Adversarial Coordination and Public Information Design: Online Supplement

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Abstract

This document contains the proofs of Examples 2 and 3 in the main text (Sections S1 and S2), as well as an extended discussion of the role played by the multiplicity of the receivers and their exogenous private information for the (sub)optimality of monotone rules (Section S3). All numbered items contain the prefix "S." Any numbered reference without the prefix "S" refers to an item in the main text.

S1. Proof of Example 2 in Main Text

Proof of Example 2. The proof is in two steps. Step 1 characterizes the threshold θ^*_σ defining the optimal deterministic monotone rule, whereas Step 2 constructs the non-monotone policy that strictly improves over the optimal deterministic monotone one.

Step 1. The primitives in this example satisfy the conditions in Theorem 2 in the main text. This means that, given any signal s disclosed by any policy Γ, MARP is in threshold strategies, which in turn implies that the default outcome is monotone in θ.

Next recall that, for any default threshold $\theta \in [0,1]$, the corresponding signal threshold $x^*_\sigma(\theta)$ is implicitly defined by $P_\sigma\left(x^*_\sigma(\theta)|\theta\right) = \theta$. Using the fact that, for any $\theta \in [-K, 1+K]$ and $x \in [\theta-\sigma, \theta+\sigma]$, $P_\sigma(x|\theta) = (x - \theta + \sigma)/2\sigma$, we have that $x^*_\sigma(\theta) = (1+2\sigma)\theta - \sigma$.

For any $\hat{\theta} \in [0,1]$, let $\Gamma^{\hat{\theta}} \equiv \{\{0,1\}, \pi^{\hat{\theta}}\}$ be the deterministic monotone policy with cutoff $\hat{\theta}$. Next, let $V^{\Gamma^{\hat{\theta}}}_\sigma(\cdot)$ be the function defined by $V^{\Gamma^{\hat{\theta}}}_\sigma(\theta) \equiv U^{\Gamma^{\hat{\theta}}}_\sigma\left(x^*_\sigma(\theta), 1|x^*_\sigma(\theta)\right)$ for any $\theta \in [\hat{\theta}/(1+2\sigma), 1]$. This function represents the expected payoff differential between investing and not investing of the marginal investor with signal $x^*_\sigma(\theta)$, when each investor invests if and only if their signal is above $x^*_\sigma(\theta)$ (and hence default occurs if, and only if, fundamentals are below θ), the quality of the investors' signal is σ, and the policy $\Gamma^{\hat{\theta}}$ announces that $s = 1$, thus revealing that $\theta \geq \hat{\theta}$. Note that, for any $0 \leq \theta < \hat{\theta}/(1+2\sigma)$, $x^*_\sigma(\theta) + \sigma < \hat{\theta}$, which implies that the signal $x^*_\sigma(\theta)$ is not consistent with the event that fundamentals are above $\hat{\theta}$. Equivalently, when $\theta \geq \hat{\theta}$, the lowest possible signal that an individual may receive is $\hat{\theta} - \sigma$. When each investor invests if and only if $x > \hat{\theta} - \sigma$, default occurs if and only if $\theta \leq \hat{\theta}/(1+2\sigma)$. Hence, the lowest default threshold that is consistent with the policy $\Gamma^{\hat{\theta}}$ is $\hat{\theta}/(1+2\sigma)$. The function $V^{\Gamma^{\hat{\theta}}}_\sigma(\theta)$ is thus defined only for $\theta \in [\hat{\theta}/(1+2\sigma), 1]$.

The cutoff θ^*_σ characterizing the optimal deterministic monotone policy is given by

$$\theta^*_\sigma = \inf\{\hat{\theta} \in [0,1] : V^{\Gamma^{\hat{\theta}}}_\sigma(\theta) \geq 0 \;\text{ for all } \theta \in [\hat{\theta}/(1+2\sigma), 1]\}. \tag{S1}$$

Claim S1. *For any $\hat{\theta} \in [0,1]$, $V^{\Gamma^{\hat{\theta}}}_\sigma(\cdot)$ has a unique minimizer. Letting*

$$\theta^{\min}_\sigma(\hat{\theta}) \equiv \arg\,min_{\theta \in [\hat{\theta}/(1+2\sigma), 1]} V^{\Gamma^{\hat{\theta}}}_\sigma(\theta),$$

we have that $\theta^{\min}_\sigma(\hat{\theta})$ satisfies $x^*_\sigma(\theta^{\min}_\sigma(\hat{\theta})) - \sigma = \hat{\theta}$.

Proof of Claim S1. Clearly, for any $\theta \in [\hat{\theta}/(1+2\sigma), \hat{\theta}]$, $V^{\Gamma^{\hat{\theta}}}_\sigma(\theta) = g$. This is because when each investor invests if and only if $x > x^*_\sigma(\theta)$ default occurs only for fundamentals below θ. Hence the announcement that $\theta > \hat{\theta}$ reveals to the marginal investor with signal $x^*_\sigma(\theta)$ that default will not occur.

Next, observe that for any $\theta \in (\hat{\theta}, (\hat{\theta}+2\sigma)/(1+2\sigma)]$, $x^*_\sigma(\theta) - \sigma < \hat{\theta}$, implying that[1]

$$V_\sigma^{\Gamma^{\hat{\theta}}}(\theta) = g - (g+|b|)\,\mathbb{P}_\sigma[\tilde{\theta} \leq \theta | \tilde{\theta} \geq \hat{\theta}; x^*_\sigma(\theta)] = g - (g+|b|)\,\frac{\theta - \hat{\theta}}{(1+2\sigma)\,\theta - \hat{\theta}},$$

which is strictly decreasing in θ. Finally, note that, for any $\theta \in ((\hat{\theta}+2\sigma)/(1+2\sigma), 1]$, $x^*_\sigma(\theta) - \sigma > \hat{\theta}$, implying that

$$V_\sigma^{\Gamma^{\hat{\theta}}}(\theta) = g - (g+|b|)\,\mathbb{P}_\sigma[\tilde{\theta} \leq \theta | \tilde{\theta} \geq \hat{\theta}; x^*_\sigma(\theta)] = g + (g+|b|)(\theta - 1),$$

which is strictly increasing in θ. Hence, $V_\sigma^{\Gamma^{\hat{\theta}}}(\cdot)$ has a single minimizer over $[\hat{\theta}/(1+2\sigma), 1]$. The latter is equal to $\theta_\sigma^{min}(\hat{\theta}) = (\hat{\theta}+2\sigma)/(1+2\sigma)$ and is such that $x^*_\sigma(\theta_\sigma^{min}(\hat{\theta})) - \sigma = \hat{\theta}$. $\square$

Next, let $\Gamma^{\theta^*_\sigma} \equiv (\{0,1\}, \pi^{\theta^*_\sigma})$ be the optimal deterministic monotone policy (with cut-off $\hat{\theta} = \theta^*_\sigma$). Using the characterization of θ^*_σ in (S1), we thus have that, under $\Gamma^{\theta^*_\sigma}$, at the point $\theta_\sigma^{min}(\theta^*_\sigma)$ at which $V_\sigma^{\Gamma^{\theta^*_\sigma}}$ reaches its minimum, $V_\sigma^{\Gamma^{\theta^*_\sigma}}(\theta_\sigma^{min}(\theta^*_\sigma)) = 0$. Using the fact that

$$V_\sigma^{\Gamma^{\theta^*_\sigma}}\left(\theta_\sigma^{min}(\theta^*_\sigma)\right) = g - (g+|b|)\,\frac{\theta_\sigma^{min}(\theta^*_\sigma) - \theta^*_\sigma}{(1+2\sigma)\,\theta_\sigma^{min}(\theta^*_\sigma) - \theta^*_\sigma},$$

we then have that $\theta^*_\sigma = (1+2\sigma)\frac{|b|}{g+|b|} - 2\sigma$. Next, let $\Gamma_\emptyset$ be the no-disclosure policy and note that, for any $\theta \in [0,1]$,

$$V_\sigma^{\Gamma_\emptyset}(\theta) = g - (g+|b|)\,\mathbb{P}_\sigma[\tilde{\theta} \leq \theta | x^*_\sigma(\theta)] = g + (g+|b|)(\theta - 1),$$

which is increasing in θ and has a unique zero at $\theta = |b|/(g+|b|) \equiv \theta^{MS}$.

This means that, in the absence of any disclosure, under the unique rationalizable strategy profile (and hence under MARP), each agent invests if and only if $x > x^*_\sigma\left(\theta^{MS}\right)$, and default occurs if and only if fundamentals are below θ^{MS}. The results above then imply that the optimal deterministic policy $\Gamma^{\theta^*_\sigma}$ is defined by a threshold $\theta^*_\sigma = (1+2\sigma)\,\theta^{MS} - 2\sigma = x^*_\sigma\left(\theta^{MS}\right) - \sigma$ that coincides with the left end-point of the support of the posterior beliefs of each agent

[1] The notation $\mathbb{P}_\sigma\left[\tilde{\theta} \leq \theta | \tilde{\theta} \geq \hat{\theta}; x\right]$ stands for the probability that an investor with signal x assigns to the event that $\tilde{\theta} \leq \theta$ when the quality of his exogenous signal is parametrized by σ and the policy reveals that $\tilde{\theta} \geq \hat{\theta}$.

with signal $x^*_\sigma\left(\theta^{MS}\right)$. In fact, for any truncation point $\hat{\theta} < x^*_\sigma\left(\theta^{MS}\right) - \sigma$, there exists θ close to θ^{MS} such that $V^{\Gamma^{\hat{\theta}}}_\sigma(\theta) < 0$ implying that refraining from investing for all $x < x^*_\sigma\left(\theta^{MS}\right)$ is rationalizable in the continuation game following the announcement that $\theta \geq \hat{\theta}$, implying that the policy $\Gamma^{\hat{\theta}}$ fails to satisfy PCP. Similarly, for any truncation point $\hat{\theta} > x^*_\sigma(\theta^{MS}) - \sigma$, $V^{\Gamma^{\hat{\theta}}}_\sigma(\theta)$ reaches its minimum at $\theta^{min}_\sigma(\hat{\theta}) > \theta^{MS}$ and is such that $V^{\Gamma^{\hat{\theta}}}_\sigma(\theta^{min}_\sigma(\hat{\theta})) = V^{\Gamma_\emptyset}_\sigma(\theta^{min}_\sigma(\hat{\theta})) > V^{\Gamma_\emptyset}_\sigma(\theta^{MS}) = 0$, where the inequality follows from the monotonicity of $V^{\Gamma_\emptyset}_\sigma(\cdot)$. Hence, $\theta^*_\sigma = x^*_\sigma\left(\theta^{MS}\right) - \sigma$.

Step 2. Having characterized the optimal deterministic monotone policy $\Gamma^{\theta^*_\sigma}$, we now show that, when σ is small, there exists another policy Γ that also satisfies PCP and guarantees no default for a larger set of fundamentals than $\Gamma^{\theta^*_\sigma}$.

Let $\sigma^\# \equiv \frac{\theta^{MS}}{2(1-\theta^{MS})} > 0$. For any $\sigma \in (0, \sigma^\#)$, $\theta^*_\sigma = (1+2\sigma)\,\theta^{MS} - 2\sigma > 0$. For any $\sigma, \delta, \gamma > 0$ small, let $\theta''_\sigma(\delta,\gamma) \equiv x^*_\sigma\left(\theta^{MS} - \delta\right) - \sigma = (1+2\sigma)(\theta^{MS} - \delta) - 2\sigma$ and $\theta'_\sigma(\delta,\gamma) \equiv \theta''_\sigma(\delta,\gamma) - \gamma$. Note that, for any $\sigma \in (0,\sigma^\#)$, $\delta > 0$ and $\gamma > 0$ can be chosen so that $0 < \theta'{}_\sigma(\delta,\gamma) < \theta''_\sigma(\delta,\gamma) < \theta^*_\sigma$.

Consider the non-monotone deterministic policy $\Gamma_{\delta,\gamma} \equiv \left\{\{0,1\}, \pi_{\delta,\gamma}\right\}$ given by

$$\pi_{\delta,\gamma}(1|\theta) \equiv \mathbf{1}\left\{\theta \in \left[\theta'{}_\sigma(\delta,\gamma), \theta''_\sigma(\delta,\gamma)\right] \cup [\theta^*_\sigma, \infty)\right\}.$$

We show that, for any $\sigma \in (0,\sigma^\#)$, there exit $\delta, \gamma > 0$ such that (i) $0 \leq \theta'{}_\sigma(\delta,\gamma) < \theta''_\sigma(\delta,\gamma) < \theta^*_\sigma$, and (ii) $V^{\Gamma_{\delta,\gamma}}_\sigma(\theta) \geq 0$ for all $\theta > \theta'{}_\sigma(\delta,\gamma)/(1+2\sigma)$, with $V^{\Gamma_{\delta,\gamma}}_\sigma(\theta) = 0$ only for $\theta = \theta^{MS}$.[2]

First observe that, for any $\sigma \in (0,\sigma^\#)$, $\delta \in \left(0, \theta^{MS} - \frac{2\sigma}{1+2\sigma}\right)$ and

$$0 < \gamma \leq (1+2\sigma)\left(\theta^{MS} - \delta\right) - 2\sigma \equiv R_0\left(\delta, \theta^{MS}, \sigma\right)$$

guarantee that $0 \leq \theta'{}_\sigma(\delta,\gamma) < \theta''_\sigma(\delta,\gamma) < \theta^*_\sigma$.[3]

Next note that, for any (σ,δ,γ) with $\sigma \in (0,\sigma^\#)$, $\delta \in \left(0, \theta^{MS} - 2\sigma/(1+2\sigma)\right)$ and $0 < \gamma \leq R_0\left(\delta,\theta^{MS},\sigma\right)$, $V^{\Gamma_{\delta,\gamma}}_\sigma(\theta) = V^{\Gamma^{\theta^*_\sigma}}_\sigma(\theta)$ for all $\theta \in \left[\theta^{MS} - \delta, 1\right]$. Indeed, for any $\theta \in \left[\theta^{MS} - \delta, 1\right]$, $x^*_\sigma(\theta) - \sigma > \theta''_\sigma(\delta,\gamma)$ implying that the the posterior beliefs of the marginal investor with signal $x^*_\sigma(\theta)$ under the policy $\Gamma_{\delta,\gamma}$ coincide with those under the policy $\Gamma^{\theta^*_\sigma}$.

[2]Consistently with the notation above, $V^{\Gamma_{\delta,\gamma}}_\sigma(\theta)$ is the expected payoff of the marginal investor with signal $x^*_\sigma(\theta)$ when the policy $\Gamma_{\delta,\gamma}$ announces that $s = 1$ and the quality of the agents' exogenous signals is parametrized by σ. For any $\theta < \theta'{}_\sigma(\delta,\gamma)/(1+2\sigma)$, $x^*_\sigma(\theta) + \sigma < \theta'$, which implies that the signal $x^*_\sigma(\theta)$ is not consistent with the event that fundamentals are above $\theta'{}_\sigma(\delta,\gamma)$. Equivalently, because the lowest signal that is consistent with $\theta \in [\theta'{}_\sigma(\delta,\gamma), \theta''_\sigma(\delta,\gamma)] \cup [\theta^*_\sigma, \infty)$ is $\theta'{}_\sigma(\delta,\gamma) - \sigma$, the lowest default threshold is $\theta'{}_\sigma(\delta,\gamma)/(1+2\sigma)$.

[3]Observe that $\sigma \in (0,\sigma^\#)$ implies that $\theta^{MS} - 2\sigma/(1+2\sigma) > 0$. In turn, $\delta \in \left(0, \theta^{MS} - 2\sigma/(1+2\sigma)\right)$ implies that $0 < \theta''_\sigma(\delta,\gamma) < \theta^*_\sigma$ and that $R_0\left(\delta,\theta^{MS},\sigma\right) > 0$. Finally, that $0 < \gamma \leq R_0\left(\delta,\theta^{MS},\sigma\right)$ implies that $0 \leq \theta'{}_\sigma(\delta,\gamma) < \theta''_\sigma(\delta,\gamma)$.

Let $\theta^{\sharp}_{\sigma}(\delta,\gamma)$ be such that $x^*_\sigma\left(\theta^{\sharp}_{\sigma}(\delta,\gamma)\right)-\sigma=\theta^{'}_{\sigma}(\delta,\gamma)$. Dropping the arguments of $\theta^{\sharp}_{\sigma}(\delta,\gamma)$, $\theta^{'}_{\sigma}(\delta,\gamma)$ and $\theta^{''}_{\sigma}(\delta,\gamma)$ to ease the notation, we have that

$$\theta'=\theta''-\gamma=x^*_\sigma\left(\theta^{MS}-\delta\right)-\sigma-\gamma=(1+2\sigma)\left(\theta^{MS}-\delta\right)-2\sigma-\gamma.$$

From the definition of $\hat{\theta}$ we have that $x^*_\sigma(\hat{\theta})-\sigma=(1+2\sigma)\,\theta^{\sharp}-2\sigma=\theta'$. Combining the above two results we obtain that $\theta^{\sharp}=\theta^{MS}-\delta-\gamma/\left(1+2\sigma\right)$. Fixing $\sigma\in(0,\sigma^{\#})$, note that, for $\delta,\gamma>0$ small, $\theta^{\sharp}\geq\theta^*_\sigma$. Specifically, for any $\sigma\in(0,\sigma^{\#})$ and any $0<\delta<2\sigma\left(1-\theta^{MS}\right)$, $\theta^{\sharp}\geq\theta^*_\sigma$ if and only if

$$\gamma\leq(1+2\sigma)\left(2\sigma\left(1-\theta^{MS}\right)-\delta\right)\equiv R_1\left(\delta,\theta^{MS},\sigma\right).$$

Next, observe that, for any $\theta\in\left[\theta^{\sharp},\theta^{MS}-\delta\right)$,

$$\begin{aligned}V_\sigma^{\Gamma_{\delta,\gamma}}(\theta) &= g-(g+|b|)\,\mathbb{P}_\sigma[\tilde{\theta}\leq\theta|\tilde{\theta}\in[x^*_\sigma(\theta)-\sigma,\theta'']\cup[\theta^*_\sigma,\infty);x^*_\sigma(\theta)]\\ &= g-(g+|b|)\left(\theta''-\theta^*_\sigma+2\sigma\left(1-\theta\right)\right)/\left(\theta''-\theta^*_\sigma+2\sigma\right),\end{aligned}$$

which is strictly increasing in θ. Similarly, for any $\theta\in\left[\theta^*_\sigma,\theta^{\sharp}\right)$,

$$\begin{aligned}V_\sigma^{\Gamma_{\delta,\gamma}}(\theta) &= g-(g+|b|)\,\mathbb{P}_\sigma[\tilde{\theta}\leq\theta|\tilde{\theta}\in[\theta',\theta'']\cup[\theta^*_\sigma,\infty);x^*_\sigma(\theta)]\\ &= g-(g+|b|)\,\frac{\theta-\theta^*_\sigma+\gamma}{x^*_\sigma(\theta)+\sigma-\theta^*_\sigma+\gamma}=g-(g+|b|)\,\frac{\theta-\theta^*_\sigma+\gamma}{(1+2\sigma)\,\theta-\theta^*_\sigma+\gamma},\end{aligned}$$

which is strictly deceasing for any $\gamma\leq\theta^*_\sigma$. Note that $\theta'\geq0$ requires that $\gamma\leq\theta^*_\sigma$. Next, note that, for $\theta\in[\theta'',\theta^*_\sigma)$,

$$\begin{aligned}V_\sigma^{\Gamma_{\delta,\gamma}}(\theta) &= g-(g+|b|)\,\mathbb{P}_\sigma[\tilde{\theta}\leq\theta|\tilde{\theta}\in[\theta',\theta'']\cup[\theta^*_\sigma,\infty);x^*_\sigma(\theta)]\\ &= g-(g+|b|)\,\frac{\gamma}{x^*_\sigma(\theta)+\sigma-\theta^*_\sigma+\gamma}=g-(g+|b|)\,\frac{\gamma}{(1+2\sigma)\,\theta-\theta^*_\sigma+\gamma},\end{aligned}$$

and, therefore, $V_\sigma^{\Gamma_{\delta,\gamma}}(\cdot)$ is increasing over the range $[\theta'',\theta^*_\sigma)$. Finally, for $\theta\in[\theta',\theta'')$, we have that

$$\begin{aligned}V_\sigma^{\Gamma_{\delta,\gamma}}(\theta) &= g-(g+|b|)\,\mathbb{P}_\sigma[\tilde{\theta}\leq\theta|\tilde{\theta}\in[\theta',\theta'']\cup[\theta^*_\sigma,\infty);x^*_\sigma(\theta)]\\ &= g-(g+|b|)\,\frac{\theta-\theta'}{x^*_\sigma(\theta)+\sigma-\theta^*_\sigma+\gamma}=g-(g+|b|)\,\frac{\theta-\theta'}{(1+2\sigma)\,\theta-\theta^*_\sigma+\gamma}.\end{aligned}$$

Hence $V_\sigma^{\Gamma_{\delta,\gamma}}(\cdot)$ is decreasing over $[\theta',\theta'')$ if $(1+2\sigma)\,\theta'=x^*_\sigma(\theta')+\sigma>\theta^*_\sigma$. Using the fact that $\theta'=\theta''-\gamma$, together with the fact that $\theta''=x^*_\sigma\left(\theta^{MS}-\delta\right)-\sigma$ and $\theta^*_\sigma=(1+2\sigma)\,\theta^{MS}-2\sigma$, we have that $(1+2\sigma)\,\theta' > \theta^*_\sigma$ if

$$\gamma \;\; < \;\; 2\sigma\left[(1+2\sigma)\,\theta^{MS}-2\sigma\right]/\,(1+2\sigma)-(1+2\sigma)\,\delta \equiv R_2\left(\delta,\theta^{MS},\sigma\right).$$

Lastly, observe that, for any $\theta \in [\theta'/(1+2\sigma),\theta']$, $V_\sigma^{\Gamma_{\delta,\gamma}}(\theta)=g$.

We thus have that the function $V_\sigma^{\Gamma_{\delta,\gamma}}$ is such that (1) $V_\sigma^{\Gamma_{\delta,\gamma}}(\theta)\geq 0$ for all $\theta \geq \theta'/(1+2\sigma)$, and (2) $V_\sigma^{\Gamma_{\delta,\gamma}}(\theta)=0$ only if $\theta=\theta^{MS}$, if and only if the following conditions hold: (a) $V_\sigma^{\Gamma_{\delta,\gamma}}\left(\theta^\sharp\right)>0$, and (b) $V_\sigma^{\Gamma_{\delta,\gamma}}(\theta'')>0$. Requiring that $V_\sigma^{\Gamma_{\delta,\gamma}}\left(\theta^\sharp\right)>0$ is equivalent to

$$\begin{aligned} g-(g+|b|)\left(\theta^\sharp-\theta^*_\sigma+\gamma\right)/\left(x^*_\sigma\left(\theta^\sharp\right)+\sigma-\theta^*_\sigma+\gamma\right) \;&>\; 0 \\ \Leftrightarrow \theta^{MS}\left(\theta^*_\sigma-\gamma\right)-\left((1+2\sigma)\,\theta^{MS}-2\sigma\right)\theta^\sharp \;&>\; 0. \end{aligned}$$

Recall that $\theta^*_\sigma=(1+2\sigma)\,\theta^{MS}-2\sigma$. Using the fact that $\theta^\sharp=\theta^{MS}-\delta-\frac{\gamma}{1+2\sigma}$, we conclude that a sufficient condition for $V_\sigma^{\Gamma_{\delta,\gamma}}\left(\theta^\sharp\right)>0$ is that

$$\begin{aligned} \left(\theta^{MS}-\delta-\gamma/\,(1+2\sigma)\right)\theta^*_\sigma \;&<\; \theta^{MS}\left(\theta^*_\sigma-\gamma\right) \\ \Leftrightarrow \gamma \;&<\; \delta\,(1+2\sigma)\left((1+2\sigma)\,\theta^{MS}-2\sigma\right)/\,(2\sigma)\equiv R_3\left(\delta,\theta^{MS},\sigma\right). \end{aligned}$$

Next, observe that $V_\sigma^{\Gamma_{\delta,\gamma}}(\theta'')>0$ is equivalent to

$$\begin{aligned} \gamma \;&<\; \left(1-\theta^{MS}\right)\left((1+2\sigma)\,\theta''-\theta^*_\sigma+\gamma\right) \\ \Leftrightarrow \gamma \;&<\; \left(\frac{1-\theta^{MS}}{\theta^{MS}}\right)\left((1+2\sigma)\left[(1+2\sigma)\left(\theta^{MS}-\delta\right)-2\sigma\right]-(1+2\sigma)\,\theta^{MS}+2\sigma\right)\equiv R_4\left(\delta,\theta^{MS},\sigma\right). \end{aligned}$$

We conclude that, for any $\sigma\in(0,\sigma^\#)$, (i) $0\leq {\theta'}_\sigma(\delta,\gamma)<\theta''_\sigma(\delta,\gamma)<\theta^*_\sigma$, and (ii) $V_\sigma^{\Gamma_{\delta,\gamma}}(\theta)\geq 0$ for all $\theta>{\theta'}_\sigma(\delta,\gamma)/(1+2\sigma)$, with $V_\sigma^{\Gamma_{\delta,\gamma}}(\theta)=0$ only for $\theta=\theta^{MS}$, if

$$0<\delta<\min\left\{\theta^{MS}-\frac{2\sigma}{1+2\sigma},2\sigma\left(1-\theta^{MS}\right),\frac{2\sigma\left[(1+2\sigma)\,\theta^{MS}-2\sigma\right]}{(1+2\sigma)^2},\frac{2\sigma}{1+2\sigma}\left[\theta^{MS}-\frac{2\sigma}{1+2\sigma}\right]\right\}\equiv\varsigma\left(\theta^{MS},\sigma\right)$$

and $0<\gamma<\min_{0\leq i\leq 4}R_i\left(\delta,\theta^{MS},\sigma\right)$. Note that $\sigma<\sigma^\#$ implies that $\varsigma\left(\theta^{MS},\sigma\right)>0$, whereas $\delta<\varsigma\left(\theta^{MS},\sigma\right)$ implies that $\min_{0\leq i\leq 4}R_i\left(\delta,\theta^{MS},\sigma\right)>0$. Finally note that, for any $\sigma\in(0,\sigma^\#)$, and any $\theta\geq\theta'_\sigma(\delta,\gamma)$, the payoff $V_\sigma^{\Gamma_{\delta,\gamma}}(\theta)$ is continuous in the threshold θ^*_σ. Hence there exists a policy Γ whose rule π is given by $\pi\,(1|\theta)\equiv\mathbf{1}\left\{\theta\in[{\theta'}_\sigma(\delta,\gamma),\theta''_\sigma(\delta,\gamma)]\cup[\theta^*_\sigma+\varepsilon,\infty)\right\}$ with $\varepsilon>0$ arbitrarily small, such that Γ strictly improves over $\Gamma^{\theta^*_\sigma}$ and is such that $V_\sigma^{\Gamma}(\theta)>0$ for all $\theta>{\theta'}_\sigma(\delta,\gamma)/(1+2\sigma)$, implying that Γ satisfies PCP. Q.E.D.

S2. Proof of Example 3 in Main Text

Preliminaries. For any $\theta \in (0,1)$, any $\sigma \in \mathbb{R}_+$, let $x^*_\sigma(\theta)$ be the critical signal threshold such that, when all agents invest for $x > x^*_\sigma(\theta)$ and refrain from investing for $x < x^*_\sigma(\theta)$, default occurs if and only if the fundamentals are below θ. Note that, in this example $x^*_\sigma(\theta) \equiv \theta + \sigma\Phi^{-1}(\theta)$, where Φ is the cdf of the standard Normal distribution and ϕ its density. Also let $x^*_\sigma(0) \equiv -\infty$ and $x^*_\sigma(1) \equiv +\infty$. For any $(\theta_0, \hat{\theta}, \sigma) \in (0,1) \times \mathbb{R} \times \mathbb{R}_+$, let $\psi(\theta_0, \hat{\theta}, \sigma)$ denote the payoff from investing of an investor with private signal $x^*_\sigma(\theta_0)$, when default occurs if and only if $\theta \leq \theta_0$, the policy reveals that $\theta \geq \hat{\theta}$, and the precision of private information is σ^{-2}. Then let $\hat{\sigma} \equiv \inf\{\sigma \in \mathbb{R}_+ : \ \psi(\theta_0,0,\sigma) > 0 \ \text{all} \ \theta_0 \in (0,1)\}$ if $\{\sigma \in \mathbb{R}_+ : \ \psi(\theta_0,0,\sigma) > 0 \ \text{all} \ \theta_0 \in (0,1)\} \neq \emptyset$ and else $\hat{\sigma} = +\infty$.[4] Then let $\Psi(\sigma) \equiv \inf_{\theta_0 \in (0,1)} \psi(\theta_0,0,\sigma)$ and note that $\lim_{\sigma \to 0^+} \ \Psi(\sigma) < 0$, implying that $\hat{\sigma} > 0$. For any $\sigma \in \mathbb{R}_+$ for which $\psi(\theta_0,0,\sigma) > 0$ for all $\theta_0 \in (0,1)$, the policy maker can avoid default for every $\theta > 0$ by using the monotone rule $\pi(\theta) = \mathbf{1}\{\theta > 0\}$. This case is uninteresting. Hereafter, we thus confine attention to the case in which $\sigma < \hat{\sigma}$.

Let $U^\Gamma_\sigma(x, 1|x)$ denote the payoff from investing of an agent with signal x who expects all other agents to invest if and only if their signal exceeds x , when the precision of private information is σ^{-2}, and the policy Γ announces that $s = 1$. Also let

$$U^\Gamma_\sigma\left(x^*_\sigma(0), 1|x^*_\sigma(0)\right) \equiv \lim_{x \to -\infty} U^\Gamma_\sigma(x, 1|x)$$

and

$$U^\Gamma_\sigma\left(x^*_\sigma(1), 1|x^*_\sigma(1)\right) \equiv \lim_{x \to +\infty} U^\Gamma_\sigma(x, 1|x).$$

Now let $\mathbb{G}_\sigma$ denote the set of deterministic binary policies $\Gamma = (\{0,1\}, \pi)$ such that, $\pi(\theta) = 0$ for all $\theta \leq 0$, $\pi(\theta) = 1$ for all $\theta > 1$ and $U^\Gamma_\sigma(x, 1|x) \geq 0$ for all $x \in \mathbb{R}$.[5] From the proofs of Theorems 1 and 2, observe that, given any σ, any deterministic binary policy Γ satisfying PCP and such that $\pi(\theta) = 0$ for all $\theta \leq 0$ and $\pi(\theta) = 1$ for all $\theta > 1$ belongs in $\mathbb{G}_\sigma$. However, $\mathbb{G}_\sigma$ contains also policies that do not satisfy PCP.[6]

[4] Recall that, when the announcement that $s = 1$ reveals that $\theta \geq 0$, the unique rationalizable profile features all agents investing, irrespective of x, if and only if $\psi(\theta_0,0,\sigma) > 0$ for all $\theta_0 \in (0,1)$.

[5] We let $\pi(\theta) = 1$ (alternatively, $\pi(\theta) = 0$) denote the degenerate lottery assigning measure 1 to $s = 1$ (alternatively, $s = 0$).

[6] These are those for which there exists x such that $U^\Gamma_\sigma(x, 1|x) = 0$; when Γ announces $s = 1$, in addition to

Proof Structure. The proof is in four steps. Step 1 establishes that, when σ is small, under any policy $\Gamma = (\{0,1\}, \pi) \in \mathbb{G}_\sigma$, any interval $(\theta', \theta''] \subset \left(0, \theta^{MS}\right]$ receiving a pass grade (i.e., such that $\pi(\theta) = 1$ for all $\theta \in (\theta', \theta''])$ has a sufficiently small Lebesgue measure, with the measure vanishing as $\sigma \to 0^+$.

Step 2 then considers an *auxiliary game* G_σ in which the agents play less aggressively than under MARP. Namely, G_σ is the game in which (i) the policy maker's choice set is $\mathbb{G}_\sigma$ and (ii) given *any* policy $\Gamma \in \mathbb{G}_\sigma$, all agents invest after receiving the signal $s = 1$ and refrain from investing after receiving the signal $s = 0$.[7] We show that, when σ is small, given any policy $\Gamma \in \mathbb{G}_\sigma$ that gives a fail grade to an interval $(\theta', \theta''] \subseteq (\underline{\theta}, \theta^{MS}]$ of large Lebesgue measure, there exists another policy $\Gamma^\# \in \mathbb{G}_\sigma$ that gives a pass grade to a F-positive measure subset of $(\theta', \theta'']$, has a mesh smaller than Γ, and is such that, when agents play as in G_σ, the probability of default under $\Gamma^\#$ is strictly smaller than under Γ.

Step 3 then combines the results from Steps 1 and 2 to show that, when σ is small, given any policy $\Gamma \in \mathbb{G}_\sigma$ for which the mesh $M(\Gamma)$ of $(0, \theta^{MS}]$ is larger than ε, there exists another policy $\Gamma' \in \mathbb{G}_\sigma$ with a mesh $M(\Gamma')$ smaller than ε such that, when agents play as in G_σ, the probability of default is strictly smaller under Γ' than under Γ. Starting from $\Gamma' \in \mathbb{G}_\sigma$ one can then construct a "nearby" policy $\Gamma^* \in \mathbb{G}_\sigma$ such that the probability of default under Γ^* is arbitrarily close to that under Γ' (and hence strictly smaller than under the original policy Γ) and such that $U_\sigma^{\Gamma^*}(x, 1|x) > 0$ for all x. The last property implies that Γ^* satisfies PCP also when agents play according to MARP. The policy Γ^* thus strictly improves upon Γ also in the original game, as claimed in the main text.

Finally, step 4 closes the proof by showing how to construct the function $\mathcal{E}$ relating the noise σ in the agents' exogenous private information to the bound $\mathcal{E}(\sigma)$ on the mesh of the policies.

Step 1. We start with the following result:

Lemma S2-A. *For any $\varepsilon \in \mathbb{R}_{++}$, there exists $\sigma(\varepsilon) \in \mathbb{R}_{++}$ such that, for any $\sigma \in (0, \sigma(\varepsilon)]$, the following is true: for any policy $\Gamma = (\{0,1\}, \pi) \in \mathbb{G}_\sigma$ and any cell $(\theta', \theta''] \in D^\Gamma$ with $|\theta'' - \theta'| > \varepsilon$, necessarily $\pi(\theta) = 0$.*

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the rationalizable profile under which all agents invest, there also exists a rationalizable profile under which each agent invests if and only if his signal exceeds x.

[7] The agent's behavior is consistent with MARP only for those $\Gamma \in \mathbb{G}_\sigma$ for which, for all x, $U_\sigma^\Gamma(x, 1|x) > 0$.

Proof of Lemma S2-A. We first show (Property S2-A below) that, for any $\sigma > 0$, if the policy maker were to replace Γ with the cutoff policy $\Gamma^{\theta'}$, then for any $\theta \leq \theta''$, $U_\sigma^{\Gamma^{\theta'}}(x_\sigma^*(\theta), 1|x_\sigma^*(\theta)) \geq U_\sigma^{\Gamma}(x_\sigma^*(\theta), 1|x_\sigma^*(\theta))$.[8] Next, we show (Property S2-B below) that, for any $\theta > \theta'$, as σ goes to zero, $U_\sigma^{\Gamma^{\theta'}}(x_\sigma^*(\theta), 1|x_\sigma^*(\theta))$ converges uniformly to $\int_0^1 u(\theta, A)\mathrm{d}A$. Because $\int_0^1 u(\theta, A)\mathrm{d}A < 0$ for $\theta < \theta^{MS}$, the above two properties imply that, for σ small, $U_\sigma^{\Gamma}(x_\sigma^*(\theta), 1|x_\sigma^*(\theta) < 0$ for some $\theta \in (\theta', \theta'']$, and hence that $\Gamma \notin \mathbb{G}_\sigma$. The result in the lemma then follows by contrapositive.

Property S2-A. *For any policy $\Gamma = (\{0,1\}, \pi) \in \mathbb{G}_\sigma$ and any cell $(\theta', \theta''] \in D^\Gamma$ such that $\pi(\theta) = 1$ for all $\theta \in (\theta', \theta'']$, $U_\sigma^{\Gamma}(x_\sigma^*(\theta), 1|x_\sigma^*(\theta)) \leq U_\sigma^{\Gamma^{\theta'}}(x_\sigma^*(\theta), 1|x_\sigma^*(\theta))$ for all $\theta \leq \theta''$.*

Proof of Property S2-A. The proof follows from Results S2-A-1 and S2-A-2 below.

Result S2-A-1. *Pick any policy $\Gamma = (\{0,1\}, \pi) \in \mathbb{G}_\sigma$. Given the partition $D^\Gamma \equiv \left\{d_i = \left(\underline{\theta}_i, \bar{\theta}_i\right] : i = 1, ..., N\right\}$ of $(0, \theta^{MS}]$ induced by Γ, take any cell $d_i = \left(\underline{\theta}_i, \bar{\theta}_i\right]$ for which $\pi(\theta) = 1$ for all $\theta \in d_i$. Let $\Gamma_L^i = \{\{0,1\}, \pi_L^i\} \in \mathbb{G}_\sigma$ be the policy constructed as follows: (a) $\pi_L^i(\theta) = 0$ for all $\theta \leq \underline{\theta}_i$; and (b) $\pi_L^i(\theta) = \pi(\theta)$ for all $\theta > \underline{\theta}_i$. Then, for all $\theta \in [0,1]$, $U_\sigma^{\Gamma_L^i}(x_\sigma^*(\theta), 1|x_\sigma^*(\theta)) \geq U_\sigma^{\Gamma}(x_\sigma^*(\theta), 1|x_\sigma^*(\theta))$.*

Proof of Result S2-A-1. Note that, under the new policy, $\pi_L^i(\theta) = \pi(\theta) \times \mathbf{1}\{\theta > \underline{\theta}_i\}$. The posterior beliefs $\Lambda_\sigma^{\Gamma_L^i}(\cdot|x, 1)$ about θ of an agent with exogenous signal x and endogenous signal $s = 1$ under the new policy Γ_L^i thus dominate, in the FOSD sense, the analogous beliefs $\Lambda_\sigma^\Gamma(\cdot|x, 1)$ under the original policy Γ.[9] The result then follows from the fact that, given any default threshold θ, the payoff differential from investing when the fundamentals are equal to $\tilde{\theta}$ and default occurs if and only if $\tilde{\theta} \leq \theta$ is nondecreasing in $\tilde{\theta}$. *End of Proof of Result S2-A-1.*

Result S2-A-2. *Pick any policy $\Gamma = \{\{0,1\}, \pi\} \in \mathbb{G}_\sigma$. Given the partition $D^\Gamma \equiv \left\{d_i = \left(\underline{\theta}_i, \bar{\theta}_i\right] : i = 1, ..., N\right\}$ of $(0, \theta^{MS}]$* induced by Γ, *take any cell $d_i = \left(\underline{\theta}_i, \bar{\theta}_i\right]$, $i \geq 2$, for which $\pi(\theta) = 1$ for all $\theta \in d_i$. Let $\Gamma_R^i = \{\{0,1\}, \pi_R^i\} \in \mathbb{G}_\sigma$ be the policy constructed from Γ as follows: (a) $\pi_R^i(\theta) = \pi(\theta)$ for all $\theta \leq \underline{\theta}_i$; and (b) $\pi_R^i(\theta) = 1$ for all $\theta > \underline{\theta}_i$. Then, for all $\theta \leq \bar{\theta}_i$, $U_\sigma^{\Gamma_R^i}(x_\sigma^*(\theta), 1|x_\sigma^*(\theta)) \geq U_\sigma^{\Gamma}(x_\sigma^*(\theta), 1|x_\sigma^*(\theta))$.*

Proof of Result S2-A-2. Let $\Theta^1 \equiv \{\theta \in \Theta : \pi(\theta) = 1\}$ and $\Theta_i^0 \equiv \{\theta \in (\underline{\theta}_i, 1] : \pi(\theta) = 0\}$.

[8] For any $\hat{\theta} \in [0,1]$, $\Gamma^{\hat{\theta}} = (\{0,1\}, \pi^{\hat{\theta}})$ is the deterministic monotone policy with cut-off $\hat{\theta}$.

[9] No matter the shape of the beliefs $\Lambda_\sigma^\Gamma(\cdot|x, 1)$, the announcement that $\theta > \underline{\theta}_i$ is always "good news" in the sense of Milgrom (1981) and hence $\Lambda_\sigma^{\Gamma_L^i}(\cdot|x, 1) \succ_{FOSD} \Lambda_\sigma^\Gamma(\cdot|x, 1)$.

For any $\theta^{\#} \leq \bar{\theta}_i$, and any x,

$$\begin{aligned}
\Lambda_\sigma^{\Gamma_R^i}\left(\theta^{\#}|x,1\right) &= \mathbb{P}[\theta \leq \theta^{\#}|x, \theta \in (\Theta^1 \cup \Theta_i^0)] = \frac{\mathbb{P}[\theta \leq \theta^{\#} \wedge \theta \in (\Theta^1 \cup \Theta_i^0)|x]}{\mathbb{P}[\theta \in (\Theta^1 \cup \Theta_i^0)|x]} \\
&= \frac{\mathbb{P}[\theta \leq \theta^{\#} \wedge \theta \in \Theta^1|x]}{\mathbb{P}[\theta \in (\Theta^1 \cup \Theta_i^0)|x]} + \frac{\mathbb{P}[\theta \leq \theta^{\#} \wedge \theta \in \Theta_i^0|x]}{\mathbb{P}[\theta \in (\Theta^1 \cup \Theta_i^0)|x]} = \frac{\mathbb{P}[\theta \leq \theta^{\#} \wedge \theta \in \Theta^1|x]}{\mathbb{P}[\theta \in (\Theta^1 \cup \Theta_i^0)|x]} \\
&\leq \mathbb{P}[\theta \leq \theta^{\#}|x, \theta \in \Theta^1] = \Lambda_\sigma^\Gamma(\theta^{\#}|x,1).
\end{aligned}$$

The first equality follows from the fact that, under the new policy Γ_R^i, the signal $s = 1$ carries the same information as the announcement that $\theta \in (\Theta^1 \cup \Theta_i^0)$. The third equality follows from the fact that $\Theta^1 \cap \Theta_i^0 = \emptyset$. The fourth equality follows from the fact that Θ_i^0 contains only fundamentals above $\bar{\theta}_i$ and that $\theta^{\#} \leq \bar{\theta}_i$. The inequality follows from the fact that $\mathbb{P}\left[\theta \in (\Theta^1 \cup \Theta_i^0)\,|x\right] \geq \mathbb{P}\left[\theta \in \Theta^1|x\right]$ along with the definition of conditional probability. The last equality follows from the fact that, under the original policy Γ, the signal $s = 1$ carries the same information as the announcement that $\theta \in \Theta^1$. Given the above inequality, and the fact that $b < 0 < g$, we then have that, for any $\theta \leq \bar{\theta}_i$,

$$\begin{aligned}
U_\sigma^\Gamma(x_\sigma^*(\theta), 1|x_\sigma^*(\theta)) &= b \cdot \Lambda_\sigma^\Gamma\left(\theta|x_\sigma^*(\theta), 1\right) + g \cdot \left(1 - \Lambda_\sigma^\Gamma\left(\theta|x_\sigma^*(\theta), 1\right)\right) \\
&\leq b \cdot \Lambda_\sigma^{\Gamma_R^i}\left(\theta|x_\sigma^*(\theta), 1\right) + g \cdot \left(1 - \Lambda_\sigma^{\Gamma_R^i}\left(\theta|x_\sigma^*(\theta), 1\right)\right) = U_\sigma^{\Gamma_R^i}(x_\sigma^*(\theta), 1|x_\sigma^*(\theta)).
\end{aligned}$$

End of Proof of Result S2-A-2.

Property S2-A follows from Results S2-A-1 and S2-A-2, by taking the cell $d_i = (\theta', \theta'']$. □

Now, fix $\varepsilon \in (0, \theta^{MS})$. For any $\theta^* \in [0, \theta^{MS} - \varepsilon]$, let Γ^{θ^*} be the monotone rule with cut-off θ^*. For any $\theta^* \in [0, \theta^{MS} - \varepsilon]$, any $\sigma \in \mathbb{R}_{++}$, let

$$H_\sigma(\theta^*; \varepsilon) \equiv \inf_{\theta \in [\theta^*, \theta^* + \varepsilon]} U_\sigma^{\Gamma^{\theta^*}}(x_\sigma^*(\theta), 1|x_\sigma^*(\theta)).$$

Note that $U_\sigma^{\Gamma^{\theta^*}}(x_\sigma^*(\theta), 1|x_\sigma^*(\theta))$ is continuous in $(\theta^*, \theta, \sigma)$ over $[0,1]^2 \times (0, \hat{\sigma}]$. From Berge's Maximum Theorem, $H_\sigma(\theta^*; \varepsilon)$ is thus continuous in (θ^*, σ) over $[0, \theta^{MS} - \varepsilon] \times (0, \hat{\sigma}]$.

For all $\theta^* \in [0, \theta^{MS} - \varepsilon]$, all $\theta \in (\theta^*, \theta^* + \varepsilon]$, $\lim_{\sigma \to 0^+} U_\sigma^{\Gamma^{\theta^*}}(x_\sigma^*(\theta), 1|x_\sigma^*(\theta)) = \int_0^1 u(\theta, A)\mathrm{d}A$. Because $\int_0^1 u(\theta, A)\mathrm{d}A$ is strictly increasing in θ and equal to zero at $\theta = \theta^{MS}$, for any $\theta^* \in [0, \theta^{MS} - \varepsilon]$, $H_{0^+}(\theta^*; \varepsilon) \equiv \lim_{\sigma \to 0^+} H_\sigma(\theta^*; \varepsilon) = \lim_{\sigma \to 0^+} \lim_{\theta \to \theta^{*+}} U_\sigma^{\Gamma^{\theta^*}}(x_\sigma^*(\theta), 1|x_\sigma^*(\theta)) = \int_0^1 u(\theta^*, A)\mathrm{d}A$. We show next that $H_\sigma(\cdot; \varepsilon)$ converges uniformly to the limit function $H_{0^+}(\cdot; \varepsilon)$ over $[0, \theta^{MS} - \varepsilon]$.

Property S2-B. *Fix $\varepsilon \in (0, \theta^{MS})$. For any $\epsilon < \varepsilon$, there exists $\sigma'(\epsilon) > 0$ such that, for* *any $\sigma \leq \sigma'(\epsilon)$, and any $\theta^* \in [0, \theta^{MS} - \varepsilon]$, $|H_\sigma(\theta^*; \varepsilon) - H_{0^+}(\theta^*; \varepsilon)| < \epsilon$.*

Proof of Property S2-B. The limit function $H_{0^+}(\cdot; \varepsilon)$ is *uniformly* continuous over $[0, \theta^{MS} - \varepsilon]$. As a consequence, there exists $\delta > 0$ such that for any $\theta, \tilde{\theta} \in [0, \theta^{MS} - \varepsilon]$, with $|\tilde{\theta} - \theta| \leq \delta$, necessarily $|H_{0^+}(\tilde{\theta}; \varepsilon) - H_{0^+}(\theta; \varepsilon)| < \epsilon/2$. Next, let $D_\delta \equiv \{(\underline{\theta}_i, \bar{\theta}_i] : i = 1, ..., N\}$, $N \in \mathbb{N}$, be any interval partition of $(0, \theta^{MS} - \varepsilon]$ with the property that every cell $(\underline{\theta}_i, \bar{\theta}_i] \in D_\delta$ is such that $|\bar{\theta}_i - \underline{\theta}_i| \leq \delta$. For any $i = 1, ..., N$, any $\sigma > 0$, let $\hat{\theta}^i_\sigma \equiv \sup\{\arg\max_{\theta \in [\underline{\theta}_i, \bar{\theta}_i]} H_\sigma(\theta; \varepsilon)\}$. That $H_\sigma(\theta; \varepsilon)$ is continuous in (σ, θ) implies that the hypothesis of Berge's Maximum Theorem hold and, hence, the correspondence $\arg\max_{\theta \in [\underline{\theta}_i, \bar{\theta}_i]} H_\sigma(\theta; \varepsilon)$ is compact-valued and upper semi-continuous in σ. As a result, for any $\sigma > 0$, $\hat{\theta}^i_\sigma = \max\{\arg\max_{\theta \in [\underline{\theta}_i, \bar{\theta}_i]} H_\sigma(\theta; \varepsilon)\}$. Moreover, $\lim_{\sigma \to 0^+} H_\sigma(\hat{\theta}^i_\sigma; \varepsilon) = H_{0^+}(\hat{\theta}^i_{0^+}; \varepsilon)$, where $\hat{\theta}^i_{0^+} \equiv \lim_{\sigma \to 0^+} \hat{\theta}^i_\sigma$.

For any $\theta^* \in [0, \theta^{MS} - \varepsilon]$, let $(\underline{\theta}_j, \bar{\theta}_j] \in D_\delta$ be the partition cell containing θ^*. Then,

$$\begin{gathered} H_\sigma(\theta^*; \varepsilon) - H_{0^+}(\theta^*; \varepsilon) \leq H_\sigma(\hat{\theta}^j_\sigma; \varepsilon) - H_{0^+}(\theta^*; \varepsilon) \\ = H_\sigma(\hat{\theta}^j_\sigma; \varepsilon) - H_{0^+}(\hat{\theta}^j_{0^+}; \varepsilon) + H_{0^+}(\hat{\theta}^j_{0^+}; \varepsilon) - H_{0^+}(\theta^*; \varepsilon) < H_\sigma(\hat{\theta}^j_\sigma; \varepsilon) - H_{0^+}(\hat{\theta}^j_{0^+}; \varepsilon) + \epsilon/2 < \epsilon \end{gathered}$$

for all $\sigma < \bar{\sigma}_j(\epsilon)$, for some $\bar{\sigma}_j(\epsilon) > 0$. The first inequality is by definition of $\hat{\theta}^i_\sigma$. The second inequality follows from the fact that $|\hat{\theta}^j_{0^+} - \theta^*| < \delta$. The last inequality follows from the fact that $\lim_{\sigma \to 0^+} H_\sigma(\hat{\theta}^j_\sigma) = H_{0^+}(\hat{\theta}^j_{0^+})$. Similar arguments imply that $H_\sigma(\theta^*; \varepsilon) - H_{0^+}(\theta^*; \varepsilon) > -\epsilon$ for all $\sigma < \underline{\sigma}_j(\epsilon)$, for some $\underline{\sigma}_j(\epsilon) > 0$.

Now let $\sigma'(\epsilon) \equiv \min\{\min_{i \in N}\{\bar{\sigma}_i(\epsilon)\}, \min_{i \in N}\{\underline{\sigma}_i(\epsilon)\}\}$. For any $\sigma \leq \sigma'(\epsilon)$, and any $\theta^* \in [0, \theta^{MS} - \varepsilon]$, we thus have that $|H_\sigma(\theta^*; \varepsilon) - H_{0^+}(\theta^*; \varepsilon)| < \epsilon$, thus proving that $H_\sigma(\cdot; \varepsilon)$ converges uniformly to $H_{0^+}(\cdot; \varepsilon)$ as $\sigma \to 0^+$. This completes the proof of Property S2-B. $\square$

Next, given $\varepsilon \in (0, \theta^{MS})$, pick an arbitrary $\eta \in (\int_0^1 u(\theta^{MS} - \varepsilon, A)\mathrm{d}A, 0)$. Because $H_{0^+}(\theta^*; \varepsilon) \leq \eta$ for all $\theta^* \in [0, \theta^{MS} - \varepsilon]$, and because $H_\sigma(\cdot; \varepsilon)$ converges uniformly to $H_{0^+}(\cdot; \varepsilon)$, there exists $\sigma(\varepsilon) > 0$ such that, for any $\sigma < (\varepsilon)$, and any $\theta^* \in [0, \theta^{MS} - \varepsilon]$, $H_\sigma(\theta^*; \varepsilon) \leq \eta < 0$. Therefore, for any $\sigma < \sigma(\varepsilon)$, and any deterministic monotone policy Γ^{θ^*} with cut-off $\theta^* \in [0, \theta^{MS} - \varepsilon]$, there exists $\theta \in [\theta^*, \theta^* + \varepsilon]$ such that $U^{\Gamma^{\theta^*}}_\sigma(x^*_\sigma(\theta), 1 | x^*_\sigma(\theta)) \leq \eta$.

Together, Properties S2-A and S2-B then imply that, for any $\sigma < \sigma(\varepsilon)$, and any policy Γ such that $\pi(\theta) = 1$ for all $\theta \in (\theta', \theta'']$ for some $(\theta', \theta''] \in D^\Gamma$ with $|\theta'' - \theta'| > \varepsilon$, necessarily $U^\Gamma_\sigma(x^*_\sigma(\theta), 1 | x^*_\sigma(\theta)) < 0$ for some $\theta \in (\theta', \theta'']$. Hence $\Gamma \notin \mathbb{G}_\sigma$. The claim in Lemma S2-A then follows by contrapositive. This completes the proof of Lemma S2-A. $\blacksquare$

Step 2. Next, we show that, for any policy $\Gamma = (\{0,1\},\pi) \in \mathbb{G}_\sigma$ that gives a fail grade to an interval $(\theta', \theta''] \subseteq (0, \theta^{MS}]$ of large Lebesgue measure, there exists another policy $\Gamma^{\#} \in \mathbb{G}_\sigma$ with a mesh $M(\Gamma^{\#}) < M(\Gamma)$ such that, when agents play as in G_σ, the probability of default under $\Gamma^{\#}$ is strictly smaller than under Γ. The result follows from Lemmas S2-B, S2-C and S2-D below.

Lemma S2-B. *For any* $\Gamma = (\{0,1\},\pi) \in \mathbb{G}_\sigma$ *such that* $\inf_{\theta\in[0,1]}\ U_\sigma^\Gamma\left(x_\sigma^*\left(\theta\right), 1|x_\sigma^*\left(\theta\right)\right) > 0$, *there exists another policy* $\hat{\Gamma} = (\{0,1\},\hat{\pi}) \in \mathbb{G}_\sigma$, *with* $M(\hat{\Gamma}) \leq M(\Gamma)$, *such that, in the auxiliary game* G_σ, *the probability of default under* $\hat{\Gamma}$ *is strictly smaller than under* Γ.

Proof of Lemma S2-B. That $\inf_{\theta\in[0,1]}\ U_\sigma^\Gamma\left(x_\sigma^*\left(\theta\right), 1|x_\sigma^*\left(\theta\right)\right) > 0$ implies that, starting from $\Gamma = (\{0,1\},\pi)$, one can construct another policy $\hat{\Gamma} = (\{0,1\},\hat{\pi})$ sufficiently close to Γ (in the L_1 norm) and such that $\hat{\pi}(\theta) \geq \pi(\theta)$ for all θ, with the inequality strict over some positive F-measure set $(\tilde{\theta}', \tilde{\theta}'') \subseteq (0,1]$, and such that (a) $\hat{\pi}(\theta) = 0$ for all $\theta \leq 0$, (b) $\hat{\pi}(\theta) = 1$ for all $\theta > 1$, (c) $U_\sigma^{\hat{\Gamma}}(x, 1|x) \geq 0$ all x, and (d) $M(\hat{\Gamma}) \leq M(\Gamma)$. By definition of $\mathbb{G}_\sigma$, $\hat{\Gamma} \in \mathbb{G}_\sigma$. That, in the auxiliary game G_σ, the probability of default under $\hat{\Gamma}$ is strictly smaller than under Γ, then follows from the fact that all agents invest when they receive the signal $s = 1$. This completes the proof of Lemma S2-B. ■

For any $\sigma > 0$, and any policy $\Gamma = (\{0,1\},\pi) \in \mathbb{G}_\sigma$, $U_\sigma^\Gamma\left(x_\sigma^*\left(\cdot\right), 1|x_\sigma^*\left(\cdot\right)\right)$ is continuous over $[0,1]$. Hence $\inf_{\theta\in[0,1]}\ U_\sigma^\Gamma\left(x_\sigma^*\left(\theta\right), 1|x_\sigma^*\left(\theta\right)\right) = \min_{\theta\in[0,1]}\ U_\sigma^\Gamma\left(x_\sigma^*\left(\theta\right), 1|x_\sigma^*\left(\theta\right)\right)$.

Lemma S2-C. *Let* $\Gamma = (\{0,1\},\pi) \in \mathbb{G}_\sigma$ *be such that* $min_{\theta\in[0,1]}\ U_\sigma^\Gamma\left(x_\sigma^*\left(\theta\right), 1|x_\sigma^*\left(\theta\right)\right) = 0$. *For any* $\theta_\sigma^\sharp \in \arg min_{\theta\in[0,1]}\ U_\sigma^\Gamma\left(x_\sigma^*\left(\theta\right), 1|x_\sigma^*\left(\theta\right)\right)$, *there exists* $\gamma_\sigma^\Gamma > 0$ *such that* $\pi\left(\theta\right) = 1$ *for* $F-$*almost all* $\theta \in \left(\theta_\sigma^\sharp - \gamma_\sigma^\Gamma, \theta_\sigma^\sharp\right)$.

Proof of Lemma S2-C. The proof is by contraposition. Suppose there exists $\delta > 0$ such that $\pi\left(\theta\right) = 0$ for F-almost all $\theta \in \left(\theta_\sigma^\sharp - \delta, \theta_\sigma^\sharp\right)$. Observe that the sign of

$$U_\sigma^\Gamma\left(x_\sigma^*\left(\theta_\sigma^\sharp - \delta\right), 1|x_\sigma^*\left(\theta_\sigma^\sharp - \delta\right)\right)$$

is the same as the sign of

$$b\int_{-\infty}^{\theta_\sigma^\sharp - \delta} \phi\left(\left(x_\sigma^*\left(\theta_\sigma^\sharp - \delta\right) - \theta\right)/\sigma\right)\pi\left(\theta\right)\mathrm{d}F(\theta) + g\int_{\theta_\sigma^\sharp - \delta}^{+\infty} \phi\left(\left(x_\sigma^*\left(\theta_\sigma^\sharp - \delta\right) - \theta\right)/\sigma\right)\pi\left(\theta\right)\mathrm{d}F(\theta).$$

Next observe that

$$
\begin{aligned}
0 &= U^{\Gamma}_{\sigma}\left(x^*_{\sigma}\left(\theta^{\sharp}_{\sigma}\right),1|x^*_{\sigma}\left(\theta^{\sharp}_{\sigma}\right)\right)\int_{-\infty}^{+\infty}\phi\left(\left(x^*_{\sigma}\left(\theta^{\sharp}_{\sigma}\right)-\theta\right)/\sigma\right)\pi\left(\theta\right)\mathrm{d}F(\theta) \\
&= \int_{-\infty}^{\infty}\left(b\mathbf{1}\left\{\theta\le\theta^{\sharp}_{\sigma}\right\}+g\mathbf{1}\left\{\theta>\theta^{\sharp}_{\sigma}\right\}\right)\phi\left(\left(x^*_{\sigma}\left(\theta^{\sharp}_{\sigma}\right)-\theta\right)/\sigma\right)\pi\left(\theta\right)\mathrm{d}F(\theta) \\
&> \int_{-\infty}^{\infty}\left(b\mathbf{1}\left\{\theta\le\theta^{\sharp}_{\sigma}\right\}+g\mathbf{1}\left\{\theta>\theta^{\sharp}_{\sigma}\right\}\right)\phi\left(\left(x^*_{\sigma}\left(\theta^{\sharp}_{\sigma}-\delta\right)-\theta\right)/\sigma\right)\pi\left(\theta\right)\mathrm{d}F(\theta) \\
&= \int_{-\infty}^{\infty}\left(b\mathbf{1}\left\{\theta\le\theta^{\sharp}_{\sigma}-\delta\right\}+g\mathbf{1}\left\{\theta>\theta^{\sharp}_{\sigma}-\delta\right\}\right)\phi\left(\left(x^*_{\sigma}\left(\theta^{\sharp}_{\sigma}-\delta\right)-\theta\right)/\sigma\right)\pi\left(\theta\right)\mathrm{d}F(\theta) \\
&= U^{\Gamma}_{\sigma}\left(x^*_{\sigma}\left(\theta^{\sharp}_{\sigma}-\delta\right),1|x^*_{\sigma}\left(\theta^{\sharp}_{\sigma}-\delta\right)\right)\int_{-\infty}^{+\infty}\phi\left(\left(x^*_{\sigma}\left(\theta^{\sharp}_{\sigma}-\delta\right)-\theta\right)/\sigma\right)\pi\left(\theta\right)\mathrm{d}F(\theta)
\end{aligned}
$$

The first equality follows from the assumptions of the lemma. The second equality follows from the definition of the function $U^{\Gamma}_{\sigma}\left(x^*_{\sigma}\left(\theta^{\sharp}_{\sigma}\right),1|x^*_{\sigma}\left(\theta^{\sharp}_{\sigma}\right)\right)$. The inequality follows from the monotonicity of $x^*_{\sigma}\left(\cdot\right)$, the fact that $\phi\left((x-\theta)/\sigma\right)$ is log-supermodular in (x,θ), and Property SCB in the proof of Theorem 2 in the main text. The third equality follows from the fact that $\pi\left(\theta\right)=0$ for F-almost all $\theta\in\left(\theta^{\sharp}_{\sigma}-\delta,\theta^{\sharp}_{\sigma}\right)$. The last equality follows from the definition of the function $U^{\Gamma}_{\sigma}(x^*_{\sigma}(\theta^{\sharp}_{\sigma}-\delta),1|x^*_{\sigma}(\theta^{\sharp}_{\sigma}-\delta))$. Hence, $U^{\Gamma}_{\sigma}(x^*_{\sigma}(\theta^{\sharp}_{\sigma}-\delta),1|x^*_{\sigma}(\theta^{\sharp}_{\sigma}-\delta))<0$, thus contradicting the assumption that $\Gamma\in\mathbb{G}_{\sigma}$. This completes the proof of Lemma S2-C. ■

Lemma S2-D. *For any $\varepsilon>0$, there exists $\sigma^{\#}(\varepsilon)\in(0,\hat{\sigma})$ such that, for any $\sigma\in(0,\sigma^{\#}(\varepsilon)]$, and any policy $\Gamma=(\{0,1\},\pi)\in\mathbb{G}_{\sigma}$ for which there exists $(\theta',\theta'']\in D^{\Gamma}$ such that (a) $|\theta''-\theta'|>\varepsilon$ and (b) $\pi(\theta)=0$ for all $\theta\in(\theta',\theta'']$, there exists another policy $\Gamma^{\#}=\left(\{0,1\},\pi^{\#}\right)\in\mathbb{G}_{\sigma}$, with $M(\Gamma^{\#})\le M(\Gamma)$, such that, in the auxiliary game G_{σ}, the probability of default under $\Gamma^{\#}$ is strictly smaller than under Γ.*

Proof of Lemma S2-D. For any $\theta\in(0,1)$, $\lim_{\sigma\to 0^+}x^*_{\sigma}\left(\theta\right)\equiv x^*_{0^+}\left(\theta\right)=\theta$. Furthermore, for any $\varepsilon\in(0,\min\{\theta^{MS},1-\theta^{MS}\})$, the function $x^*_{0^+}:\left[\frac{\varepsilon}{4},1-\frac{\varepsilon}{4}\right]\to\mathbb{R}$ is uniformly continuous. Hence, for any $\delta<\varepsilon/4$, there exists $\tilde{\sigma}\left(\delta\right)>0$ such that, for any $\sigma\in(0,\tilde{\sigma}(\delta)]$, and any $\theta\in\left[\frac{\varepsilon}{4},1-\frac{\varepsilon}{4}\right]$, we have that $|x^*_{\sigma}\left(\theta\right)-\theta|\le\delta$.[10] In turn, this implies that, for any $\varepsilon>0$ small, there exists $\sigma^{\#}(\varepsilon)\in(0,\hat{\sigma}]$ such that, for any $\sigma\in(0,\sigma^{\#}(\varepsilon)]$, and any $(\theta',\theta'']\in D^{\Gamma}$ such that $|\theta''-\theta'|>\varepsilon$, we have that, for any $\theta\in[\theta'',1-\frac{\varepsilon}{4}]$, $|\theta-x^*_{\sigma}\left(\theta\right)|<|(\theta'+\theta'')/2-x^*_{\sigma}\left(\theta\right)|$. Likewise, for any $\theta\in[\varepsilon/4,\theta']$, and any $\hat{\theta}\ge\theta''$, we have that $|\theta-x^*_{\sigma}\left(\theta\right)|<|x^*_{\sigma}\left(\theta\right)-\hat{\theta}|$ when

[10]The proof for the existence of a sequence $\left\{x^*_{\sigma_n}(\cdot)\right\}_n$ with domain $\left[\frac{\varepsilon}{4},1-\frac{\varepsilon}{4}\right]$ converging uniformly to its limit function $x^*_{0^+}(\cdot)$ follows from the same arguments that establish the uniform convergence of $\{H_{\sigma_n}(\cdot)\}_n$ to $H_{0^+}(\cdot)$ in Step 1.

$\sigma \in (0, \sigma^{\#}(\varepsilon)]$.

Next, pick any policy $\Gamma = (\{0,1\}, \pi) \in \mathbb{G}_\sigma$ for which there exists $d \equiv (\theta', \theta''] \in D^\Gamma$ such that (a) $|\theta'' - \theta'| > \varepsilon$ and (b) $\pi(\theta) = 0$ for all $\theta \in (\theta', \theta'']$. If $\min_{\theta \in [0,1]} U_\sigma^\Gamma \left(x_\sigma^*(\theta), 1 | x_\sigma^*(\theta)\right) > 0$, the result follows directly from Lemma S2-B. Thus assume that $\min_{\theta \in [0,1]} U_\sigma^\Gamma \left(x_\sigma^*(\theta), 1 | x_\sigma^*(\theta)\right) = 0$.

Suppose that $\min_{\theta \in [\theta'',1]} U_\sigma^\Gamma \left(x_\sigma^*(\theta), 0 | x_\sigma^*(\theta)\right) > 0$. By Lemma S2-C, $U_\sigma^\Gamma \left(x_\sigma^*(\theta), 1 | x_\sigma^*(\theta)\right) > 0$ for all $\theta \in (\theta', \theta'']$. Hence, $\min_{\theta \in [\theta',1]} U_\sigma^\Gamma \left(x_\sigma^*(\theta), 1 | x_\sigma^*(\theta)\right) > 0$.

Below we show that, starting from Γ, we can then construct a policy $\Gamma^\eta \in \mathbb{G}_\sigma$, with $M(\Gamma^\eta) \leq M(\Gamma)$ such that, when agents play as in G_σ, the probability of default under Γ^η is strictly smaller than under Γ. Γ^η is obtained from Γ by giving a pass grade to a positive-measure interval of types in the middle of $(\theta', \theta'']$. Formally, take $\eta \in (0, (\theta'' - \theta')/2)$ and let $\Gamma^\eta = (\{0,1\}, \pi^\eta)$ be the policy whose rule π^η is given by (a) $\pi^\eta(\theta) = \pi(\theta)$ for all $\theta \notin [(\theta' + \theta'')/2, (\theta' + \theta'')/2 + \eta]$, and (b) $\pi^\eta(\theta) = 1$ for all $\theta \in [(\theta' + \theta'')/2, (\theta' + \theta'')/2 + \eta]$. Below we show that $U^{\Gamma^\eta} \left(x_\sigma^*(\theta), 1 | x_\sigma^*(\theta)\right) \geq 0$ for all $\theta \in [0,1]$. To see this, let $\Theta^1 \equiv \{\theta \in \Theta : \pi(\theta) = 1\}$ be the collection of fundamentals receiving a pass grade under the original policy Γ. For any $\theta \in [0, \theta']$, and any x,

$$
\begin{aligned}
&\Lambda_\sigma^{\Gamma^\eta}(\theta | x, 1) = \mathbb{P}[\tilde{\theta} \leq \theta | x, \tilde{\theta} \in (\Theta^1 \cup [(\theta' + \theta'')/2, (\theta' + \theta'')/2 + \eta])] \\
&= \tfrac{\mathbb{P}[\tilde{\theta} \leq \theta \wedge \tilde{\theta} \in \Theta^1 | x]}{\mathbb{P}[\tilde{\theta} \in (\Theta^1 \cup [(\theta' + \theta'')/2, (\theta' + \theta'')/2 + \eta]) | x]} \leq \mathbb{P}[\tilde{\theta} \leq \theta | x, \tilde{\theta} \in \Theta^1] = \Lambda_\sigma^\Gamma(\theta | x, 1).
\end{aligned}
$$

The first equality follows from the fact that, under Γ^η, the signal $s = 1$ carries the same information as the announcement that $\tilde{\theta} \in (\Theta^1 \cup [(\theta' + \theta'')/2, (\theta' + \theta'')/2 + \eta])$. The inequality follows from the fact that $\mathbb{P}\left[\tilde{\theta} \in (\Theta^1 \cup [(\theta' + \theta'')/2, (\theta' + \theta'')/2 + \eta]) | x\right] > \mathbb{P}\left[\tilde{\theta} \in \Theta^1 | x\right]$. The last equality follows from fact that, under the original policy Γ, the signal $s = 1$ carries the same information as the announcement that $\tilde{\theta} \in \Theta^1$.

Given the above inequality, and the fact that, $b < 0 < g$, we then have that, for any $\theta \in [0, \theta']$,

$$
\begin{aligned}
&U_\sigma^\Gamma(x_\sigma^*(\theta), 1 | x_\sigma^*(\theta)) = b \cdot \Lambda_\sigma^\Gamma(\theta | x_\sigma^*(\theta), 1) + g \cdot \left[1 - \Lambda_\sigma^\Gamma(\theta | x_\sigma^*(\theta), 1)\right] \\
&\leq b \cdot \Lambda_\sigma^{\Gamma^\eta}(\theta | x_\sigma^*(\theta), 1) + g \cdot \left[1 - \Lambda_\sigma^{\Gamma^\eta}(\theta | x_\sigma^*(\theta), 1)\right] = U_\sigma^{\Gamma^\eta}(x_\sigma^*(\theta), 1 | x_\sigma^*(\theta)).
\end{aligned}
$$

Hence $U^{\Gamma^\eta} \left(x_\sigma^*(\theta), 1 | x_\sigma^*(\theta)\right) \geq 0$, all $\theta \leq \theta'$. That $\min_{\theta \in [\theta',1]} U_\sigma^\Gamma \left(x_\sigma^*(\theta), 1 | x_\sigma^*(\theta)\right) > 0$, along with the continuity of $U_\sigma^{\Gamma^\eta} \left(x_\sigma^*(\theta), 1 | x_\sigma^*(\theta)\right)$ in η implies that $\min_{\theta \in [0,1]} U_\sigma^{\Gamma^\eta} \left(x_\sigma^*(\theta), 1 | x_\sigma^*(\theta)\right) \geq 0$ for η small. Hence $\Gamma^\eta \in \mathbb{G}_\sigma$.

Next, consider the more interesting case in which $\min_{\theta\in[\theta'',1]} U_\sigma^\Gamma\left(x_\sigma^*\left(\theta\right),0|x_\sigma^*\left(\theta\right)\right) = 0$. Let $\theta_\sigma^\# \equiv \inf\left\{\theta \geq \theta'' : U_\sigma^\Gamma(x_\sigma^*(\theta),1|x_\sigma^*(\theta)) = 0\right\}$. An implication of Lemma S2-C is that that $\theta_\sigma^\# > \theta''$. Also let $(\theta''',\theta'''']\subset[0,1]$ be the first interval to the immediate right of $(\theta',\theta'']$ such that $\pi(\theta)=1$ for all $\theta\in(\theta''',\theta'''']$ and let $\hat{\theta}=\min\left\{\theta'''',\theta_\sigma^\#\right\}$.[11]

Now, pick $\xi>0$ small and let $\delta(\xi)$ be implicitly defined by

$$F((\theta'+\theta'')/2+\xi)-F((\theta'+\theta'')/2)=F((\theta'''+\hat{\theta})/2+\delta(\xi))-F((\theta'''+\hat{\theta})/2). \tag{S2}$$

Consider the policy $\Gamma^\xi=(\{0,1\},\pi^\xi)$ defined by (a) $\pi^\xi(\theta)=\pi(\theta)$ for all $\theta\notin[(\theta'+\theta'')/2,(\theta'+\theta'')/2+\xi]\cup[(\theta'''+\hat{\theta})/2,(\theta'''+\hat{\theta})/2+\delta(\xi)]$, (b) $\pi^\xi(\theta)=1$ for all $\theta\in[(\theta'+\theta'')/2,(\theta'+\theta'')/2+\xi]$, and (c) $\pi^\xi(\theta)=0$ for all $\theta\in[(\theta'''+\hat{\theta})/2,(\theta'''+\hat{\theta})/2+\delta(\xi)]$. Below we establish that, when $\xi>0$ is small, such a policy is such that $\min_{\theta\in[0,1]}U^{\Gamma^\xi}\left(x_\sigma^*\left(\theta\right),1|x_\sigma^*\left(\theta\right)\right)>0$ and hence $\Gamma^\xi\in\mathbb{G}_\sigma$. To see this, for any arbitrary policy $\tilde{\Gamma}=\left\{\left\{0,1\right\},\widetilde{\pi}\right\}$, any $\theta\in[0,1]$, let

$$V_\sigma^{\tilde{\Gamma}}\left(\theta\right)\equiv U_\sigma^{\tilde{\Gamma}}\left(x_\sigma^*\left(\theta\right),1|x_\sigma^*\left(\theta\right)\right)p_\sigma^{\tilde{\Gamma}}\left(x_\sigma^*\left(\theta\right),1\right),$$

where, for any x, $p_\sigma^{\tilde{\Gamma}}(x,1)\equiv\int_\Theta\tilde{\pi}(\theta)p_\sigma(x|\theta)\mathrm{d}F(\theta)$, with $p_\sigma(x|\theta)\equiv\frac{1}{\sigma}\phi((x-\theta)/\sigma)$.

By definition of $\theta_\sigma^\#$, we must have that, for all θ, $0=V_\sigma^\Gamma\left(\theta_\sigma^\#\right)\leq V_\sigma^\Gamma\left(\theta\right)$. Next, for any $\xi>0$, define $\varphi_R\left(\xi\right)\equiv\min_{\theta\in[\theta'',1]}V_\sigma^{\Gamma^\xi}\left(\theta\right)$. Let $\overline{u}(\tilde{\theta},\theta)\equiv g\mathbf{1}\{\tilde{\theta}>\theta\}+b\mathbf{1}\{\tilde{\theta}\leq\theta\}$ and note that, for any θ,

$$V_\sigma^{\Gamma^\xi}\left(\theta\right)=V_\sigma^\Gamma\left(\theta\right)+\int_{(\theta'+\theta'')/2}^{(\theta'+\theta'')/2+\xi}\overline{u}(\tilde{\theta},\theta)p_\sigma(x_\sigma^*\left(\theta\right)|\tilde{\theta})\mathrm{d}F(\tilde{\theta})-\int_{(\theta'''+\hat{\theta})/2}^{(\theta'''+\hat{\theta})/2+\delta(\xi)}\overline{u}(\tilde{\theta},\theta)p_\sigma(x_\sigma^*\left(\theta\right)|\tilde{\theta})\mathrm{d}F(\tilde{\theta}).$$

Using the envelope theorem, we have that, for any $\theta_\sigma^\xi\in\arg\min_{\theta\in[\theta'',1]}V_\sigma^{\Gamma^\xi}\left(\theta\right)$,

$$\begin{aligned}
\varphi_R'\left(\xi\right) &= f\left((\theta'+\theta'')/2+\xi\right)\overline{u}\left((\theta'+\theta'')/2+\xi,\theta_\sigma^\xi\right)p_\sigma\left(x_\sigma^*(\theta_\sigma^\xi)|(\theta'+\theta'')/2+\xi\right)\\
&\quad -f\left((\theta'''+\hat{\theta})/2+\delta(\xi)\right)\overline{u}((\theta'''+\hat{\theta})/2+\delta(\xi),\theta_\sigma^\xi)p_\sigma(x_\sigma^*(\theta_\sigma^\xi)|(\theta'''+\hat{\theta})/2+\delta(\xi))\delta'(\xi)\\
&= f\left((\theta'+\theta'')/2+\xi\right)[\overline{u}((\theta'+\theta'')/2+\xi,\theta_\sigma^\xi)p_\sigma(x_\sigma^*(\theta_\sigma^\xi)|(\theta'+\theta'')/2+\xi)\\
&\quad -\overline{u}((\theta'''+\hat{\theta})/2+\delta(\xi),\theta_\sigma^\xi)p_\sigma(x_\sigma^*(\theta_\sigma^\xi)|(\theta'''+\hat{\theta})/2+\delta(\xi))],
\end{aligned}$$

where the second equality uses the implicit function theorem applied to (S2) to obtain that

[11] The existence of such an interval follows from the fact that $\pi(\theta)=1$ in a left neighborhood of $\theta_\sigma^\#$ by virtue of Lemma S2-C. Also observe that, when $\theta''<\theta^{MS}$, such an interval is adjacent to $(\theta',\theta'']$ and hence $\theta'''=\theta''$.

$\delta'(\xi) = f\left((\theta' + \theta'')/2 + \xi\right)/f((\theta''' + \hat{\theta})/2 + \delta(\xi))$. As a consequence,

$$\begin{aligned} \lim\nolimits_{\xi \to 0^+} \varphi'_R(\xi) = f\left((\theta' + \theta'')/2\right) [\overline{u}\left((\theta' + \theta'')/2, \theta^{\#}_{\sigma}\right) p_\sigma\left(x^*_\sigma(\theta^{\#}_\sigma)|(\theta' + \theta'')/2\right) \\ - \overline{u}((\theta''' + \hat{\theta})/2, \theta^{\#}_\sigma) p_\sigma(x^*_\sigma(\theta^{\#}_\sigma)|(\theta''' + \hat{\theta})/2)]. \end{aligned} \tag{S3}$$

That $\sigma < \sigma^{\#}(\varepsilon)$ implies that $|x^*_\sigma\left(\theta^{\#}_\sigma\right) - (\theta''' + \hat{\theta})/2| < |x^*_\sigma\left(\theta^{\#}_\sigma\right) - (\theta' + \theta'')/2|$. That $p_\sigma(x|\theta)$ is single-peaked in turn implies that $p_\sigma(x^*_\sigma(\theta^{\#}_\sigma)|(\theta' + \theta'')/2) < p_\sigma(x^*_\sigma(\theta^{\#}_\sigma)|(\theta''' + \hat{\theta})/2)$ and hence that

$$\begin{aligned} \overline{u}\left((\theta' + \theta'')/2, \theta^{\#}_\sigma\right) p_\sigma\left(x^*_\sigma\left(\theta^{\#}_\sigma\right)|(\theta' + \theta'')/2\right) - \overline{u}\left((\theta''' + \hat{\theta})/2, \theta^{\#}_\sigma\right) p_\sigma\left(x^*_\sigma\left(\theta^{\#}_\sigma\right)|(\theta''' + \hat{\theta})/2\right) \\ = b \cdot \left(p_\sigma\left(x^*_\sigma\left(\theta^{\#}_\sigma\right)|(\theta' + \theta'')/2\right) - p_\sigma\left(x^*_\sigma\left(\theta^{\#}_\sigma\right)|(\theta''' + \hat{\theta})/2\right)\right) > 0. \end{aligned}$$

Thus, $\lim_{\xi \to 0^+} \varphi'_R(\xi) > 0$. By continuity of $U^{\Gamma^\xi}_\sigma\left(x^*_\sigma(\theta), 1|x^*_\sigma(\theta)\right)$ in ξ, we then have that, for $\xi > 0$ small, $\min_{\theta \in [\theta'', 1]} U^{\Gamma^\xi}_\sigma\left(x^*_\sigma(\theta), 1|x^*_\sigma(\theta)\right) > 0$.

Next, we prove that, under the policy Γ^ξ, $\min_{\theta \in [0, \theta'']} U^{\Gamma^\xi}_\sigma\left(x^*_\sigma(\theta), 1|x^*_\sigma(\theta)\right) > 0$. For any $\xi > 0$, define $\varphi_L(\xi) \equiv \min_{\theta \in [0, \theta']} V^{\Gamma^\xi}_\sigma(\theta)$. Arguments similar to those used above to compute $\lim_{\xi \to 0^+} \varphi'_R(\xi)$ imply that, for any $\theta^{\#\#}_\sigma \in \arg\min_{\theta \in [0, \theta']} V^{\Gamma}_\sigma(\theta)$, when $\sigma \leq \sigma^{\#}(\varepsilon)$,

$$\begin{aligned} \lim_{\xi \to 0^+} \varphi'_L(\xi) &= f\left((\theta' + \theta'')/2\right) [\overline{u}((\theta' + \theta'')/2, \theta^{\#\#}_\sigma) p_\sigma(x^*_\sigma(\theta^{\#\#}_\sigma)|(\theta' + \theta'')/2) \\ &\quad - \overline{u}((\theta''' + \hat{\theta})/2, \theta^{\#\#}_\sigma) p_\sigma(x^*_\sigma(\theta^{\#\#}_\sigma)|(\theta''' + \hat{\theta})/2)] \\ &= f\left((\theta' + \theta'')/2\right) g \left[p_\sigma(x^*_\sigma(\theta^{\#\#}_\sigma)|(\theta' + \theta'')/2) - p_\sigma(x^*_\sigma(\theta^{\#\#}_\sigma)|(\theta''' + \hat{\theta})/2))\right] > 0. \end{aligned}$$

The first equality follows from steps analogous to those used to establish (S3). The second equality follows from the fact that, by assumption $\theta^{\#\#}_\sigma \leq \theta'$. The inequality is a consequence of the fact that, for $\sigma \leq \sigma^{\#}(\varepsilon)$, $|x^*_\sigma\left(\theta^{\#\#}_\sigma\right) - (\theta' + \theta'')/2| < |x^*_\sigma\left(\theta^{\#\#}_\sigma\right) - (\theta''' + \hat{\theta})/2)|$, which, together with the fact that the noise distribution is single-peaked, implies that

$$p_\sigma(x^*_\sigma(\theta^{\#\#}_\sigma)|(\theta' + \theta'')/2) > p_\sigma(x^*_\sigma(\theta^{\#\#}_\sigma)|(\theta''' + \hat{\theta})/2).$$

Hence, for $\xi > 0$ small, $U^{\Gamma^\xi}_\sigma\left(x^*_\sigma(\theta), 1|x^*_\sigma(\theta)\right) > 0$ for all $\theta \in [0, \theta']$. Furthermore, by Lemma S2-C, $U^{\Gamma}_\sigma\left(x^*_\sigma(\theta), 1|x^*_\sigma(\theta)\right) > 0$ for all $\theta \in (\theta', \theta'']$. Hence, provided that ξ is small, the continuity of $U^{\Gamma^\xi}_\sigma\left(x^*_\sigma(\theta), 1|x^*_\sigma(\theta)\right)$ in ξ implies that $U^{\Gamma^\xi}_\sigma\left(x^*_\sigma(\theta), 1|x^*_\sigma(\theta)\right) > 0$ also for $\theta \in (\theta', \theta'']$. Combining all the properties above, we thus conclude that, for $\xi > 0$ small, $\min_{\theta \in [0,1]} U^{\Gamma^\xi}_\sigma\left(x^*_\sigma(\theta), 1|x^*_\sigma(\theta)\right) > 0$. Hence $\Gamma^\xi \in \mathbb{G}_\sigma$.

By construction, $M(\Gamma^\xi) < M(\Gamma)$. Furthermore, when agents play according to G_σ, the probability of default under Γ^{ξ} is the same as under Γ. Lemma S2-B then implies that, starting from Γ^{ξ}, one can construct a policy $\Gamma^{\#} \in \mathbb{G}_{\sigma}$, close to Γ^{ξ} in the L_1 norm, such that (1) $M(\Gamma^{\#}) \leq M(\Gamma^{\xi})$ and (2), when agents play according to G_{σ}, the probability of default under $\Gamma^{\#}$ is strictly smaller than under Γ. This completes the proof of Lemma S2-D. ■

Step 3. Steps 1 and 2 imply that there exists a function $\bar{\sigma} : \left(0, \min\{\theta^{MS}, 1-\theta^{MS}\}\right) \to \mathbb{R}_{++}$, with $\bar{\sigma}(\varepsilon) \leq \min\{\sigma(\varepsilon), \sigma^{\#}(\varepsilon)\}$ for all $\varepsilon \in \left(0, \min\{\theta^{MS}, 1-\theta^{MS}\}\right)$ and with $\bar{\sigma}(\varepsilon) \to 0^{+}$ as $\varepsilon \to 0^{+}$, such that the following is true: For any $\varepsilon \in \left(0, \min\{\theta^{MS}, 1-\theta^{MS}\}\right)$, any $\sigma \in (0, \bar{\sigma}(\varepsilon)]$, and any policy $\Gamma = (\{0,1\}, \pi) \in \mathbb{G}_{\sigma}$ with $M(\Gamma) > \varepsilon$, there exists another policy $\Gamma' = (\{0,1\}, \pi') \in \mathbb{G}_{\sigma}$ with $M(\Gamma') \leq \varepsilon$ such that, when the agents play as in the auxiliary game G_{σ}, the probability of default under Γ' is strictly smaller than under Γ.[12]

Furthermore, the arguments establishing Lemma S2-D reveal that the policy Γ' can be constructed so that $U_{\sigma}^{\Gamma'}(x, 1|x) > 0$ for all x. The policy Γ' thus satisfies PCP also when agents play according to MARP. The claim in the Example then follows by taking $\Gamma^{*} = \Gamma'$ with Γ' satisfying the above properties.

Step 4. We now complete the proof by showing how to construct the function $\mathcal{E}$ in the example. Let (ε_n) be a non-increasing sequence satisfying $\lim_{n\to\infty} \varepsilon_n = 0$. For each $n \in \mathbb{N}$, then let $\sigma_n = \overline{\sigma}(\varepsilon_n)$, with the function $\overline{\sigma}(\cdot)$ as defined in Step 3. The results in Steps 1-3 above imply that, given $(\varepsilon_n, \sigma_n)$, there exist strictly decreasing subsequences $(\tilde{\varepsilon}_n)$ and $(\tilde{\sigma}_n)$ satisfying $\lim_{n\to\infty} \tilde{\varepsilon}_n = \lim_{n\to\infty} \tilde{\sigma}_n = 0$ such that, for any $n \in \mathbb{N}$, the conclusions in Step 3 hold for $\varepsilon = \tilde{\varepsilon}_n$ and $\bar{\sigma}(\varepsilon_n) = \tilde{\sigma}_n$. Then let $\bar{\sigma} = \tilde{\sigma}_0 > 0$ and $\mathcal{E} : (0, \bar{\sigma}] \to \mathbb{R}_{+}$ be the function defined by $\mathcal{E}(\sigma) = \varepsilon_n$ for all $\sigma \in (\sigma_{n+1}, \sigma_n]$. The result in the example then follows from Steps 1-3, by letting $\mathcal{E}(\cdot)$ be the function so constructed. Q.E.D.

[12] Observe that the thresholds $\sigma(\varepsilon)$ and $\sigma^{\#}(\varepsilon)$ identified in Steps 1 and 2 above are invariant to the initial policy Γ. The same arguments used to arrive at a policy $\Gamma^{\#}$ with mesh $M(\Gamma^{\#}) < M(\Gamma)$ can then be iterated till one arrives at a policy Γ' with mesh $M(\Gamma') \leq \varepsilon$.

S3. Multiplicity of Receivers and of Exogenous Private Information

Single Receiver

To appreciate the role that the multiplicity of the receivers plays for the results in the main text, consider the following variant of the economy of Section 2 in the main text.

Timing. At $t = 0$, the policy maker chooses a disclosure policy $\Gamma = \{S, \pi\}$ that, for each fundamental θ, sends a signal s from a distribution $\pi(\theta) \in \Delta(S)$.[13]At $t = 1$, a single receiver with signal x drawn from a log-supermodular distribution $p(x|\theta)$ has to decide whether to take a "friendly" action, $a = 1$, or an adversarial" action, $a = 0$.

Payoffs. The policy maker's payoff is equal to $W > 0$ in case of no default and $L < 0$ in case of default. Default occurs if and only if $\theta \leq 1 - a$. Hence,

$$U^{PM}(\theta, a) \equiv W \cdot \mathbf{1}(\theta > 1 - a) + L \cdot \mathbf{1}(\theta \leq 1 - a).$$

As for the receiver's payoff, we consider two cases. The first one corresponds to a market where the receiver's payoff is aligned with the policy maker's payoff, as in the baseline model of Section 2. The second case, instead, corresponds to a market where the receiver's payoff is misaligned with the policy maker's payoff, over the critical region of fundamentals $(0, 1)$ where the fate of the bank depends on the receiver's behavior.

Case 1: Aligned Preferences

The receiver's payoff differential between taking the friendly action (interpreted as "pledging funds" to the bank) and the adversarial action (interpreted as "refraining from pledging") is given by $u^I(\theta, 1) - u^I(\theta, 0) = b < 0$ in case of default, and by $u^I(\theta, 1) - u^I(\theta, 0) = g > 0$ in case of no default, with $g > 0 > b$. In this case, it is immediate to see that the following pass/fail policy is optimal: the policy maker gives a pass to all banks with fundamentals $\theta > 0$ and a fail to all banks with fundamentals $\theta \leq 0$.

Case 2: Misaligned Preferences

[13] Here we accommodate for stochastic disclosure rules, although this is not essential to the results.

The receiver's payoff differential between taking the friendly action (interpreted as "refraining from speculating" against the bank) and the adversarial action (interpreted as "speculating" against the bank) is given by $u^I(\theta, 1) - u^I(\theta, 0) = -g < 0$ for any $\theta \le 1$, and by $u^I(\theta, 1) - u^I(\theta, 0) = -b > 0$ for any $\theta > 1$, with $g > 0 > b$. That is, the receiver obtains a payoff equal to 0 when she abstains from speculating against the bank (the friendly action). When, instead, she speculates against the bank (the adversarial action), she obtains a payoff equal to $g > 0$ in case speculation is successful (i.e., in case of default) and a payoff equal to $b < 0$ in case the bank survives the attack.

We start by showing that, in this case, Assumptions 2 and 3 in Guo & Shmaya (2019) are satisfied. In fact, note that, for any realization $x \in \mathbb{R}$ of the receiver's signal, the ratio between the receiver's and the sender's payoff differential is equal to

$$\varphi(\theta) \equiv \frac{u^I(\theta, 1) - u^I(\theta, 0)}{U^{PM}(\theta, 1) - U^{PM}(\theta, 0)} = \begin{cases} -\infty & \theta \le 0 \\ \frac{-g}{W-L} & \theta \in (0, 1] \\ +\infty & \theta > 1 \end{cases}$$

and is increasing in θ, which implies that Assumption 2 in Guo & Shmaya (2019) holds. That Assumption 3 also holds follows from noting that the receiver's payoff differential changes from negative to positive at $\theta = 1$, for any $x \in \mathbb{R}$. By virtue of Theorem 3.1 in Guo & Shmaya (2019), the optimal policy is thus a deterministic cutoff mechanism that recommends to take action $a = 1$ on intervals $(\underline{\pi}(x), \bar{\pi}(x)) \subset \Theta$, with $\underline{\pi}(x)$ decreasing in x, and $\bar{\pi}(x)$ increasing in x.

Next observe, when there is a continuum of receivers with the same payoffs as the representative speculator above, under an adversarial/robust design, the optimal policy satisfies the same properties as in Theorems 1-3 in the baseline model of Section 2 in the main body, despite the misalignment in payoffs. This is because, under MARP, independently of whether the payoffs are aligned or misaligned, all agents play the adversarial action unless it is iteratively dominant for them to play the friendly action, exactly as in the baseline model.

The optimal policy with a single receiver is thus fundamentally different from the optimal policy with multiple receivers. First, with a single receiver, the optimal policy cannot be implemented with a simple pass/fail announcement. It requires sending multiple (in fact a continuum) of grades. Each grade is associated with a different cut-off $x^*(s)$ such that, given the announced grade s, the receiver plays the friendly action only if $x > x^*(s)$. With multiple receivers, instead, when $p(x|\theta)$ is log-supermodular, as assumed here, the optimal policy is a simple pass/fail test (Theorem 2* in the main text).

Second, observe that, with a single receiver, the optimal policy has the *interval structure*. That is, for any x, the optimal policy induces the receiver to play the friendly action over an interval $(\underline{\pi}(x), \bar{\pi}(x)) \subset \Theta$ of states. With multiple receivers, instead, the optimal policy has the interval structure only when it is monotone (in this case, the interval is $(\underline{\pi}(x), \bar{\pi}(x)) = (\theta^*, +\infty)$ for all x).

We conclude that the structure of the optimal policy with a single (privately informed) receiver is fundamentally different from the one with multiple (privately informed) receivers.

Multiple Receivers with No Exogenous Private Information

Next, consider an economy with a continuum of receivers, of measure 1, but assume that they do not possess any exogenous private information. All receivers share the policy maker's prior F about θ. As in the main body, denote by $A \in [0,1]$ the aggregate action and let $u^I(\theta, A)$ denote the representative agent's payoff differential between action $a = 1$ and action $a = 0$, when fundamentals are θ and the aggregate action is A.

When agents do not possess exogenous private information, the optimal policy is a monotone binary policy, irrespective of whether the agents' payoffs are aligned or not with the policy maker's payoff. To see this, denote by $\mu_s^\pi \in \Delta(\Theta)$ the common posterior generated by the observation of signal realization $s \in S$ under the policy π. When agents play according to MARP, the only way the policy maker can induce an agent to play the friendly action $a = 1$ is to convince him that the friendly action is strictly dominant for him. That is, each agent plays $a = 1$ at s if, and only if,[14]

$$\int_\Theta u^I(\theta, 0)\, \mu_s^\pi(\mathrm{d}\theta) > 0.$$

As a result, under the adversarial/robust design, the game with multiple receivers who possess

[14] To see this, note that, because the game is supermodular, when the above inequality is reversed, playing the aggressive action $a = 0$ becomes a best response to the conjecture that everyone else plays the aggressive action.

no exogenous private information is isomorphic to a game with a single receiver with payoff differential equal to $u^I(\theta, 0)$. That the optimal policy in such a case is monotone follows from Mensch (2021) and Inostroza (2023).

The optimal policy is thus again fundamentally different from the optimal policy for the economy with multiple receivers possessing heterogenous private information.

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