982)). When none of the capacity constraints binds (i.e., $M \geq n^A \cdot n^B$, and $m_l^m \geq n^{-k}$, all $l \in N^k$,

$k = A, B$), the platform's problem can be viewed as a collection of $n^A \cdot n^B$ separate two-armed bandit problems, one for each potential pair of agents, with the reward from matching the pair (i, j) given by $S_{ijt}^{m;\beta}$ and the reward from activating the "safe arm" identically equal to zero. It is again well known that the solution to each such problems consists in activating the risky arm if the index $S_{ijt}^{I;\beta} > 0$ and the safe arm if $S_{ijt}^{I;\beta} < 0$.

Now fix the weights β and denote by $\tilde{\chi}$ a matching rule that maximizes the continuation weighted surplus at all histories, and by $\tilde{\chi}^{-l,k}$ a matching rule that does so in the absence of agent l from side $k \in \{A, B\}$ (equivalently, that maximizes continuation weighted surplus when the myopic score of any match that involves agent l from side k is identically equal to zero, in which case $\tilde{\chi}^{-l,k}$ can be assumed to never implement any match involving agent l). From Step 1 above such rules are index rules.

Denote by $\tilde{\psi}_{t\geq 1} \equiv (\tilde{\psi}_s)_{s\geq 1}$ the collection of payment functions defined by Condition (15) in the main text, when the matching rule is $\tilde{\chi}$. Henceforth, the weighted surpluses W_t and $W_t^{-l,k}$, as well as the resulting marginal and flow contributions to weighted surplus, as defined in the main text, unless otherwise specified, are with respect to the matching rules $\tilde{\chi}$ and $\tilde{\chi}^{-l,k}$, respectively. Note that, because the weights β are held fixed, to ease the notation we drop them from all functions below, when there is no risk of confusion.

Lemma 1. *Consider an auction in which the matching rule is $\tilde{\chi}$ and the payments from period 1 onwards are given by $\tilde{\psi}_{t\geq 1}$, as defined by Condition (15) in the main text. In such an auction, participating and following truthful strategies (i.e.,* in each period, selecting the membership status corresponding to the true vertical type and then bidding truthfully the myopic values for all matches) *constitutes a periodic ex-post continuation equilibrium, after any period-t history, $t \geq 1$.*

Proof of Lemma 1. We show that, in the continuation game that starts in period $t \geq 1$, irrespective of the history of past play, of the true vertical type profile θ, and of the history of past and current horizontal types $\varepsilon^t \equiv (\varepsilon_s)_{s=1}^t$, any agent $l \in N^k$, $k = A, B$, who expects all other agents to participate and follow truthful strategies from period t (included) onwards, finds it optimal to do the same.

Consider agent l from side A (the problem for any agent from side B is similar). Suppose that the true profile of vertical types is θ, the true profile of period-t match values is v_t, the profile of period-0 membership choices is θ_0, and the history of past matches is x^{t-1}. Denote by

$$\mathbb{E}^{\tilde{\lambda}[\tilde{\chi}]|\theta, v_t, x^{t-1}; (\hat{\theta}_{lt}^A, b_{lt}^A)} \left[R_{lt+1}^A(\theta_0, \theta_{t+1}, b_{t+1}, x^t) \right]$$

the expected contribution of agent l to continuation weighted surplus from period $t+1$ onwards, when, in period t, the agent selects the period-t membership status $\hat{\theta}_{lt}^A$, submits the period-t bids b_{lt}^A, follows a truthful strategy from period $t+1$ onwards, and expects all other agents to follow truthful strategies at all periods $s \geq t$.[1] Note that, when the agent follows the truthful strategy also in period

[1] The stochastic process $\tilde{\lambda}[\tilde{\chi}]|\theta, v_t, x^{t-1}; (\hat{\theta}_{lt}^A, b_{lt}^A)$ is here over future matches, bids and membership choices, under

t (i.e., when $\hat{\theta}^A_{lt} = \theta^A_l$ and $b^k_{lt} = v^A_{lt}$), then

$$\tilde{\lambda}[\tilde{\chi}]|\theta, v_t, x^{t-1}; (\theta^A_l, v^A_{lt}) = \lambda[\tilde{\chi}]|\theta, v_t, x^{t-1},$$

where the process $\lambda[\tilde{\chi}]|\theta, v_t, x^{t-1}$ is as defined in the main text.

Since the agent can revise his membership status in any of the subsequent periods, any deviation from the truthful strategy in period t can be corrected in period $t+1$. This means that, to prove the result, it suffices to show that the agent prefers to follow the truthful strategy from period t onwards than deviating in period t and then reverting to the truthful strategy from period $t+1$ onwards.

Under the proposed auction rules, when the agent follows the truthful strategy from period $t+1$ onwards, his continuation payoff from period $t+1$ onwards is given by

$$\frac{1}{\beta^A_l(\theta^A_{l0})} R^A_{lt+1}(\theta_0, \theta_{t+1}, b_{t+1}, x^t).$$

Therefore, it is enough to show that, for any period-t selection $(\hat{\theta}^A_{lt}, b^A_{lt})$,

$$\begin{aligned}
&\sum_{j\in N^B} v^A_{ljt}\tilde{\chi}_{ljt}\left(\theta_0, (\theta^A_l, \theta^{-l,A}), (v^A_{lt}, v^{-l,A}_t), x^{t-1}\right) - \tilde{\psi}^A_{lt}\left(\theta_0, (\theta^A_l, \theta^{-l,A}), (v^A_{lt}, v^{-l,A}_t), x^{t-1}\right)\\
&\quad + \frac{\delta}{\beta^A_l(\theta^A_{l0})}\mathbb{E}^{\lambda[\tilde{\chi}]|\theta^A_l, \theta^{-l,A}, v^A_{lt}, v^{-l,A}_t, x^{t-1}}\left[R^A_{lt+1}(\theta_0, \theta_{t+1}, b_{t+1}, x^t)\right]\\
&\quad \geq \sum_{j\in N^B} v^A_{ljt}\tilde{\chi}_{ljt}\left(\theta_0, (\hat{\theta}^A_{lt}, \theta^{-l,A}), (b^A_{lt}, v^{-l,A}_t), x^{t-1}\right) - \tilde{\psi}^A_{lt}\left(\theta_0, (\hat{\theta}^A_{lt}, \theta^{-l,A}), (b^A_{lt}, v^{-l,A}_t), x^{t-1}\right)\\
&\qquad + \frac{\delta}{\beta^A_l(\theta^A_{l0})}\mathbb{E}^{\tilde{\lambda}[\tilde{\chi}]|\theta^A_l, \theta^{-l,A}, v^A_{lt}, v^{-l,A}_t, x^{t-1}; (\hat{\theta}^A_{lt}, b^A_{lt})}\left[R^A_{lt+1}(\theta_0, \theta_{t+1}, b_{t+1}, x^t)\right]. \qquad \text{(A.2)}
\end{aligned}$$

The left hand side of the above inequality can be rewritten in terms of the functions W_t and $W^{-l,A}_t$ as follows:

$$\frac{1}{\beta^A_l(\theta^A_{l0})}\left[W_t\left(\theta_0, (\theta^A_l, \theta^{-l,A}), (v^A_{lt}, v^{-l,A}_t), x^{t-1}\right) - W^{-l,A}_t\left(\theta_0, (\theta^A_l, \theta^{-l,A}), (v^A_{lt}, v^{-l,A}_t), x^{t-1}\right)\right]. \qquad \text{(A.3)}$$

That is, agent l's expected continuation payoff when he follows the truthful strategy from period t onward is equal to his expected contribution to the maximal continuation weighted surplus, scaled by the weight $\beta^A_l(\theta^A_{l0})$. It then suffices to show that (A.3) is weakly greater than the right hand side of (A.2).

the rule $\tilde{\chi}$, when agent l's period-t choices are $(\hat{\theta}^A_{lt}, b^A_{lt})$, the true profile of vertical types is θ, the true profile of period-t horizontal types is ε_t, with ε_t obtained from θ and v_t using (A.1), the history of past matches is x^{t-1}, the agent plans to follow a truthful strategy from $t+1$ onwards, and all other agents follow truthful strategies from period t onwards.

Next, note that the flow contribution r_{lt}^k can be rewritten as

$$\begin{aligned} r_{lt}^k(\theta_0,\theta_t,b_t,x^{t-1}) &= \sum_{i\in N^A}\sum_{j\in N^B} S_{ijt}^{m;\beta}\left(\theta_0,\theta_t,b_t,x^{t-1}\right)\cdot\tilde{\chi}_{ijt}(\theta_0,\theta_t,b_t,x^{t-1}) \\ &\quad - \sum_{i\in N^A}\sum_{j\in N^B} S_{ijt}^{m;\beta}\left(\theta_0,\theta_t,b_t,x^{t-1}\right)\tilde{\chi}_{ijt}^{-l,k}(\theta_0,\theta_t,b_t,x^{t-1}) \\ &\quad + \delta\mathbb{E}^{\lambda[\tilde{\chi}]|\theta_t,b_t,x^{t-1}}\left[W_{t+1}^{-l,k}(\theta_0,\theta_{t+1},b_{t+1},x^t)\right] \\ &\quad - \delta\mathbb{E}^{\lambda[\tilde{\chi}^{-l,k}]|\theta_t,b_t,x^{t-1}}\left[W_{t+1}^{-l,k}(\theta_0,\theta_{t+1},b_{t+1},x^t)\right]. \end{aligned} \tag{A.4}$$

Using (A.4), we can rewrite the period-t payment in the right-hand side of (A.2) as follows:

$$\begin{aligned} &\tilde{\psi}_{lt}^A\left(\theta_0,(\hat{\theta}_{lt}^A,\theta^{-l,A}),(b_{lt}^A,v_t^{-l,A}),x^{t-1}\right) \\ &= \sum_{j\in N^B} b_{ljt}^A\tilde{\chi}_{ljt}\left(\theta_0,(\hat{\theta}_{lt}^A,\theta^{-l,A}),(\hat{b}_{lt}^A,v_t^{-l,A}),x^{t-1}\right) - \frac{1}{\beta_l^A(\theta_{l0}^A)}r_{lt}^A\left(\theta_0,(\hat{\theta}_{lt}^A,\theta^{-l,A}),(\hat{b}_{lt}^A,v_t^{-l,A}),x^{t-1}\right) \\ &\quad = \sum_{j\in N^B} b_{ljt}^A\tilde{\chi}_{ljt}\left(\theta_0,(\hat{\theta}_{lt}^A,\theta^{-l,A}),(b_{lt}^A,v_t^{-l,A}),x^{t-1}\right) \\ &\quad - \frac{1}{\beta_l^A(\theta_{l0}^A)}\sum_{i\in N^A}\sum_{j\in N^B} S_{ijt}^{m;\beta}\left(\theta_0,(\hat{\theta}_{lt}^A,\theta^{-l,A}),(b_{lt}^A,v_t^{-l,A}),x^{t-1}\right)\cdot\tilde{\chi}_{ijt}\left(\theta_0,(\hat{\theta}_{lt}^A,\theta^{-l,A}),(b_{lt}^A,v_t^{-l,A}),x^{t-1}\right) \\ &\quad - \frac{\delta}{\beta_l^A(\theta_{l0}^A)}\mathbb{E}^{\tilde{\lambda}[\tilde{\chi}]|\hat{\theta}_{lt}^A,\theta_{-l}^A,b_{lt}^A,v_t^{-l,A},x^{t-1}}\left[W_{t+1}^{-l,A}(\theta_0,\theta_{t+1},b_{t+1},x^t)\right] \\ &\quad + \frac{1}{\beta_l^A(\theta_{l0}^A)}W_t^{-l,A}\left(\theta_0,(\hat{\theta}_{lt}^A,\theta^{-l,A}),(b_{lt}^A,v_t^{-l,A}),x^{t-1}\right) \\ &\quad = -\frac{1}{\beta_l^A(\theta_{l0}^A)}\sum_{i\in N^A\setminus\{l\}}\sum_{j\in N^B} S_{ijt}^{m;\beta}\left(\theta_0,(\hat{\theta}_{lt}^A,\theta^{-l,A}),(b_{lt}^A,v_t^{-l,A}),x^{t-1}\right)\cdot\tilde{\chi}_{ijt}\left(\theta_0,(\hat{\theta}_{lt}^A,\theta^{-l,A}),(b_{lt}^A,v_t^{-l,A}),x^{t-1}\right) \\ &\quad - \frac{1}{\beta_l^A(\theta_{l0}^A)}\sum_{j\in N^B}\left(\beta_j^B(\theta_{j0}^B)v_{ljt}^B - c_{ljt}(x^{t-1})\right)\cdot\tilde{\chi}_{ljt}\left(\theta_0,(\hat{\theta}_{lt}^A,\theta^{-l,A}),(b_{lt}^A,v_t^{-l,A}),x^{t-1}\right) \\ &\quad - \frac{\delta}{\beta_l^A(\theta_{l0}^A)}\mathbb{E}^{\tilde{\lambda}[\tilde{\chi}]|\hat{\theta}_{lt}^A,\theta^{-l,A},b_{lt}^A,v_t^{-l,A},x^{t-1}}\left[W_{t+1}^{-l,A}(\theta_0,\theta_{t+1},b_{t+1},x^t)\right] \\ &\quad + \frac{1}{\beta_l^A(\theta_{l0}^A)}W_t^{-l,A}\left(\theta_0,(\hat{\theta}_{lt}^A,\theta^{-l,A}),(b_{lt}^A,v_t^{-l,A}),x^{t-1}\right). \end{aligned}$$

Furthermore, note that

$$\begin{aligned} &\mathbb{E}^{\tilde{\lambda}[\tilde{\chi}]|\theta_l^A,\theta^{-l,A},v_{lt}^A,v_t^{-l,A},x^{t-1};(\hat{\theta}_{lt}^A,b_{lt}^A)}\left[R_{lt+1}^A(\theta_0,\theta_{t+1},b_{t+1},x^t)\right] \\ &\quad = \mathbb{E}^{\tilde{\lambda}[\tilde{\chi}]|\theta_l^A,\theta^{-l,A},v_{lt}^A,v_t^{-l,A},x^{t-1};(\hat{\theta}_{lt}^A,b_{lt}^A)}\left[W_{t+1}(\theta_0,\theta_{t+1},b_{t+1},x^t) - W_{t+1}^{-l,A}(\theta_0,\theta_{t+1},b_{t+1},x^t)\right] \\ &\quad = \mathbb{E}^{\tilde{\lambda}[\tilde{\chi}]|\theta_l^A,\theta^{-l,A},v_{lt}^A,v_t^{-l,A},x^{t-1};(\hat{\theta}_{lt}^A,b_{lt}^A)}\left[W_{t+1}(\theta_0,\theta_{t+1},b_{t+1},x^t)\right] \\ &\qquad - \mathbb{E}^{\lambda[\tilde{\chi}]|\hat{\theta}_{lt}^A,\theta^{-l,A},b_{lt}^A,v_t^{-l,A},x^{t-1}}\left[W_{t+1}^{-l,A}(\theta_0,\theta_{t+1},b_{t+1},x^t)\right], \end{aligned}$$

where the last equality uses the fact that, given x^t, $W_{t+1}^{-l,A}(\theta_0,\theta_{t+1},b_{t+1},x^t)$ is invariant to agent l's period-t bids and that the period-t decisions x_t are invariant to the agent's *true* types. Therefore, the right hand side of the inequality (A.2) is equal to

$$\frac{1}{\beta_l^A(\theta_{l0}^A)}\sum_{i\in N^A}\sum_{j\in N^B} S_{ijt}^{m;\beta}\left(\theta_0,\theta,v_t,x^{t-1}\right)\cdot\tilde{\chi}_{ijt}\left(\theta_0,(\hat{\theta}_{lt}^A,\theta^{-l,A}),(\hat{b}_{lt}^A,v_t^{-l,A}),x^{t-1}\right)$$
$$+\frac{\delta}{\beta_l^A(\theta_{l0}^A)}\mathbb{E}^{\tilde{\lambda}[\tilde{\chi}]|\theta_l^A,\theta^{-l,A},v_{lt}^A,v_t^{-l,A},x^{t-1};(\hat{\theta}_{lt}^A,b_{lt}^A)}\left[W_{t+1}(\theta_0,\theta_{t+1},b_{t+1},x^t)\right]$$
$$-\frac{1}{\beta_l^A(\theta_{l0}^A)}W_t^{-l,A}\left(\theta_0,(\hat{\theta}_{lt}^A,\theta^{-l,A}),(b_{lt}^A,v_t^{-l,A}),x^{t-1}\right).$$

Since, again, $W_t^{-l,A}$ is invariant to agent l's period-t bids, to establish that the inequality in (A.2) holds, it suffices to show that

$$W_t(\theta_0,\theta,v_t,x^{t-1})\geq\sum_{i\in N^A}\sum_{j\in N^B} S_{ijt}^{m;\beta}\left(\theta_0,\theta,v_t,x^{t-1}\right)\cdot\tilde{\chi}_{ijt}\left(\theta_0,(\hat{\theta}_{lt}^A,\theta^{-l,A}),(\hat{b}_{lt}^A,v_t^{-l,A}),x^{t-1}\right)$$
$$+\mathbb{E}^{\tilde{\lambda}[\tilde{\chi}]|\theta_l^A,\theta^{-l,A},v_{lt}^A,v_t^{-l,A},x^{t-1};(\hat{\theta}_{lt}^A,\hat{b}_{lt}^A)}\left[W_{t+1}(\theta_0,\theta_{t+1},b_{t+1},x^t)\right]. \tag{A.5}$$

The inequality in (A.5) follows from the fact that the matching rule $\tilde{\chi}$ maximizes continuation weighted surplus.

That it is (periodic ex-post) optimal for each agent to participate at all periods $t\geq 1$, and after all histories, follows from the fact that each agent's continuation payoff under truthful strategies coincides with his expected contribution to continuation weighted surplus, which is always nonnegative, scaled by a strictly positive weight.

The arguments above therefore establish that, when the matching rule is $\tilde{\chi}$ and the payments from period 1 onwards are $\tilde{\psi}_{\geq 1}$, participating and following truthful strategies constitutes a periodic ex-post continuation equilibrium, after any period-t history, $t\geq 1$. This completes the proof of the lemma. ■

Step 2. We now show that, when the period-0 membership fees are as in (6) in the main text, participating in period zero and selecting a membership status equal to the true vertical type is optimal for any individual who expects all other agents to do the same, irrespective of the individual's beliefs about the other agents' vertical types.

Let

$$\tilde{D}_l^k(\theta_{l0}^k,\theta;\chi)\equiv\begin{cases}\mathbb{E}^{\lambda[\chi]|\theta_{l0}^k,\theta,}\left[\sum_{t=1}^{\infty}\delta^t\sum_{h\in N^B}\varepsilon_{lht}^A\chi_{lht}\right] & \text{if } k=A\\ \mathbb{E}^{\lambda[\chi]|\theta_{l0}^k,\theta}\left[\sum_{t=1}^{\infty}\delta^t\sum_{h\in N^A}\varepsilon_{hlt}^B\chi_{hlt}\right] & \text{if } k=B\end{cases},$$

denote the match quality that agent $l\in N^k$ from side $k=A,B$ expects under the rule χ when the true profile of vertical types is $\theta\in\Theta$, the agent selects the membership status θ_{l0}^k in period zero and then conforms to the truthful strategy from period $t=1$ onwards, and all agents other than l (from side k) follow truthful strategies at each period. Note that $\lambda[\chi]|\theta$ denotes the stochastic process over matches, bids, and membership choices, when the true vertical types are θ, all agents other than agent l from side A follow truthful strategies from period $t=0$ onwards and agent l from side k chooses θ_{l0}^k in period 0 and then follows a truthful strategy from $t=1$ onwards. Also note that $\tilde{D}_l^k(\theta_l^k,(\theta_l^k,\theta^{-l,k});\chi) = D_l^k(\theta;\chi)$, with the function $D_l^k(\theta;\chi)$ as defined in Condition (7) in the main text— hereafter we highlight the dependence of the function $D_l^k(\theta;\chi)$ on the matching rule χ to avoid possible confusion.

For any agent $i \in N^A$ (the arguments for the side-B agents are analogous), let

$$\hat{U}_i^A(\theta) \equiv \mathbb{E}^{\lambda[\chi]|\theta}\left[\sum_{t=1}^{\infty}\delta^t \sum_{j\in N^B}\theta_i^A\varepsilon_{ijt}^A\chi_{ijt}(\theta,\theta_t,b_t,x^{t-1})\right] - \mathbb{E}^{\lambda[\chi]|\theta}\left[\sum_{t=0}^{\infty}\delta^t\psi_{it}^A(\theta,\theta_t,b_t,x^{t-1})\right]$$

denote the payoff that the agent expects in the matching auctions defined by the rules (χ,ψ) when the true vertical type profile is θ and all agents follow truthful strategies at all periods.[2]

Let $\tilde{\chi}$ be any matching rule that maximizes continuation weighted surplus after all histories, and $\tilde{\psi} = (\tilde{\psi}_0, \tilde{\psi}_{\geq 1})$ the associated payment rule, as defined by Conditions (6) and (15) in the main text. Note that, for each agent $l \in N^k$, each profile θ of true vertical types,

$$\mathbb{E}^{\lambda[\tilde{\chi}]|\theta}\left[\tilde{\psi}_{l0}^k(\theta) + \sum_{t=1}^{\infty}\delta^t\tilde{\psi}_{lt}^k(\theta,\theta_t,b_t,x^{t-1})\right] = \theta_l^k D_l^k(\theta;\tilde{\chi}) - \int_{\underline{\theta}_l^k}^{\theta_l^k} D_l^k((y,\theta^{-l,k});\tilde{\chi})dy - L_l^k,$$

which guarantees that, when all agents follow truthful strategies in each period, including period zero, the period-zero expected payoffs are given by

$$\hat{U}_l^k(\theta) = \int_{\underline{\theta}_l^k}^{\theta_l^k} D_l^k((y,\theta^{-l,k});\tilde{\chi})dy + L_l^k. \tag{A.6}$$

The next lemma shows that any matching rule $\tilde{\chi}$ that maximizes continuation weighted surplus at all histories satisfies certain monotonicity conditions which play a central role in establishing the optimality of truthful strategies at period zero.

Lemma 2. *Suppose $\tilde{\chi}$ maximizes continuation weighted surplus W_t at all histories. For all $l \in N^k$, $k = A, B$, the following monotonicities hold: (i) $\tilde{D}_l^k\left(\theta_{l0}^k,\theta;\tilde{\chi}\right)$ is non-decreasing in θ_{l0}^k, all $\theta \in \Theta$; (ii) $D_l^k\left((\theta_l^k,\theta^{-l,k});\tilde{\chi}\right)$ is non-decreasing in θ_l^k, all $\theta^{-l,k} \in \Theta^{-l,k}$.*

Proof of Lemma 2. Consider an arbitrary agent $i \in N^A$ from side A (the arguments for the side-B agents are analogous) and fix the profile of types $\theta^{-i,A}$ for the other agents. We prove claim (ii) first.

Claim (ii). Take any pair of types $\theta_i^A, \hat{\theta}_i^A \in \Theta_i^A$, with $\theta_i^A < \hat{\theta}_i^A$. That $\tilde{\chi}$ maximizes continuation

[2] The dependence of the payoff on the mechanism (χ,ψ) is omitted for convenience.

weighted surplus implies that[3]

$$
\begin{aligned}
&\mathbb{E}\Bigg[\sum_{t=1}^{\infty}\delta^t \sum_{r\in N^A\setminus\{i\}}\sum_{j\in N^B}\left(\beta_r^A(\theta_r^A)b_{rjt}^A+\beta_j^B(\theta_j^B)b_{rjt}^B-c_{rjt}(x^{t-1})\right)\\
&\qquad\cdot\tilde{\chi}_{rjt}\left((\hat{\theta}_i^A,\theta^{-i,A}),(\hat{\theta}_i^A,\theta^{-i,A}),\left(\left(\hat{\theta}_i^A\varepsilon_{ijt}^A\right)_{j\in N^B},b_t^{-i,A}\right),x^{t-1}\right)|\hat{\theta}_i^A,\theta^{-i,A}\Bigg]\\
&+\mathbb{E}\Bigg[\sum_{t=1}^{\infty}\delta^t\sum_{j\in N^B}\left(\beta_i^A(\hat{\theta}_i^A)\left(\hat{\theta}_i^A\varepsilon_{ijt}^A\right)+\beta_j^B(\theta_j^B)b_{ijt}^B-c_{ijt}(x^{t-1})\right)\\
&\qquad\cdot\tilde{\chi}_{ijt}\left((\hat{\theta}_i^A,\theta^{-i,A}),(\hat{\theta}_i^A,\theta^{-i,A}),\left(\left(\hat{\theta}_i^A\varepsilon_{ijt}^A\right)_{j\in N^B},b_t^{-i,A}\right),x^{t-1}\right)|\hat{\theta}_i^A,\theta^{-i,A}\Bigg]\\
&\geq\mathbb{E}\Bigg[\sum_{t=1}^{\infty}\delta^t \sum_{r\in N^A\setminus\{i\}}\sum_{j\in N^B}\left(\beta_r^A(\theta_r^A)b_{rjt}^A+\beta_j^B(\theta_j^B)b_{rjt}^B-c_{rjt}(x^{t-1})\right)\\
&\qquad\cdot\tilde{\chi}_{rjt}\left((\theta_i^A,\theta^{-i,A}),(\theta_i^A,\theta^{-i,A}),\left(\left(\theta_i^A\varepsilon_{ijt}^A\right)_{j\in N^B},b_t^{-i,A}\right),x^{t-1}\right)|(\theta_i^A,\theta^{-i,A})\Bigg]\\
&+\mathbb{E}\Bigg[\sum_{t=1}^{\infty}\delta^t\sum_{j\in N^B}\left(\beta_i^A(\hat{\theta}_i^A)\left(\hat{\theta}_i^A\varepsilon_{ijt}^A\right)+\beta_j^B(\theta_j^B)b_{ijt}^B-c_{ijt}(x^{t-1})\right)\\
&\qquad\cdot\tilde{\chi}_{ijt}\left((\theta_i^A,\theta^{-i,A}),(\theta_i^A,\theta^{-i,A}),\left(\left(\theta_i^A\varepsilon_{ijt}^A\right)_{j\in N^B},b_t^{-i,A}\right),x^{t-1}\right)|(\theta_i^A,\theta^{-i,A})\Bigg].
\end{aligned}
$$

The left-hand side of the above inequality is the expected weighted surplus — under the weights $(\beta_i^A(\hat{\theta}_i^A),\beta^{-i,A}(\theta^{-i,A}))$ — when all agents follow truthful strategies from period $t=0$ onward and the true profile of vertical types is $(\hat{\theta}_i^A,\theta^{-i,A})$. The right-hand side of the inequality is the expected weighted surplus — under the same weights $(\beta_i^A(\hat{\theta}_i^A),\beta^{-i,A}(\theta^{-i,A}))$ — when, at the same profile of true vertical types $(\hat{\theta}_i^A,\theta^{-i,A})$, all agents other than agent i from side A follow truthful strategies in all periods whereas agent i from side A perfectly replicates the behavior of type θ_i^A in each period (that is, he selects the membership status θ_i^A at $t=0$ and then, at each subsequent period, given the true horizontal types $\varepsilon_{it}^A\equiv(\varepsilon_{ijt}^A)_{j=1,\ldots,n^B}$, submits bids equal to $b_{ijt}^A=\theta_i^A\varepsilon_{ijt}^A$, all $j\in N^B$). The expectations are with respect to the horizontal types given the vertical types. The inequality follows from the fact that, holding the weights $(\beta_i^A(\hat{\theta}_i^A),\beta^{-i,A}(\theta^{-i,A}))$ fixed in the computation of the flow surpluses, at each period, given the bids $\left(\left(\hat{\theta}_i^A\varepsilon_{ijt}^A\right)_{j\in N^B},b_t^{-i,A}\right)$, the matches $\tilde{\chi}_t\left((\hat{\theta}_i^A,\theta^{-i,A}),(\hat{\theta}_i^A,\theta^{-i,A}),\left(\left(\hat{\theta}_i^A\varepsilon_{ijt}^A\right)_{j\in N^B},b_t^{-i,A}\right)\right)$ implemented when the membership status selected by agent i leads to the same weight $\beta_i^A(\hat{\theta}_i^A)$ used to compute the flow surpluses maximize the continuation weighted surplus. Note that, when type $\hat{\theta}_i^A$ replicates the same behavior of type θ_i^A in each period, the expectation over the horizontal types when the true profile of vertical types is $(\hat{\theta}_i^A,\theta^{-i,A})$ is the same as when the true profile of vertical types is $(\theta_i^A,\theta^{-i,A})$. This follows from the

[3]To ease the notation, we drop from the inequality below the various measures under which the expectations are taken. The expectations are over future bids, the selection of future membership statuses, and matches, under the processes induced by the agents playing according to the strategies described in the text after the inequality.

independence of the horizontal types from the vertical ones. The reason why, in the right-hand side of the above inequality we condition on $(\theta_i^A, \theta^{-i,A})$, despite the true state being $(\hat{\theta}_i^A, \theta^{-i,A})$, is that this facilitates the comparison with the inequality we establish below by inverting the roles of θ_i^A and $\hat{\theta}_i^A$.

Importantly, the above inequality is not to be confused with agent i's incentive-compatibility constraints. As explained in the main text, the monotonicity of match quality in the lemma does not follow from standard arguments in screening models which are based on the combination of incentive compatibility with the supermodularity of the agents' payoffs.

Next, inverting the role of $\hat{\theta}_i^A$ and θ_i^A, we have that

$$\begin{aligned}
&\mathbb{E}\Bigg[\sum_{t=1}^{\infty}\delta^t \sum_{r\in N^A\setminus\{i\}}\sum_{j\in N^B}\left(\beta_r^A(\theta_r^A)b_{rjt}^A+\beta_j^B(\theta_j^B)b_{rjt}^B-c_{rjt}(x^{t-1})\right)\\
&\qquad\cdot\tilde{\chi}_{rjt}\left((\theta_i^A,\theta^{-i,A}),(\theta_i^A,\theta^{-i,A}),\left(\left(\theta_i^A\varepsilon_{ijt}^A\right)_{j\in N^B},b_t^{-i,A}\right),x^{t-1}\right)|(\theta_i^A,\theta^{-i,A})\Bigg]\\
&+\mathbb{E}\Bigg[\sum_{t=1}^{\infty}\delta^t\sum_{j\in N^B}\left(\beta_i^A(\theta_i^A)\left(\theta_i^A\varepsilon_{ijt}^A\right)+\beta_j^B(\theta_j^B)b_{ijt}^B-c_{ijt}(x^{t-1})\right)\\
&\qquad\cdot\tilde{\chi}_{ijt}\left((\theta_i^A,\theta^{-i,A}),(\theta_i^A,\theta^{-i,A}),\left(\left(\theta_i^A\varepsilon_{ijt}^A\right)_{j\in N^B},b_t^{-i,A}\right),x^{t-1}\right)|(\theta_i^A,\theta^{-i,A})\Bigg]\\
&\geq\mathbb{E}\Bigg[\sum_{t=1}^{\infty}\delta^t \sum_{r\in N^A\setminus\{i\}}\sum_{j\in N^B}\left(\beta_r^A(\theta_r^A)b_{rjt}^A+\beta_j^B(\theta_j^B)b_{rjt}^B-c_{rjt}(x^{t-1})\right)\\
&\qquad\cdot\tilde{\chi}_{rjt}\left((\hat{\theta}_i^A,\theta^{-i,A}),(\hat{\theta}_i^A,\theta^{-i,A}),\left(\left(\hat{\theta}_i^A\varepsilon_{ijt}^A\right)_{j\in N^B},b_t^{-i,A}\right),x^{t-1}\right)|(\hat{\theta}_i^A,\theta_{-i}^A)\Bigg]\\
&+\mathbb{E}\Bigg[\sum_{t=1}^{\infty}\delta^t\sum_{j\in N^B}\left(\beta_i^A(\theta_i^A)\left(\theta_i^A\varepsilon_{ijt}^A\right)+\beta_j^B(\theta_j^B)b_{ijt}^B-c_{ijt}(x^{t-1})\right)\\
&\qquad\cdot\tilde{\chi}_{ijt}\left((\hat{\theta}_i^A,\theta^{-i,A}),(\hat{\theta}_i^A,\theta_{-i}^A),\left(\left(\hat{\theta}_i^A\varepsilon_{ijt}^A\right)_{j\in N^B},b_t^{-i,A}\right),x^{t-1}\right)|(\hat{\theta}_i^A,\theta_{-i}^A)\Bigg].
\end{aligned}$$

Combining the last two inequalities, we have that

$$\left(\beta_i^A(\hat{\theta}_i^A)\hat{\theta}_i^A-\beta_i^A(\theta_i^A)\theta_i^A\right)\cdot\left(D_i^A((\theta_{-i}^A,\hat{\theta}_i^A);\tilde{\chi})-D_i^A((\theta_{-i}^A,\theta_i^A);\tilde{\chi})\right)\geq 0.$$

Because $\beta_i^A(\cdot)$ is strictly positive and non-decreasing, it must be that

$$D_i^A((\theta_{-i}^A,\hat{\theta}_i^A);\tilde{\chi})\geq D_i^A((\theta_{-i}^A,\theta_i^A);\tilde{\chi}).$$

Claim (i). Again, because $\tilde{\chi}$ maximizes continuation weighted surplus after any history, we have that for any $\theta \in \Theta$, any $\hat{\theta}_i^A, \theta_i^A \in \Theta_i^A$,[4]

$$\mathbb{E}\Bigg[\sum_{t=1}^{\infty}\delta^t \sum_{r\in N^A\setminus\{i\}}\sum_{j\in N^B}\left(\beta_r^A(\theta_r^A)b_{rjt}^A+\beta_j^B(\theta_j^B)b_{rjt}^B-c_{rjt}(x^{t-1})\right)$$
$$\cdot\,\tilde{\chi}_{rjt}\left((\hat{\theta}_i^A,\theta^{-i,A}),(\theta_i^A,\theta^{-i,A}),\left(\left(\theta_i^A\varepsilon_{ijt}^A\right)_{j\in N^B},b_t^{-i,A}\right),x^{t-1}\right)|(\theta_i^A,\theta^{-i,A})\Bigg]$$
$$+\,\mathbb{E}\Bigg[\sum_{t=1}^{\infty}\delta^t \sum_{j\in N^B}\left(\beta_i^A(\hat{\theta}_i^A)\left(\theta_i^A\varepsilon_{ijt}^A\right)+\beta_j^B(\theta_j^B)b_{ijt}^B-c_{ijt}(x^{t-1})\right)$$
$$\cdot\,\tilde{\chi}_{ijt}\left((\hat{\theta}_i^A,\theta^{-i,A}),(\theta_i^A,\theta^{-i,A}),\left(\left(\theta_i^A\varepsilon_{ijt}^A\right)_{j\in N^B},b_t^{-i,A}\right),x^{t-1}\right)|(\theta_i^A,\theta^{-i,A})\Bigg]$$
$$\geq\mathbb{E}\Bigg[\sum_{t=1}^{\infty}\delta^t \sum_{r\in N^A\setminus\{i\}}\sum_{j\in N^B}\left(\beta_r^A(\theta_r^A)b_{rjt}^A+\beta_j^B(\theta_j^B)b_{rjt}^B-c_{rjt}(x^{t-1})\right)$$
$$\cdot\,\tilde{\chi}_{rjt}\left((\theta_i^A,\theta^{-i,A}),(\theta_i^A,\theta^{-i,A}),\left(\left(\theta_i^A\varepsilon_{ijt}^A\right)_{j\in N^B},b_t^{-i,A}\right),x^{t-1}\right)|(\theta_i^A,\theta_{-i}^A)\Bigg]$$
$$+\,\mathbb{E}\Bigg[\sum_{t=1}^{\infty}\delta^t \sum_{j\in N^B}\left(\beta_i^A(\hat{\theta}_i^A)\left(\theta_i^A\varepsilon_{ijt}^A\right)+\beta_j^B(\theta_j^B)b_{ijt}^B-c_{ijt}(x^{t-1})\right)$$
$$\cdot\,\tilde{\chi}_{ijt}\left((\theta_i^A,\theta^{-i,A}),(\theta_i^A,\theta_{-i}^A),\left(\left(\theta_i^A\varepsilon_{ijt}^A\right)_{j\in N^B},b_t^{-i,A}\right),x^{t-1}\right)|(\theta_i^A,\theta_{-i}^A)\Bigg].$$

The left-hand side of the above inequality is the expected weighted surplus when the true vertical types are $(\theta_i^A, \theta^{-i,A})$, the weights used to compute the flow surpluses are given by $(\beta_i^A(\hat{\theta}_i^A), \beta^{-i,A}(\theta^{-i,A}))$, all agents other than agent i from side A follow truthful strategies, and agent i selects the membership status $\hat{\theta}_i^A$ and then bids truthfully at all periods. The right-hand side of the above inequality is the expected weighted surplus under the same true vertical types $(\theta_i^A, \theta^{-i,A})$, when the weights used to compute the flow surpluses continue to be given by $(\beta_i^A(\hat{\theta}_i^A), \beta^{-i,A}(\theta^{-i,A}))$, and all agents, including agent i from side A, follow truthful strategies at all periods. Once again, the inequality follows from the fact that, when the weights used to evaluate the flow surpluses are $(\beta_i^A(\hat{\theta}_i^A), \beta^{-i,A}(\theta^{-i,A}))$, expected weighted surplus is higher when the membership status selected by agent i in period zero leads to same weight $\beta_i^A(\hat{\theta}_i^A)$ used to evaluate the flow surpluses.

[4] Again, the various expectations are under the processes induced by the agents playing according to the strategies described in the text after the inequality.

Inverting the roles of $\hat{\theta}_i^A$ and θ_i^A, we also have that

$$\mathbb{E}\Bigg[\sum_{t=1}^{\infty}\delta^t \sum_{r\in N^A\setminus\{i\}}\sum_{j\in N^B}\left(\beta_r^A(\theta_r^A)b_{rjt}^A+\beta_j^B(\theta_j^B)b_{rjt}^B-c_{rjt}(x^{t-1})\right)$$
$$\cdot\,\tilde{\chi}_{rjt}\left((\theta_i^A,\theta^{-i,A}),(\theta_i^A,\theta^{-i,A}),\left(\left(\theta_i^A\varepsilon_{ijt}^A\right)_{j\in N^B},b_t^{-i,A}\right),x^{t-1}\right)|(\theta_i^A,\theta^{-i,A})\Bigg]$$
$$+\,\mathbb{E}\Bigg[\sum_{t=1}^{\infty}\delta^t\sum_{j\in N^B}\left(\beta_i^A(\theta_i^A)\left(\theta_i^A\varepsilon_{ijt}^A\right)+\beta_j^B(\theta_j^B)b_{ijt}^B-c_{ijt}(x^{t-1})\right)$$
$$\cdot\,\tilde{\chi}_{ijt}\left((\theta_i^A,\theta^{-i,A}),(\theta_i^A,\theta^{-i,A}),\left(\left(\theta_i^A\varepsilon_{ijt}^A\right)_{j\in N^B},b_t^{-i,A}\right),x^{t-1}\right)|(\theta_i^A,\theta^{-i,A})\Bigg]$$
$$\geq\mathbb{E}\Bigg[\sum_{t=1}^{\infty}\delta^t\sum_{r\in N^A\setminus\{i\}}\sum_{j\in N^B}\left(\beta_r^A(\theta_r^A)b_{rjt}^A+\beta_j^B(\theta_j^B)b_{rjt}^B-c_{rjt}(x^{t-1})\right)$$
$$\cdot\,\tilde{\chi}_{rjt}\left((\hat{\theta}_i^A,\theta^{-i,A}),(\theta_i^A,\theta^{-i,A}),\left(\left(\theta_i^A\varepsilon_{ijt}^A\right)_{j\in N^B},b_t^{-i,A}\right),x^{t-1}\right)|(\theta_i^A,\theta_{-i}^A)\Bigg]$$
$$+\,\mathbb{E}\Bigg[\sum_{t=1}^{\infty}\delta^t\sum_{j\in N^B}\left(\beta_i^A(\theta_i^A)\left(\theta_i^A\varepsilon_{ijt}^A\right)+\beta_j^B(\theta_j^B)b_{ijt}^B-c_{ijt}(x^{t-1})\right)$$
$$\cdot\,\tilde{\chi}_{ijt}\left((\hat{\theta}_i^A,\theta^{-i,A}),(\theta_i^A,\theta_{-i}^A),\left(\left(\theta_i^A\varepsilon_{ijt}^A\right)_{j\in N^B},b_t^{-i,A}\right),x^{t-1}\right)|(\theta_i^A,\theta_{-i}^A)\Bigg].$$

Combining the above two inequalities, we have that, for any $\theta\in\Theta$, any $\hat{\theta}_{i0}^A,\theta_{i0}^A\in\Theta_i^A$,

$$\left(\beta_i^A(\hat{\theta}_{i0}^A)-\beta_i^A(\theta_{i0}^A)\right)\cdot\theta_i^A\cdot\left(\tilde{D}_i^A(\hat{\theta}_{i0}^A,\theta;\tilde{\chi})-\tilde{D}_i^A(\theta_{i0}^A,\theta;\tilde{\chi})\right)\geq 0.$$

Because $\theta_i^A>0$ and $\beta_i^A(\cdot)$ is strictly positive and non-decreasing, it must be that $\tilde{D}_i^A(\cdot,\theta;\tilde{\chi})$ is non-decreasing in θ_{i0}^A.[5] This completes the proof of the lemma. ■

Next, for any agent $i\in N^A$ (the arguments for the side-B agents are analogous), let

$$\tilde{U}_i^A(\theta_{i0}^A;\theta)\equiv\mathbb{E}^{\lambda[\chi]|\theta}\left[\sum_{t=1}^{\infty}\delta^t\sum_{j\in N^B}\theta_i^A\varepsilon_{ijt}^A\chi_{ijt}((\theta_{i0}^A,\theta^{-i,A}),\theta_t,b_t,x^{t-1})\right]$$
$$-\,\mathbb{E}^{\lambda[\chi]|\theta}\left[\sum_{t=0}^{\infty}\delta^t\psi_{it}^A((\theta_{i0}^A,\theta^{-i,A}),\theta_t,b_t,x^{t-1})\right]$$

denote the agent's expected payoff, when the true vertical type profile is θ, the agent chooses the membership status θ_{i0}^A in period zero, he follows a truthful strategy from period $t=1$ onwards, and all other agents follow truthful strategies from period $t=0$ onwards.[6] Note that $\tilde{U}_i^A(\theta_i^A;(\theta_i^A,\theta^{-i,A}))=\hat{U}_i^A(\theta_i^A,\theta^{-i,A})$, where $\hat{U}$ is as in (A.6).

From Step 1, participating and following truthful strategies is a periodic ex-post continuation

[5] Note that, because the only influence of period-0 membership statuses on match quality is through their impact on the weights β, if $\beta_i^A(\hat{\theta}_{i0}^A)=\beta_i^A(\theta_{i0}^A)$, then $\tilde{D}_i^A(\hat{\theta}_{i0}^A,\theta;\tilde{\chi})=\tilde{D}_i^A(\theta_{i0}^A,\theta;\tilde{\chi})$.

[6] Once again, the dependence of the payoff on (χ,ψ) is omitted for convenience.

equilibrium starting from any period-1 history (including those reached off path, by deviations in period zero). Standard arguments can then be used to show that the following envelope condition must be satisfied for all $l \in N^k$, $k = A, B$, all $\theta_{l0}^k \in \Theta_l^k$, all $\theta \in \Theta$,[7]

$$\tilde{U}_l^k(\theta_{l0}^k; \theta) = \tilde{U}_l^k(\theta_{l0}^k; (\theta_{l0}^k, \theta^{-l,k})) + \int_{\theta_{l0}^k}^{\theta_l^k} \tilde{D}_l^k(\theta_{l0}^k, (y, \theta^{-l,k}); \tilde{\chi}) dy. \tag{A.7}$$

The payoff that agent $l \in N^k$ from side $k = A, B$ obtains by selecting the membership status θ_{l0}^k when the true profile of vertical types is θ is thus given by

$$\begin{aligned}
\tilde{U}_l^k(\theta_{l0}^k; \theta) &= \tilde{U}_l^k(\theta_{l0}^k; (\theta_{l0}^k, \theta^{-l,k})) + \int_{\theta_{l0}^k}^{\theta_l^k} \tilde{D}_l^k(\theta_{l0}^k, (y, \theta^{-l,k}); \tilde{\chi}) dy \\
&\leq \tilde{U}_l^k(\theta_{l0}^k; (\theta_{l0}^k, \theta^{-l,k})) + \int_{\theta_{l0}^k}^{\theta_l^k} \tilde{D}_l^k(y, (y, \theta^{-l,k}); \tilde{\chi}) dy = \hat{U}_l^k(\theta_{l0}^k, \theta^{-l,k}) + \int_{\theta_{l0}^k}^{\theta_l^k} D_l^k((y, \theta^{-l,k}); \tilde{\chi}) dy \\
&= \hat{U}_l^k(\theta),
\end{aligned}$$

where the first equality follows from (A.7), the inequality follows from part (i) in Lemma 2, and the other equalities follow from (A.6) and the definition of the interim expected payoffs. Hence, given θ, the agent is better off following a truthful strategy from period zero onwards than deviating in period zero and then following a truthful strategy from period one onwards.

Finally, since for any $l \in N^k$, $k = A, B$, any $\theta \in \Theta$, D_i^k is uniformly bounded by E_i^k, participation constraints are satisfied when $L_i^k \geq (\bar{\theta}_i^k - \underline{\theta}_i^k) E_i^k$.

Combining the results in Step 2 with those in Step 1, we thus have that, when L_l^k is large enough, all $l \in N^k$, $k = A, B$, participating in the auctions and following a truthful strategy is a periodic ex-post equilibrium in the entire game. This completes the proof of Part 1. Q.E.D.

Part 2. We want to show that the following is true: "*When the weights are those in Theorem 2 and each agent's expected match quality under truthful strategies, as defined in Section 3, is such that $D_l^k((\underline{\theta}_l^k, \theta^{-l,k}); \beta^P) \geq 0$, all $l \in N^k$, $k = A, B$, and $\theta^{-l,k} \in \Theta^{-l,k}$, then the matching auctions in which (a) the scores are the indexes $S_{ijt}^{I;\beta^P}$ and (b) the payments are those in (6) and (15) in the main text, with weights β^P and with $L_l^k = 0$, all $l \in N^k$, $k = A, B$, are profit-maximizing*"

Proof of Part 2. The arguments parallel those establishing Theorem 2 in the main text. Consider any feasible mechanism Γ and any BNE σ of the game induced by Γ. Denote by $\hat{\chi} = (\hat{\chi}_t(\theta, \varepsilon^t))_{t=1}^{\infty}$

[7] Condition (A.7) follows from the fact that, in a fictitious problem where agent l's period-0 membership status is exogenously fixed at θ_{l0}^k, and else the agent is free to choose any strategy he wants from $t = 1$ onwards, the payoff $\tilde{U}_l^k(\theta_{l0}^k; \theta)$ the agent expects by following a truthful strategy from $t = 1$ onwards coincides with the value function for the aforementioned fictitious problem (this follows directly from the fact that a truthful strategy maximizes the agent's payoff starting from any history). Condition (A.7) then follows from the above observation along with the fact that the value function of the aforementioned fictitious problem is Lipschitz continuous in the agent's true vertical type θ_l^k, with derivative equal to $\tilde{D}_l^k(\theta_{l0}^k, (\theta_l^k, \theta^{-l,k})$.

and $\hat{\psi} = (\hat{\psi}_t(\theta, \varepsilon^t))_{t=0}^{\infty}$ the matching and payment rules, as a function of the true state, induced by σ in Γ.

The platform's profits under (Γ, σ) are equal to

$$\mathbb{E}^{\lambda[\hat{\chi}]}\left[\sum_{k=A,B}\sum_{l\in N^k}\sum_{t=0}^{\infty}\delta^t\hat{\psi}_{lt}^k(\theta,\varepsilon^t)-\sum_{t=1}^{\infty}\delta^t\sum_{i\in N^A}\sum_{j\in N^B}c_{ijt}(\hat{\chi}^{t-1}(\theta,\varepsilon^{t-1}))\hat{\chi}_{ijt}(\theta,\varepsilon^t)\right], \tag{A.8}$$

where $\lambda[\hat{\chi}]$ denotes the endogenous process over vertical and horizontal types under the matching rule $\hat{\chi}$ induced by the strategies σ in Γ. Alternatively, (A.8) can be rewritten as follows:

$$\mathbb{E}^{\lambda[\hat{\chi}]}\left[\sum_{t=1}^{\infty}\sum_{i\in N^A}\sum_{j\in N^B}\delta^t\left(\left(\theta_i^A\varepsilon_{ijt}^A+\theta_j^B\varepsilon_{ijt}^B-c_{ijt}(\hat{\chi}^{t-1}(\theta,\varepsilon^{t-1}))\right)\hat{\chi}_{ijt}(\theta,\varepsilon^t)\right)-\sum_{k=A,B}\sum_{l\in N^k}U_l^k(\theta_l^k)\right], \tag{A.9}$$

where $U_l^k(\theta_l^k)$ denotes the period-0 interim expected payoff of agent $l \in N^k$ from side $k = A, B$ when his true vertical type is θ_l^k, under the equilibrium σ in the mechanism Γ. Note that we denote the interim payoffs by U_l^k to differentiate them from the interim payoff functions $\hat{U}_l^k$ in the matching auction.

The period-0 participation constraints are satisfied if for all $l \in N^k$, $k = A, B$, $\theta_l^k \in \Theta_l^k$, $U_l^k(\theta_l^k) \geq 0$.

Following steps similar to those in Pavan, Segal and Toikka (2014, Theorem 1), one can show that the period-0 (interim) expected payoff of each agent $l \in N^k$, $k = A, B$ must satisfy the following envelope condition

$$U_l^k(\theta_l^k) = U_l^k(\underline{\theta}_l^k) + \int_{\underline{\theta}_l^k}^{\theta_l^k}\mathbb{E}\left[D_l^k(\theta;\hat{\chi})|y\right]dy, \tag{A.10}$$

where the expectation is taken over the entire profile of vertical types θ given agent l's own vertical type. This envelope condition, together with integration by parts, yields the following representation of the platform's profits,

$$\mathbb{E}^{\lambda[\hat{\chi}]}\Bigg[\sum_{t=1}^{\infty}\delta^t\sum_{i\in N^A}\sum_{j\in N^B}\Bigg(\Big(1-\frac{1-F_i^A(\theta_i^A)}{f_i^A(\theta_i^A)\theta_i^A}\Big)\theta_i^A\varepsilon_{ijt}^A+\Big(1-\frac{1-F_j^B(\theta_j^B)}{f_j^B(\theta_j^B)\theta_j^B}\Big)\theta_j^B\varepsilon_{ijt}^B$$
$$-c_{ijt}(\hat{\chi}^{t-1}(\theta,\varepsilon^{t-1}))\Bigg)\hat{\chi}_{ijt}(\theta,\varepsilon^t)\Bigg]-\sum_{k=A,B}\sum_{l\in N^k}U_l^k(\underline{\theta}_l^k). \tag{A.11}$$

The first term, which is only a function of the matching rule $\hat{\chi}$, is the *expected dynamic virtual surplus* (DVS) generated by the matching rule $\hat{\chi}$. Clearly, because such a representation applies to the matching rule generated by *any* BNE of *any* mechanism, the above representation applies also to the state-contingent matching rules generated by the truthful strategies in the matching auctions.

Now observe that, when the weights are given by $\beta^P \equiv (\beta_l^{k,P}(\cdot))_{l\in N^k}^{k=A,B}$, and either (i) $M = 1$, or (ii) $M \geq n^A \cdot n^B$ and $m_l^m \geq n^{-k}$, all $l \in N^k$, $k = A, B$, the matches implemented under the truthful equilibria of the matching auction with scores $S_{ijt}^{I;\beta^P}$ maximize DVS. This is because (a) these matches have been shown to maximize the continuation weighted surplus W_t for any vector of strictly positive and non-decreasing weights $\beta = (\beta_l^k(\cdot))_{l\in N^k, k=A,B}$ (step 1 in the proof of Part 1 above), (b) the ex-ante weighted expected surplus when the weights are given by β^P coincides with DVS, and (c) the weights β^P are strictly positive and non-decreasing.

Also note that, while we have restricted attention to (Γ, σ) that yield a deterministic matching rule $\hat{\chi}$, the platform cannot increase its profits by selecting a pair (Γ, σ) that induces a stochastic matching rule. This is because the matches under any such pair (Γ, σ) can also be induced through a deterministic direct mechanism that conditions on the type reports of a fictitious agent. The platform's profits under any such a mechanism thus continue to be given by the expression in (A.11), but with the matching rule conditioning on the behavior of such fictitious agent. This means that the platform's profits are equal to the weighted average of the platform's profits under the *deterministic* matching rules obtained by conditioning on the various reports of the fictitious agent. Because (A.11) is maximized over all possible deterministic rules under the equilibria in truthful strategies of the proposed matching auctions, we thus have that any pair (Γ, σ) yielding a stochastic matching rule can never improve upon the truthful equilibria of the proposed matching auctions when it comes to the platform's profits.

Next, let ψ^{β^P} be the payment scheme defined by Conditions (6) and (15) in the main text, when the scores are $S_{ijt}^{I;\beta^P}$ and $L_l^k = 0$, all $l \in N^k$, $k = A, B$. It is then easy to see that, in the proposed auctions, the payoff expected, in equilibrium, by the lowest vertical type of each agent is exactly equal to zero (this follows from (A.6) and the fact that, given any $\theta^{-l,k} \in \Theta^{-l,k}$, $\hat{U}_l^k(\underline{\theta}_l^k, \theta^{-l,k}) = 0$, all $l \in N^k, k = A, B$). This means that the truthful equilibria of the above matching auctions maximize both terms of (A.11). Provided all the period-0 participation constraints are satisfied (something we verify below), we then have that the platform's profits are maximized under the truthful equilibria of the proposed matching auctions.

To verify that, under the conditions in Part 2, all period-0 participation constraints are satisfied it suffices to observe that, for all $\theta^{-l,k}$, $l \in N^k$, $k = A, B$, $D_l^k((\theta_l^k, \theta^{-l,k}); \beta^P)$ is non-decreasing in θ_l^k, as established in part (ii) of Lemma 2 above. That $D_l^k((\underline{\theta}_l^k, \theta^{-l,k}); \beta^P) \geq 0$, all $l \in N^k$, $k = A, B$, all $\theta^{-l,k}$ then guarantees that $D_l^k(\theta, \hat{\beta}) \geq 0$, all $\theta \in \Theta$, $l \in N^k$, $k = A, B$. Because the period-0 interim payoffs satisfy the envelope condition

$$\hat{U}_l^k(\theta) = \int_{\underline{\theta}_l^k}^{\theta_l^k} D_l^k((y, \theta^{-l,k}); \tilde{\chi})dy$$

we then have that $\hat{U}_l^k(\theta) \geq 0$, all $\theta \in \Theta$, $l \in N^k$, $k = A, B$. This means that all the period-0 participation constraints are satisfied (in a periodic ex-post sense, i.e., for any θ, and not just in expectation over $\theta^{-l,k}$ given θ_l^k). This completes the proof of Part 2. Q.E.D.

Part 3. We want to show that the following is true: *"(i) The matching auctions in which (a) the scores are $S_{ijt}^{I;\beta^W}$, with weights β^W as defined in Theorem 3, and (b) the payments are those in (6) and (15) in the main text, with weights β^W and with L_l^k large enough, all $l \in N^k$, $k = A, B$, are welfare* *maximizing. (ii) Suppose that, when agents follow truthful strategies in the auctions with scores $S^{I;\beta^W}_{ijt}$, $D^k_l((\underline{\theta}^k_l, \theta^{-l,k}); \beta^W) \geq 0$, all $l \in N^k$, $k = A, B$, and all $\theta^{-l,k}$. Then the matching auctions in which the scores are $S^{I;\beta^W}_{ijt}$ and the payments are given by (6) and (15) in the main text, with $L^k_l = 0$, all $l \in N^k$, $k = A, B$, admit ex-post periodic equilibria in which agents participate and follow truthful strategies at all histories. Furthermore, the truthful equilibria of such auctions maximize the platform's profits over all BNE of all feasible mechanisms implementing welfare-maximizing matches and inducing the agents to join the platform in period zero."*

Proof of Part 3. The arguments parallel those in the proof of Theorem 3 in the main text. Claim (i) follows from the fact that, when $\beta = \beta^W$, the matches implemented under truthful strategies maximize the sum of all agents' expected payoffs, net of the platform's costs, at all histories (this follows directly from the arguments establishing Part 1 in the present document). Claim (ii) follows from the fact that (a) the platform's expected profits under any BNE of any feasible mechanism Γ implementing the welfare-maximizing matches and inducing all agents to join at $t = 0$ satisfy the representation in (A.11), with $U^k_l(\underline{\theta}^k_l) \geq 0$, (b) each agent's period-0 expected payoff satisfies Condition (A.10), (c) in the proposed auctions, $U^k_l(\underline{\theta}^k_l) = 0$ if, and only if, the payments defined by Conditions (6) and (15) in the main text (for $\beta = \beta^W$) are such that $L^k_l = 0$, all $l \in N^k$, $k = A, B$, and (d) when the payments are the ones defined by Conditions (6) and (15) in the main text (for $\beta = \beta^W$) with $L^k_l = 0$, all $l \in N^k$, $k = A, B$, all agents' period-0 participation constraints are satisfied, regardless of their beliefs over other agents' types, if, and only if, $D^k_l((\underline{\theta}^k_l, \theta^{-l,k}); \beta^W) \geq 0$. The latter property in turn follows from the fact that expected match quality $D^k_l((\cdot, \theta^{-l,k}); \beta^W)$ under the truthful equilibria of the proposed auctions is non-decreasing in the agents' true vertical type, which was shown in part (ii) of Lemma 2 above. This completes the proof of Part 3 in the theorem. Q.E.D.

Part 4. We want to show the following: *"Suppose all agents derive a nonnegative utility from interacting with all other agents from the opposite side (formally, $\varepsilon^k_{ijt} \geq 0$, all $(i, j) \in N^A \times N^B$, $k = A, B$, $t \geq 1$).*
(1) If none of the capacity constraints binds (i.e., if $M \geq n^A \cdot n^B$, and $m^m_l \geq n^{-k}$, all $l \in N^k$, $k = A, B$), then, for all $(i, j) \in N^A \times N^B$, all $t \geq 1$, $\chi^P_{ijt} = 1 \Rightarrow \chi^W_{ijt} = 1$.
(2) If $M = 1$, then, for all $t \geq 1$, $\sum_{(i,j) \in N^A \times N^B} \chi^W_{ijt} \geq \sum_{(i,j) \in N^A \times N^B} \chi^P_{ijt}$."

Proof of Part 4. Let $\chi^P_t(\theta, \omega)$ and $\chi^W_t(\theta, \omega)$ denote the state-contingent matches implemented in period $t \geq 1$, under the truthful equilibria of, respectively, the profit-maximizing and the welfare-maximizing auctions described in Parts 2 and 3 in the proof of Theorem 5 in the present document. Note that the arguments of these functions are the exogenous vertical types θ and the sequences of exogenous innovations $\omega \equiv (\omega^k_{ijs})^{s=1,\ldots,\infty}_{(i,j) \in N^A \times N^B, k=A,B}$ that, along with the matches implemented in previous periods generate the horizontal types ε. Because the match values are endogenous, such a representation favors the comparison of the matches sustained under the two auctions by making the "state variables" exogenous, thus eliminating the confusion that may originate from the fact that the histories of horizontal types need not coincide under the two auctions.

Similarly, let $S_{ijt}^{I;P}(\theta,\omega)$ and $S_{ijt}^{I;W}(\theta,\omega)$ denote the period-t indexes under the truthful equilibria of the profit-maximizing and the welfare-maximizing auctions, respectively. Because (θ,ω) are exogenous and time-invariant, they are dropped from all the functions χ^P, χ^W, $S^{I;P}$, and $S^{I;W}$ below.

First, observe that, because $\beta_l^{k,P}(\theta_l^k) \leq 1 = \beta_l^{k,W}(\theta_l^k)$, all $\theta_l^k \in \Theta_l^k$, $l \in N^k$, $k = A, B$, and because the evolution of the match values is time-autonomous and the horizontal types are nonnegative, for any $(i,j) \in N^A \times N^B$, any $t, \tau \geq 1$,

$$\sum\nolimits_{s=1}^{t-1} \chi_{ijs}^W = \sum\nolimits_{s=1}^{\tau-1} \chi_{ijs}^P \;\Rightarrow\; S_{ijt}^{I;W} \geq S_{ij\tau}^{I;P}. \tag{A.12}$$

Claim 1. When none of the capacity constraints binds, in each period $t \geq 1$, the matches implemented under the equilibria of the profit-maximizing auctions (alternatively, the welfare-maximizing auctions) are all those for which the index $S_{ijt}^{I;P} \geq 0$ (alternatively, $S_{ijt}^{I;W} \geq 0$). The result then follows from the fact that, for any $(i,j) \in N^A \times N^B$, $t \geq 1$, $S_{ijt}^{I;W} \geq S_{ijt}^{I;P}$. The last property, in turn, follows by induction. First observe that the property is necessarily true at $t = 1$, given (A.12) and the fact that, in period $t = 1$, the number of past interactions is necessarily the same under profit and welfare maximization. Now suppose the result holds for all $1 \leq s < t$. Note that any match for which $S_{ijt}^{I;P} \geq 0$ has been active at each preceding period $s < t$, both under profit maximization and under welfare maximization. The result then follows again from (A.12), which implies that $S_{ijt}^{I;W} \geq S_{ijt}^{I;P}$.

Claim 2. First observe that, under the equilibria of the profit-maximizing auction, if at some period $t \geq 1$, $\chi_{ijt}^P = 0$, all $(i,j) \in N^A \times N^B$, then $\chi_{ijs}^P = 0$, all $s > t$, all $(i,j) \in N^A \times N^B$. The same property holds for χ^W. Next, observe that, if matching stops at period t under profit maximization (alternatively, welfare maximization), then $S_{ijt}^{I;P} < 0$ all $(i,j) \in N^A \times N^B$ (alternatively, $S_{ijt}^{I;W} < 0$ all $(i,j) \in N^A \times N^B$). Now suppose that, under profit maximization, matching is still active in period t (meaning, there exists $(i,j) \in N^A \times N^B$ such that $\chi_{ijt}^P = 1$). Then there are two cases. (1) Either $\sum_{s=1}^{t-1} \chi_{ijs}^W = \sum_{s=1}^{t-1} \chi_{ijs}^P$, all $(i,j) \in N^A \times N^B$, in which case (A.12) implies that $S_{ijt}^{I;W} \geq S_{ijt}^{I;P}$ for all $(i,j) \in N^A \times N^B$, which implies the result. Or, (2) there exists $(i,j) \in N^A \times N^B$ such that $\sum_{s=1}^{t-1} \chi_{ijs}^W < \sum_{s=1}^{t-1} \chi_{ijs}^P$. To see this, recall that the fact that matching is still active in period t under profit maximization implies it must have been active in each preceding period as well. That $\sum_{s=1}^{t-1} \chi_{ijs}^W < \sum_{s=1}^{t-1} \chi_{ijs}^P$ in turn implies there must exist $\tau < t$ such that $\sum_{s=1}^{\tau-1} \chi_{ijs}^P = \sum_{s=1}^{t-1} \chi_{ijs}^W$, and $\chi_{ij\tau}^P = 1$. That $\chi_{ij\tau}^P = 1$ in turn implies $S_{ij\tau}^{I;P} \geq 0$. That $\sum_{s=1}^{\tau-1} \chi_{ijs}^P = \sum_{s=1}^{t-1} \chi_{ijs}^W$ in turn implies that $S_{ijt}^{I;W} \geq S_{ij\tau}^{I;P}$, where the result follows again from (A.12). From the discussion above, that $S_{ijt}^{I;W} \geq S_{ij\tau}^{I;P}$ in turn implies that matching must be active in period t also under welfare maximization. This completes the proof of Part 4 and of the theorem. Q.E.D.