(In)efficiency in Information Acquisition and Aggregation through Prices

Supplementary Material

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Abstract

This document contains extended proofs of the results in the main text. The numbering of the conditions in the supplement is independent from the one in the main text and any cross-referencing in the supplement is meant with respect to the numbering of the conditions in the supplement, not the one in the main text.

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Proof of Proposition 1.

As explained in the main text, when the traders submit affine demand schedules with parameters $(a, \hat{b}, \hat{c})$, the equilibrium price is equal to

$$p = \frac{\alpha + \beta\hat{b}}{1 + \beta\hat{c}} + \frac{\beta a}{1 + \beta\hat{c}} z \tag{1}$$

where

$$z \equiv \theta + \omega, \tag{2}$$

with $\omega \equiv f(y)\eta - u/(\beta a)$. The information about θ contained in the equilibrium price is thus the same as the one contained in a public signal whose noise ω has precision[1]

$$\tau_\omega(a) \equiv \frac{\beta^2 a^2 y \tau_u \tau_\eta}{\beta^2 a^2 \tau_u + y\tau_\eta}. \tag{3}$$

In turn, this implies that the equilibrium trades $x_i = as_i + \hat{b} - \hat{c}p$ are affine functions of the traders' exogenous private information s_i and the endogenous public information z contained in the price. That is, when the endogenous public information contained in the price is equivalent to z, a trader with private signal s_i purchases an amount of the asset equal to

$$x_i = as_i + b + cz$$

where

$$b = \hat{b} - \hat{c}\frac{\alpha + \beta\hat{b}}{1 + \beta\hat{c}} \tag{4}$$

and

$$c = -\frac{\beta a\hat{c}}{1 + \beta\hat{c}}. \tag{5}$$

For each vector $(a, \hat{b}, \hat{c})$ describing the traders' demand schedules, there exists a unique vector (a, b, c) describing the traders' equilibrium trades as a function of their (exogenous) private information, s_i, and the (endogenous) public information, z, and vice versa. Hereafter, we find it more convenient to characterize the equilibrium use of information in terms of the vector (a, b, c) describing the equilibrium trades. When the individual trades are given by $x_i = as_i + b + cz$, the aggregate trade is equal to

$$\tilde{x} = \int x_i di = a(\theta + f(y)\eta) + b + cz.$$

Using the fact that $z \equiv \theta + f(y)\eta - u/(\beta a)$, we thus have that

$$\tilde{x} = a(z + \frac{u}{\beta a}) + b + cz = (a + c)z + \frac{u}{\beta} + b.$$

Using the expression for the inverse aggregate supply function $p = \alpha - u + \beta\tilde{x}$, we then have that the equilibrium price can be expressed as follows:

$$p = \alpha + \beta b + \beta(a + c)z. \tag{6}$$

[1] To derive $\tau_\omega(a)$ we use the fact that $f(y) = 1/\sqrt{y}$.

Next, observe that

$$\begin{aligned}\mathbb{E}[\theta|I_i,p]=\mathbb{E}[\theta|s_i,z] &= \left[\begin{array}{cc} Cov(\theta,s_i) & Cov(\theta,z)\end{array}\right]\left[\begin{array}{cc} Var(s_i) & Cov(s_i,z)\\ Cov(s_i,z) & Var(z)\end{array}\right]^{-1}\left[\begin{array}{c} s_i-\mathbb{E}[s_i]\\ z-\mathbb{E}[z]\end{array}\right]\\ &= \left[\begin{array}{cc} \sigma_\theta^2 & \sigma_\theta^2\end{array}\right]\left[\begin{array}{cc} \sigma_\theta^2+\sigma_\epsilon^2 & \sigma_\theta^2+f(y)^2\sigma_\eta^2\\ \sigma_\theta^2+f(y)^2\sigma_\eta^2 & \sigma_\theta^2+\sigma_\omega^2(a)\end{array}\right]^{-1}\left[\begin{array}{c} s_i-\mathbb{E}[s_i]\\ z-\mathbb{E}[z]\end{array}\right],\end{aligned}$$

where $\sigma_\theta^2\equiv\tau_\theta^{-1}$, $\sigma_\omega^2(a)\equiv\tau_\omega(a)^{-1}$, $\sigma_\eta^2\equiv\tau_\eta^{-1}$, and $\sigma_\epsilon^2\equiv\tau_\epsilon^{-1}$. Substituting for the inverse of the variance-covariance matrix, we have that

$$\begin{aligned}\mathbb{E}[\theta|s_i,z] &= \frac{1}{(\sigma_\theta^2+\sigma_\epsilon^2)(\sigma_\theta^2+\sigma_\omega^2(a))-(\sigma_\theta^2+f(y)^2\sigma_\eta^2)^2}\times\\ &\quad \left[\begin{array}{cc} \sigma_\theta^2 & \sigma_\theta^2\end{array}\right]\left[\begin{array}{cc} \sigma_\theta^2+\sigma_\omega^2(a) & -(\sigma_\theta^2+f(y)^2\sigma_\eta^2)\\ -(\sigma_\theta^2+f(y)^2\sigma_\eta^2) & \sigma_\theta^2+\sigma_\epsilon^2\end{array}\right]\left[\begin{array}{c} s_i-\mathbb{E}[s_i]\\ z-\mathbb{E}[z]\end{array}\right].\end{aligned}$$

Expanding the quadratic form, we have that

$$\begin{aligned}\mathbb{E}[\theta|s_i,z] &= \frac{\sigma_\theta^2\left(\sigma_\omega^2(a)-f(y)^2\sigma_\eta^2\right)}{(\sigma_\theta^2+\sigma_\epsilon^2)(\sigma_\theta^2+\sigma_\omega^2(a))-(\sigma_\theta^2+f(y)^2\sigma_\eta^2)^2}(s_i-\mathbb{E}[s_i])\\ &\quad+\frac{\sigma_\theta^2\left(\sigma_\epsilon^2-f(y)^2\sigma_\eta^2\right)}{(\sigma_\theta^2+\sigma_\epsilon^2)(\sigma_\theta^2+\sigma_\omega^2(a))-(\sigma_\theta^2+f(y)^2\sigma_\eta^2)^2}(z-\mathbb{E}[z]).\end{aligned}$$

Using the fact that $\mathbb{E}[s_i]=\mathbb{E}[z]=0$, and replacing σ_θ^2 with τ_θ^{-1}, $\sigma_\omega^2(a)$ with $\tau_\omega(a)^{-1}$, σ_η^2 with τ_η^{-1}, σ_ϵ^2 with τ_ϵ^{-1}, and $f(y)=1/\sqrt{y}$, we have that

$$\mathbb{E}[\theta|s_i,z] = \gamma_1(\tau_\omega(a))s_i+\gamma_2(\tau_\omega(a))z$$

where, for any τ_ω,

$$\gamma_1(\tau_\omega)\equiv\frac{\tau_\epsilon y\tau_\eta\left(y\tau_\eta-\tau_\omega\right)}{y^2\tau_\eta^2\left(\tau_\omega+\tau_\epsilon+\tau_\theta\right)-\tau_\omega\tau_\epsilon\left(\tau_\theta+2y\tau_\eta\right)} \tag{7}$$

and

$$\gamma_2(\tau_\omega)\equiv\frac{\tau_\omega\left(y^2\tau_\eta^2-\tau_\epsilon y\tau_\eta\right)}{y^2\tau_\eta^2\left(\tau_\omega+\tau_\epsilon+\tau_\theta\right)-\tau_\omega\tau_\epsilon\left(\tau_\theta+2y\tau_\eta\right)}=\left(1-\gamma_1(\tau_\omega)\frac{\tau_\theta+y\tau_\eta}{y\tau_\eta}\right)\frac{\tau_\omega}{\tau_\omega+\tau_\theta}. \tag{8}$$

Now recall that optimality requires that the equilibrium trades satisfy

$$x_i=\frac{1}{\lambda}\left(\mathbb{E}[\theta|s_i,z]-p\right).$$

Using the fact that $p=\alpha+\beta b+\beta(a+c)z$, and the characterization of $\mathbb{E}[\theta|s_i,z]$ above, we thus have that

$$x_i = \frac{1}{\lambda}\left[\gamma_1(\tau_\omega(a))s_i-(\alpha+\beta b)+\left(\gamma_2(\tau_\omega(a))-\beta(a+c)\right)z\right].$$

The sensitivity of the equilibrium trades to private information must thus satisfy

$$a=\frac{\gamma_1(\tau_\omega(a))}{\lambda}. \tag{9}$$

The sensitivity of the equilibrium trades to the endogenous public signal contained in the equilibrium price must satisfy

$$c = \frac{1}{\lambda} \left(\gamma_2(\tau_\omega(a)) - \beta(a + c) \right). \tag{10}$$

The constant b in the equilibrium trades must satisfy

$$b = -\frac{\alpha + \beta b}{\lambda}. \tag{11}$$

Replacing the expression for $\gamma_1(\tau_\omega(a))$ in (7) into (9), we thus conclude that the sensitivity a^* of the equilibrium demand schedules to the traders' private information must solve the following equation

$$a^* = \frac{1}{\lambda \Lambda(\tau_\omega(a^*))}, \tag{12}$$

where

$$\Lambda(\tau_\omega) \equiv \frac{y^2 \tau_\eta^2 \left(\tau_\omega + \tau_\epsilon + \tau_\theta\right) - \tau_\omega \tau_\epsilon \left(\tau_\theta + 2y\tau_\eta\right)}{\tau_\epsilon y \tau_\eta \left(y\tau_\eta - \tau_\omega\right)}. \tag{13}$$

Using (8), (10), and (9), we have that the sensitivity of the equilibrium trades to the endogenous public signal contained in the equilibrium price must satisfy

$$c = \frac{1}{\beta + \lambda} \left[\left(1 - \lambda a - \lambda a \frac{\tau_\theta}{y\tau_\eta} \right) \frac{\tau_\omega(a)}{\tau_\omega(a) + \tau_\theta} - \beta a \right]. \tag{14}$$

Using (11), in turn we have that the constant b in the equilibrium trades is given by

$$b = -\frac{\alpha}{\beta + \lambda}. \tag{15}$$

Finally, inverting the relationship between b and $\hat{b}$ and c and $\hat{c}$ using (4) and (5), we have that, given a^*, the values of $\hat{c}^*$ and $\hat{b}^*$ satisfy $\hat{c}^* = \hat{C}(a^*)$ and $\hat{b}^* = \hat{B}(a^*)$, where, for any a, the functions $\hat{C}$ and $\hat{B}$ are given by

$$\hat{C}(a) \equiv -\frac{\tau_\omega(a) y \tau_\eta (1 - \lambda a - \beta a) - \lambda a \tau_\theta \tau_\omega(a) - \beta a y \tau_\eta \tau_\theta}{\lambda \beta a y \tau_\eta \left(\tau_\omega(a) + \tau_\theta - \tau_\theta \tau_\omega(a)\right) + \beta \tau_\omega(a) y \tau_\eta}, \tag{16}$$

and

$$\hat{B}(a) \equiv \frac{\alpha}{\beta + \lambda} \left(\lambda \hat{C}(a) - 1 \right). \tag{17}$$

The formula for $\hat{C}(a)$ in the main text is obtained from (16) after replacing the formula for $\tau_\omega(a)$. To complete the proof, it thus suffices to show that equation (12) admits a unique solution and that such a solution satisfies $0 < a^* < 1/\lambda$. To see this, use the fact that $\tau_\epsilon = y\tau_e\tau_\eta/(\tau_e + \tau_\eta)$ to observe that this equation is equivalent to

$$\lambda \beta^2 \tau_u \left(y\tau_\eta + \tau_\theta \right) a^3 + \lambda y \left[y\tau_e\tau_\eta + \tau_\theta(\tau_e + \tau_\eta) \right] a - \tau_e \tau_\eta y^2 = 0. \tag{18}$$

Clearly, because the left-hand side is strictly increasing in a, the above cubic equation has a unique real root, which is strictly positive. Furthermore, when $a = 1/\lambda$, the left-hand side is equal to

$$\frac{\beta^2 \tau_u}{\lambda^2} \left(y\tau_\eta + \tau_\theta \right) + \tau_\theta (\tau_e + \tau_\eta) y > 0.$$

We conclude that $a^* \in (0, 1/\lambda)$. Q.E.D.

Derivation of welfare under FB allocation.

Because

$$\int_0^1 \left(x_i^2\right) di > \left(\int_0^1 x_i di\right)^2$$

we have that W is maximal when $x_i = x^o$ for all i, with

$$x^o \equiv \frac{\theta - \alpha + u}{\beta + \lambda}.$$

Q.E.D.

Derivation of welfare losses.

Ex-post welfare is equal to

$$W^o = \theta x^o - \frac{\lambda}{2}(x^o)^2 - \left(\alpha - u + \beta \frac{x^o}{2}\right) x^o = \frac{\beta + \lambda}{2}(x^o)^2.$$

It follows that

$$WL = \frac{\beta + \lambda}{2}\mathbb{E}\left[(x^o)^2\right] - \mathbb{E}\left[(\theta - \alpha + u)\,\tilde{x} - \beta\frac{\tilde{x}^2}{2} - \frac{\lambda}{2}\int_0^1 x_i^2 di\right].$$

Replacing $x^o = \frac{\theta - \alpha + u}{\beta + \lambda}$ into the above expression and using the fact that $\mathbb{E}\left[\int_0^1 x_i^2 di\right] = \mathbb{E}\left[\mathbb{E}[x_i^2|\tilde{x}]\right]$, we have that

$$\begin{aligned} WL &= \frac{\beta + \lambda}{2}\mathbb{E}\left[(x^o)^2\right] - \frac{1}{2}\mathbb{E}\left[2\left(\beta + \lambda\right)\tilde{x}x^o - \beta\tilde{x}^2 - \lambda\int_0^1 x_i^2 di\right] \\ &= \frac{\beta + \lambda}{2}\mathbb{E}\left[(x^o)^2\right] + \frac{1}{2}\mathbb{E}\left[(\beta + \lambda)\tilde{x}^2 - 2x^o\tilde{x}(\beta + \lambda) - \lambda\tilde{x}^2 + \lambda\mathbb{E}[x_i^2|\tilde{x}]\right] \\ &= \frac{\beta + \lambda}{2}\mathbb{E}[(\tilde{x} - x^o)^2] + \frac{\lambda}{2}\mathbb{E}[(x_i - \tilde{x})^2]. \end{aligned}$$

Q.E.D.

Proof of Lemma 1.

The same arguments as in the proof of Proposition 1 imply that, when the traders submit demand schedules of the form $x_i = as_i + \hat{b} - \hat{c}p$, for some $(a, \hat{b}, \hat{c})$, the trades induced by market clearing can be expressed as a function of the endogenous public information z generated by the market-clearing price by letting $x_i = as_i + b + cz$ where $z \equiv \theta + f(y)\eta - u/(\beta a)$ is the endogenous information about θ contained in the equilibrium price, and where the noise in the endogenous signal has precision $\tau_\omega(a) = \left(\beta^2 a^2 y \tau_u \tau_\eta\right) / \left(\beta^2 a^2 \tau_u + y\tau_\eta\right)$.

Furthermore, the values of b and c are given by (4) and (5). Using the above representation, we have that the aggregate volume of trade when the demand schedules are given by $(a, \hat{b}, \hat{c})$ is given by

$\tilde{x} = a(\theta + f(y)\eta) + b + cz$ and hence ex-ante welfare is given by

$$\mathbb{E}[W] = \mathbb{E}\left[(\theta - \alpha + u)\left(a(\theta + f(y)\eta) + b + cz\right) - \beta\tfrac{(a(\theta+f(y)\eta)+b+cz)^2}{2} - \int_0^1 \tfrac{\lambda}{2}\left(as_i + b + cz\right)^2 di\right].$$

Note that

$$\frac{\partial\mathbb{E}[W]}{\partial b} = \mathbb{E}\left[(\theta - \alpha + u) - \beta\left(a(\theta + f(y)\eta) + b + cz\right) - \lambda\left(as + b + cz\right)\right] = -\alpha - (\beta + \lambda)b,$$

$$\frac{\partial^2\mathbb{E}[W]}{\partial b^2} = -(\beta + \lambda) < 0,$$

$$\frac{\partial\mathbb{E}[W]}{\partial c} = \mathbb{E}\left[z\left(\theta - \alpha + u\right) - \beta\left(a(\theta + f(y)\eta) + b + cz\right)z - \lambda z\left(as + b + cz\right)\right],$$

$$\frac{\partial^2\mathbb{E}[W]}{\partial c^2} = \mathbb{E}\left[-\beta z^2 - \lambda z^2\right] < 0,$$

and $\partial^2\mathbb{E}[W]/\partial c\partial b = 0$. Hence $\mathbb{E}[W]$ is concave in b and c . For any a, the optimal values of b and c are thus given by the FOCs $\partial\mathbb{E}[W]/\partial b = 0$ and $\partial\mathbb{E}[W]/\partial c = 0$ from which we obtain that $b = -\alpha/(\beta+\lambda)$ and

$$\mathbb{E}\left[z\left(\theta + u\right) - \beta\left(a(\theta + f(y)\eta)\right)z - \beta cz^2 - \lambda azs - \lambda cz^2\right] = 0.$$

The last condition can be rewritten as

$$Cov\left[\left(\theta + u - \beta a(\theta + f(y)\eta)\right), z\right] - (\beta + \lambda)\, cVar(z) - \lambda aCov(z, s) = 0$$

from which we obtain that

$$c = \frac{Cov\left[\left(\theta + u - \beta a(\theta + f(y)\eta)\right), z\right]}{(\beta + \lambda)\, Var(z)} - \frac{\lambda aCov(z, s)}{(\beta + \lambda)\, Var(z)}.$$

Using the fact that $z \equiv \theta + f(y)\eta - \frac{u}{\beta a}$ and $s = \theta + \frac{1}{\sqrt{y}}(\eta + e)$, we have that

$$Var(z) = \frac{1}{\tau_\theta} + \frac{1}{\tau_\omega(a)} = \sigma_\theta^2 + \sigma_\omega^2(a),$$

where $\sigma_\theta^2 = 1/\tau_\theta$ and $\sigma_\omega^2(a) = 1/\tau_\omega(a)$. Furthermore,

$$\begin{aligned} Cov\left[\left(\theta + u - \beta a(\theta + f(y)\eta)\right), z\right] &= Cov\left[\left(\theta + u - \beta a(\theta + f(y)\eta)\right), \theta + f(y)\eta - \tfrac{u}{\beta a}\right] \\ &= Cov\left[\theta(1 - \beta a), \theta\right] + Cov\left[u, -\tfrac{u}{\beta a}\right] - Cov\left[\beta af(y)\eta, f(y)\eta\right] \\ &= (1 - \beta a)\sigma_\theta^2 - \tfrac{\sigma_u^2}{\beta a} - \beta af(y)^2\sigma_\eta^2, \end{aligned}$$

and $Cov\left[z, s\right] = \sigma_\theta^2 + f(y)^2\sigma_\eta^2$. Hence,

$$\begin{aligned} c &= \frac{(1 - \beta a)\sigma_\theta^2 - \frac{\sigma_u^2}{\beta a} - \beta af(y)^2\sigma_\eta^2}{(\beta + \lambda)\left(\sigma_\theta^2 + \sigma_\omega^2(a)\right)} - \frac{\lambda a(\sigma_\theta^2 + f(y)^2\sigma_\eta^2)}{(\beta + \lambda)\left(\sigma_\theta^2 + \sigma_\omega^2(a)\right)} \\ &= \frac{1}{\beta + \lambda}\left[\left(1 - \lambda a - \lambda a\frac{\tau_\theta}{y\tau_\eta}\right)\frac{\tau_\omega(a)}{\tau_\omega(a) + \tau_\theta} - \beta a\right]. \end{aligned}$$

We conclude that, given a, the optimal values for c and b are given by the same functions in (14) and (15) that characterize the parameters c and b as a function of a under the equilibrium usage of information. To go from the optimal trades to the demand schedules that implement them, it then suffices to use the functions defined by (4) and (5). We thus conclude that, for any choice of a, the optimal values of $\hat{c}^T$ and $\hat{b}^T$ are given by the functions (16) and (17), as claimed. Q.E.D.

Derivation of formula for welfare losses.

As shown above, the welfare losses can be expressed as

$$WL = \frac{\beta+\lambda}{2}\mathbb{E}[(\tilde{x}-x^o)^2] + \frac{\lambda}{2}\mathbb{E}[(x_i-\tilde{x})^2],$$

where x^0 is given by

$$x^o \equiv \frac{\theta+u-\alpha}{\beta+\lambda}. \tag{19}$$

We have also shown above that, for any vector $(a, \hat{b}, \hat{c})$ describing the demand schedules, there exists a unique vector (a, b, c) describing the induced trades $x_i = as_i + b + cz$ at the market-clearing price, and vice versa, where $z \equiv \theta + f(y)\eta - \frac{u}{\beta a}$ is the endogenous signal contained in the market-clearing price. This also means, when the traders submit the demand schedules corresponding to the vector $(a, \hat{b}, \hat{c})$, the aggregate volume of trade at the market-clearing price can be expressed as a function of (θ, η, z) as follows: $\tilde{x} = a(\theta + f(y)\eta) + b + cz$. Therefore, the dispersion of individual trades around the aggregate trade can be expressed as

$$\mathbb{E}[(x_i-\tilde{x})^2] = \mathbb{E}[a^2 f(y)^2 e_i^2] = \frac{a^2}{y\tau_e}.$$

Next, use the fact that, for any a, the optimal values of c and b are given by (14) and (15), along with the fact that $z \equiv \theta + f(y)\eta - \frac{u}{\beta a}$, and the fact that $f(y) = 1/\sqrt{y}$, to obtain that

$$\tilde{x} \quad = \quad a(\theta+f(y)\eta)+b+cz = \frac{\lambda a(\theta+f(y)\eta)+u-\alpha+\left(1-\lambda a-\lambda a\frac{\tau_\theta}{y\tau_\eta}\right)\frac{\tau_\omega(a)}{\tau_\omega(a)+\tau_\theta}z}{\beta+\lambda}.$$

Combining the expression for $\tilde{x}$ derived above with the expression for x^0 in (19), we have that

$$\mathbb{E}[(\tilde{x}-x^o)^2] \quad = \quad \mathbb{E}\left[\left(\frac{\lambda a(\theta+f(y)\eta)+u-\alpha+\left(1-\lambda a-\lambda a\frac{\tau_\theta}{y\tau_\eta}\right)\frac{\tau_\omega(a)}{\tau_\omega(a)+\tau_\theta}z}{\beta+\lambda} - \frac{\theta-\alpha+u}{\beta+\lambda}\right)^2\right].$$

Simplifying, we have that

$$\mathbb{E}[(\tilde{x}-x^o)^2] = \mathbb{E}\left[\left(\frac{\lambda a f(y)\eta}{\beta+\lambda} + \frac{\left(1-\lambda a-\lambda a\frac{\tau_\theta}{y\tau_\eta}\right)\frac{\tau_\omega(a)}{\tau_\omega(a)+\tau_\theta}(z-\theta)}{\beta+\lambda} - \frac{\left[1-\lambda a-\left(1-\lambda a-\lambda a\frac{\tau_\theta}{y\tau_\eta}\right)\frac{\tau_\omega(a)}{\tau_\omega(a)+\tau_\theta}\right]\theta}{\beta+\lambda}\right)^2\right].$$

Using the fact that $f(y) = 1/\sqrt{y}$, and that $\mathbb{E}[\omega\theta] = \mathbb{E}[\eta\theta] = 0$, we then have that

$$\begin{aligned}\mathbb{E}[(\tilde{x}-x^o)^2] \quad &= \quad \frac{\left(\left(1-\lambda a-\lambda a\frac{\tau_\theta}{y\tau_\eta}\right)\frac{\tau_\omega(a)}{\tau_\omega(a)+\tau_\theta}\right)^2}{(\beta+\lambda)^2\tau_\omega(a)} + \frac{\lambda^2a^2+2\lambda a\left(1-\lambda a-\lambda a\frac{\tau_\theta}{y\tau_\eta}\right)\frac{\tau_\omega(a)}{\tau_\omega(a)+\tau_\theta}}{(\beta+\lambda)^2 y\tau_\eta} \\ &\quad + \frac{\left(1-\lambda a-\left(1-\lambda a-\lambda a\frac{\tau_\theta}{y\tau_\eta}\right)\frac{\tau_\omega(a)}{\tau_\omega(a)+\tau_\theta}\right)^2}{(\beta+\lambda)^2\tau_\theta}.\end{aligned}$$

Replacing the expressions for $\mathbb{E}[(x_i - \tilde{x})^2]$ and $\mathbb{E}[(\tilde{x} - x^o)^2]$ derived above into the formula for the welfare losses, we then have that, for any a, when $\hat{b}$ and $\hat{c}$ are set optimally, the welfare losses can be expressed as

$$\begin{aligned} WL(a,\tau_\omega(a)) \;&=\; \frac{\left[\left(1-\lambda a-\lambda a\frac{\tau_\theta}{y\tau_\eta}\right)\frac{\tau_\omega(a)}{\tau_\omega(a)+\tau_\theta}\right]^2}{2\left(\beta+\lambda\right)\tau_\omega(a)} + \frac{\lambda^2 a^2 + 2\lambda a\left(1-\lambda a-\lambda a\frac{\tau_\theta}{y\tau_\eta}\right)\frac{\tau_\omega(a)}{\tau_\omega(a)+\tau_\theta}}{2\left(\beta+\lambda\right)y\tau_\eta} \\ &\quad + \frac{\left[1-\lambda a-\left(1-\lambda a-\lambda a\frac{\tau_\theta}{y\tau_\eta}\right)\frac{\tau_\omega(a)}{\tau_\omega(a)+\tau_\theta}\right]^2}{2\left(\beta+\lambda\right)\tau_\theta} + \frac{\lambda a^2}{2y\tau_e}. \end{aligned} \tag{20}$$

as claimed in the main text. Q.E.D.

Proof of Proposition 2.

As shown above, once b and c are set optimally as a function of a to minimize the welfare losses, the latter can be expressed as a function of a and $\tau_\omega(a)$, with the formula for $WL(a,\tau_\omega(a))$ given by (20), with $\tau_\omega(a) = (\beta^2 a^2 \tau_u \tau_\eta y)/(\beta^2 a^2 \tau_u + y\tau_\eta)$. The socially optimal level of a is thus the one that minimizes $WL(a,\tau_\omega(a))$ and is given by the FOC

$$\frac{dWL(a,\tau_\omega(a))}{da} \;=\; \frac{\partial WL(a,\tau_\omega(a))}{\partial a} + \frac{\partial WL(a,\tau_\omega(a))}{\partial \tau_\omega(a)}\frac{\partial \tau_\omega(a)}{\partial a} = 0.$$

Note that

$$\begin{aligned} \frac{\partial WL(a,\tau_\omega(a))}{\partial a} \;&=\; -\frac{\left(1-\lambda a-\lambda a\frac{\tau_\theta}{y\tau_\eta}\right)\frac{\tau_\omega(a)}{\tau_\omega(a)+\tau_\theta}\left(\lambda\frac{y\tau_\eta+\tau_\theta}{y\tau_\eta}\frac{\tau_\omega(a)}{\tau_\omega(a)+\tau_\theta}\right)}{\left(\beta+\lambda\right)\tau_\omega(a)} \\ &\quad + \frac{\lambda^2 a + \lambda\left(1-\lambda a-\lambda a\frac{\tau_\theta}{y\tau_\eta}\right)\frac{\tau_\omega(a)}{\tau_\omega(a)+\tau_\theta} - \lambda^2 a\frac{y\tau_\eta+\tau_\theta}{y\tau_\eta}\frac{\tau_\omega(a)}{\tau_\omega(a)+\tau_\theta}}{\left(\beta+\lambda\right)y\tau_\eta} \\ &\quad + \frac{\left[1-\lambda a-\left(1-\lambda a-\lambda a\frac{\tau_\theta}{y\tau_\eta}\right)\frac{\tau_\omega(a)}{\tau_\omega(a)+\tau_\theta}\right]\left(-\lambda+\lambda\left(\frac{y\tau_\eta+\tau_\theta}{y\tau_\eta}\right)\frac{\tau_\omega(a)}{\tau_\omega(a)+\tau_\theta}\right)}{\left(\beta+\lambda\right)\tau_\theta} + \frac{\lambda a}{y\tau_e}, \end{aligned}$$

and that

$$\begin{aligned} \frac{\partial WL(a,\tau_\omega(a))}{\partial \tau_\omega(a)} \;&=\; \frac{\left(1-\lambda a-\lambda a\frac{\tau_\theta}{y\tau_\eta}\right)^2}{2\left(\beta+\lambda\right)}\frac{\tau_\theta-\tau_\omega(a)}{\left(\tau_\omega(a)+\tau_\theta\right)^3} + \frac{\lambda a\left(1-\lambda a-\lambda a\frac{\tau_\theta}{y\tau_\eta}\right)}{\left(\beta+\lambda\right)y\tau_\eta}\frac{\tau_\theta}{\left(\tau_\omega(a)+\tau_\theta\right)^2} \\ &\quad - \frac{\left[1-\lambda a-\left(1-\lambda a-\lambda a\frac{\tau_\theta}{y\tau_\eta}\right)\frac{\tau_\omega(a)}{\tau_\omega(a)+\tau_\theta}\right]}{\left(\beta+\lambda\right)\tau_\theta}\left(1-\lambda a-\lambda a\frac{\tau_\theta}{y\tau_\eta}\right)\frac{\tau_\theta}{\left(\tau_\omega(a)+\tau_\theta\right)^2}. \end{aligned}$$

Also note that

$$\frac{\partial \tau_\omega(a)}{\partial a} \;=\; \frac{2\beta^2 a y^2 \tau_\eta^2 \tau_u}{\left(\beta^2 a^2 \tau_u + y\tau_\eta\right)^2}.$$

Using the expressions above, we obtain that

$$\begin{aligned}\frac{dWL(a,\tau_\omega(a))}{da} &= -\frac{\left(1-\lambda a-\lambda a\frac{\tau_\theta}{y\tau_\eta}\right)\frac{\tau_\omega(a)}{\tau_\omega(a)+\tau_\theta}\left(\lambda\frac{y\tau_\eta+\tau_\theta}{y\tau_\eta}\frac{\tau_\omega(a)}{\tau_\omega(a)+\tau_\theta}\right)}{(\beta+\lambda)\,\tau_\omega(a)}+\frac{\lambda a}{y\tau_e}+H(a)\\ &+\frac{\lambda^2 a+\lambda\left(1-\lambda a-\lambda a\frac{\tau_\theta}{y\tau_\eta}\right)\frac{\tau_\omega(a)}{\tau_\omega(a)+\tau_\theta}-\lambda^2 a\frac{y\tau_\eta+\tau_\theta}{y\tau_\eta}\frac{\tau_\omega(a)}{\tau_\omega(a)+\tau_\theta}}{(\beta+\lambda)\,y\tau_\eta}\\ &+\frac{\left[1-\lambda a-\left(1-\lambda a-\lambda a\frac{\tau_\theta}{y\tau_\eta}\right)\frac{\tau_\omega(a)}{\tau_\omega(a)+\tau_\theta}\right]\left(-\lambda+\lambda\left(\frac{y\tau_\eta+\tau_\theta}{y\tau_\eta}\right)\frac{\tau_\omega(a)}{\tau_\omega(a)+\tau_\theta}\right)}{(\beta+\lambda)\,\tau_\theta}\end{aligned}$$

where

$$\begin{aligned}H(a)\equiv\frac{\beta^2 a y^2\tau_\eta^2\tau_u}{(\beta^2a^2\tau_u+y\tau_\eta)^2}&\left\{\frac{\left(1-\lambda a-\lambda a\frac{\tau_\theta}{y\tau_\eta}\right)^2}{(\beta+\lambda)}\frac{\tau_\theta-\tau_\omega(a)}{(\tau_\omega(a)+\tau_\theta)^3}+\frac{2\lambda a\left(1-\lambda a-\lambda a\frac{\tau_\theta}{y\tau_\eta}\right)}{(\beta+\lambda)\,y\tau_\eta}\frac{\tau_\theta}{(\tau_\omega(a)+\tau_\theta)^2}\right.\\ &\left.-\frac{2\left[1-\lambda a-\left(1-\lambda a-\lambda a\frac{\tau_\theta}{y\tau_\eta}\right)\frac{\tau_\omega(a)}{\tau_\omega(a)+\tau_\theta}\right]}{(\beta+\lambda)\,\tau_\theta}\left(1-\lambda a-\lambda a\frac{\tau_\theta}{y\tau_\eta}\right)\frac{\tau_\theta}{(\tau_\omega(a)+\tau_\theta)^2}\right\}.\end{aligned}$$

Hence, the first-order-condition $dWL(a,\tau_\omega(a))/da=0$ is equivalent to

$$\begin{aligned}0 &= \lambda a\tau_\epsilon\left((y\tau_\eta+\tau_\theta)^2\frac{\tau_\omega(a)}{\tau_\omega(a)+\tau_\theta}\right)+\lambda a y\tau_\eta\tau_\epsilon\left(\tau_\omega(a)+\tau_\theta\right)-2\lambda a\tau_\epsilon\left(y\tau_\eta+\tau_\theta\right)\tau_\omega(a)\\ &+\lambda a\tau_\epsilon\frac{(\tau_\omega(a)+\tau_\theta)}{\tau_\theta}\left(y\tau_\eta-(y\tau_\eta+\tau_\theta)\frac{\tau_\omega(a)}{\tau_\omega(a)+\tau_\theta}\right)^2+\lambda a y\tau_\eta\tau_\epsilon\frac{y\tau_\eta\left(\tau_\omega(a)+\tau_\theta\right)(\beta+\lambda)}{\lambda y\tau_e}\\ &+y\tau_\eta\tau_\epsilon\frac{(\beta+\lambda)\left(\tau_\omega(a)+\tau_\theta\right)y\tau_\eta H(a)}{\lambda}-y\tau_\eta\tau_\epsilon\left(y\tau_\eta-\tau_\omega(a)\right),\end{aligned}$$

from which we obtain that

$$\begin{aligned}y\tau_\eta\tau_\epsilon\left(y\tau_\eta-\tau_\omega(a)\right) &= \lambda a\left\{y^2\tau_\eta^2\tau_\epsilon-\tau_\omega(a)\tau_\epsilon\left(\tau_\theta+2y\tau_\eta\right)+\left(\tau_\omega(a)+\tau_\theta\right)y^2\tau_\eta^2\right.\\ &\left.+y\tau_\eta\tau_\epsilon\frac{y\tau_\eta\left(\tau_\omega(a)+\tau_\theta\right)\beta}{\lambda y\tau_e}+y\tau_\eta\tau_\epsilon\frac{(\beta+\lambda)\left(\tau_\omega(a)+\tau_\theta\right)y\tau_\eta H(a)}{\lambda^2 a}\right\}.\end{aligned}$$

Using the definitions of the $\Lambda(\cdot)$, $\Delta(\cdot)$, and $\Xi(\cdot)$ functions in the main text, we then have that that a^T must solve

$$a^T=\frac{1}{\lambda}\frac{1}{\Lambda(\tau_\omega(a))+\Xi(a)+\Delta(a)}.$$

It is straightforward to verify that

$$\begin{aligned}\left.\frac{dWL(a,\tau_\omega(a))}{da}\right|_{a=\frac{1}{\lambda}} &= \frac{\lambda\tau_\theta}{(\beta+\lambda)\,y\tau_\eta(\tau_\omega(a)+\tau_\theta)}\frac{y\tau_\eta}{\beta^2a^2\tau_u+y\tau_\eta}\\ &\left(1-\frac{\beta^2a^2\tau_u}{(\beta^2a^2\tau_u+y\tau_\eta)}\times\frac{\tau_\theta}{(\tau_\omega(a)+\tau_\theta)}\right)+\frac{\lambda a}{y\tau_e}>0,\end{aligned}$$

and that

$$\left.\frac{dWL(a,\tau_\omega(a))}{da}\right|_{a=0} = \frac{\frac{\tau_\omega(a)}{\tau_\omega(a)+\tau_\theta}\left(-\lambda\frac{y\tau_\eta+\tau_\theta}{y\tau_\eta}\frac{\tau_\omega(a)}{\tau_\omega(a)+\tau_\theta}\right)}{(\beta+\lambda)\,\tau_\omega(a)} + \frac{\lambda\left(\frac{\tau_\omega(a)}{\tau_\omega(a)+\tau_\theta}\right)}{(\beta+\lambda)\,y\tau_\eta}$$
$$+\frac{\left(1-\frac{\tau_\omega(a)}{\tau_\omega(a)+\tau_\theta}\right)\left(-\lambda+\lambda\left(\frac{y\tau_\eta+\tau_\theta}{y\tau_\eta}\right)\frac{\tau_\omega(a)}{\tau_\omega(a)+\tau_\theta}\right)}{(\beta+\lambda)\,\tau_\theta}$$
$$\propto \frac{\tau_\omega(a)}{y\tau_\eta}-1=-\frac{y\tau_\eta}{\beta^2a^2\tau_u+y\tau_\eta}<0,$$

which implies that $0 < a^T < 1/\lambda$, as claimed in the proposition. Q.E.D.

Optimal sensitivity to private information when agents do not learn from prices.

In the cursed economy, each trader receives a private signal $s_i = \theta + \underbrace{f(y)\eta + f(y)e_i}_{\equiv\epsilon_i}$ and a public signal $z = \theta + \underbrace{f(y)\eta + \chi}_{\equiv\zeta}$, and believes p to be orthogonal to $\left(\theta, \eta, (e_i)_{i=0}^{i=1}\right)$. Following steps similar to those leading to Proposition 1 in the main text, we have that $\mathbb{E}[\theta|s_i, z] = \bar{\gamma}_1 s_i + \bar{\gamma}_2 z$, where

$$\bar{\gamma}_1 \equiv \frac{\tau_\epsilon y\tau_\eta\,(y\tau_\eta - \tau_\zeta)}{y^2\tau_\eta^2(\tau_\zeta+\tau_\epsilon+\tau_\theta)-\tau_\zeta\tau_\epsilon(\tau_\theta+2y\tau_\eta)}$$

and

$$\bar{\gamma}_2 \equiv \frac{y\tau_\eta\tau_\zeta\,(y\tau_\eta-\tau_\epsilon)}{y^2\tau_\eta^2(\tau_\zeta+\tau_\epsilon+\tau_\theta)-\tau_\epsilon\tau_\zeta(\tau_\theta+2y\tau_\eta)} = \left(1-\bar{\gamma}_1\frac{\tau_\theta+y\tau_\eta}{y\tau_\eta}\right)\frac{\tau_\zeta}{\tau_\zeta+\tau_\theta}.$$

Observe that the cursed-equilibrium demand schedules must satisfy

$$x_i \;=\; \frac{1}{\lambda}\left(\mathbb{E}[\theta|s_i,z]-p\right). \tag{21}$$

Now let $x_i = a^*_{exo}s_i + \hat{b}^*_{exo} - \hat{c}^*_{exo}p + \hat{d}^*_{exo}z$ denote the cursed-equilibrium demand schedules. From the derivations above, we have that $a^*_{exo} = \bar{\gamma}_1/\lambda$, $\hat{b}^*_{exo} = 0$, $\hat{c}^*_{exo} = 1/\lambda$, and $\hat{d}^*_{exo} = \bar{\gamma}_2/\lambda$. Using the formula for $\bar{\gamma}_1$ above we have that the formula for a^*_{exo} is equivalent to

$$a^*_{exo} \;=\; \frac{1}{\lambda\Lambda(\tau_\zeta)}, \tag{22}$$

as claimed in the main text.

Now suppose that, given a, the planner is constrained to choose $(\hat{b}, \hat{c}, \hat{d})$ to maintain the same relationship between a and $(\hat{b}, \hat{c}, \hat{d})$ as between a^*_{exo} and $(\hat{b}^*_{exo}, \hat{c}^*_{exo}, \hat{d}^*_{exo})$ in the cursed equilibrium. Using the fact that

$$\bar{\gamma}_2 = \left(1-\bar{\gamma}_1\frac{\tau_\theta+y\tau_\eta}{y\tau_\eta}\right)\frac{\tau_\zeta}{\tau_\zeta+\tau_\theta},$$

and the fact that $\bar{\gamma}_1 = \lambda a^*_{exo}$, we have that, in the cursed equilibrium, the relationship between a^*_{exo}

and $(\hat{b}^*_{exo}, \hat{c}^*_{exo}, \hat{d}^*_{exo})$ is given by $\hat{b}^*_{exo} = 0$, $\hat{c}^*_{exo} = 1/\lambda$, and

$$\hat{d}^*_{exo} = \frac{1}{\lambda}\left(1 - \lambda a^*_{exo}\frac{\tau_\theta + y\tau_\eta}{y\tau_\eta}\right)\frac{\tau_\zeta}{\tau_\zeta + \tau_\theta},$$

The above properties imply that, in the cursed economy, for any choice of a, the planner is constrained to select demand schedules of the form

$$x_i = \frac{1}{\lambda}\left(\lambda a s_i + \left(1 - \frac{\lambda a\left(\tau_\theta + y\tau_\eta\right)}{y\tau_\eta}\right)\frac{\tau_\zeta}{\tau_\zeta + \tau_\theta}z - p\right). \tag{23}$$

The planner then chooses a to minimize the welfare losses

$$WL = \frac{(\beta + \lambda)}{2}\mathbb{E}[(\tilde{x} - x^o)^2] + \frac{\lambda}{2}\mathbb{E}[(x_i - \tilde{x})^2]$$

under the the above demand schedules, taking into account the market-clearing condition.

Following steps similar to those in the baseline economy, and using the market-clearing condition, we have that, when the traders' demand schedules are given by (23),

$$\begin{aligned}\frac{(\beta + \lambda)}{2}\mathbb{E}[(\tilde{x} - x^o)^2] = &\frac{\left(\left(1 - \frac{\lambda a(y\tau_\eta + \tau_\theta)}{y\tau_\eta}\right)\frac{\tau_\zeta}{\tau_\zeta + \tau_\theta}\right)^2}{(\beta + \lambda)^2\tau_\zeta} + \frac{\lambda^2 a^2 + 2\lambda a\left(1 - \frac{\lambda a(y\tau_\eta + \tau_\theta)}{y\tau_\eta}\right)\frac{\tau_\zeta}{\tau_\zeta + \tau_\theta}}{(\beta + \lambda)^2 y\tau_\eta} \\ &+ \frac{\left(1 - \lambda a - \left(1 - \frac{\lambda a(y\tau_\eta + \tau_\theta)}{y\tau_\eta}\right)\frac{\tau_\zeta}{\tau_\zeta + \tau_\theta}\right)^2}{(\beta + \lambda)^2\tau_\theta}\end{aligned}$$

and

$$\frac{\lambda\mathbb{E}[(x_i - \tilde{x})^2]}{2} = \frac{\lambda a^2}{2y\tau_e}.$$

This means that, for any a, the welfare losses are equal to

$$\begin{aligned}WL = &\frac{\left[\left(1 - \frac{\lambda a(y\tau_\eta + \tau_\theta)}{y\tau_\eta}\right)\frac{\tau_\zeta}{\tau_\zeta + \tau_\theta}\right]^2}{2(\beta + \lambda)\tau_\zeta} + \frac{\lambda^2 a^2 + 2\lambda a\left(1 - \frac{\lambda a(y\tau_\eta + \tau_\theta)}{y\tau_\eta}\right)\frac{\tau_\zeta}{\tau_\zeta + \tau_\theta}}{2(\beta + \lambda)y\tau_\eta} \\ &+ \frac{\left[1 - \lambda a - \left(1 - \frac{\lambda a(y\tau_\eta + \tau_\theta)}{y\tau_\eta}\right)\frac{\tau_\zeta}{\tau_\zeta + \tau_\theta}\right]^2}{2(\beta + \lambda)\tau_\theta} + \frac{\lambda a^2}{2y\tau_e}.\end{aligned}$$

Following steps similar to those in the proof of Proposition 2, we then have that the value of a that minimizes the above welfare losses is equal to

$$a^T_{exo} = \frac{1}{\lambda}\frac{1}{\Lambda(\tau_\zeta) + \frac{\tau_\eta\beta(\tau_\zeta + \tau_\theta)}{\lambda\tau_e(y\tau_\eta - \tau_\zeta)}}$$

as claimed in the main text. Q.E.D.

Proof of Proposition 3.

We start by establishing the first two equalities. Observe that the function F given, for all a, by

$$\mathrm{F}(a) = a - \frac{1}{\lambda\Lambda(\tau_\omega(a))}$$

is strictly increasing. To see this, recall that, for any τ_ω, $\Lambda(\tau_\omega) = 1/\gamma_1(\tau_\omega)$. Then note that

$$\gamma_1(\tau_\omega) = \frac{\tau_\epsilon y\tau_\eta\left(y\tau_\eta - \tau_\omega\right)}{y^2\tau_\eta^2\left(\tau_\omega + \tau_\epsilon + \tau_\theta\right) - \tau_\omega\tau_\epsilon\left(\tau_\theta + 2y\tau_\eta\right)}$$

is decreasing in τ_ω if and only if $\tau_\eta y > \tau_\epsilon$. Because

$$\tau_\epsilon \equiv \frac{\tau_e}{\tau_e + \tau_\eta}\tau_\eta y,$$

we have that $\gamma_1(\tau_\omega)$ is decreasing in τ_ω. Because $\tau_\omega(a)$ is increasing in a, we conclude that F is strictly increasing.

Next, let F^T be the function given, for any a, by

$$\mathrm{F}^T(a) = a - \frac{1}{\lambda}\frac{1}{\Lambda(\tau_\omega(a)) + \Delta(a) + \Xi(a)}.$$

Because Δ and Ξ are both increasing, F^T is strictly increasing.

The first two equalities follow from the above monotonicities along with the fact that a^* solves $\mathrm{F}(a^*) = 0$ whereas a^T solves $\mathrm{F}^T(a^T) = 0$. Indeed, when $\Delta(a^T) + \Xi(a^T) > 0$,

$$\mathrm{F}(a^T) = \frac{1}{\lambda}\frac{1}{\Lambda(\tau_\omega(a^T)) + \Delta(a^T) + \Xi(a^T)} - \frac{1}{\lambda\Lambda(\tau_\omega(a^T))} < 0,$$

implying that $a^* > a^T$. If $a^* > a^T$, then

$$\frac{1}{\lambda\Lambda(\tau_\omega(a^*))} > \frac{1}{\lambda}\frac{1}{\Lambda(\tau_\omega(a^T)) + \Delta(a^T) + \Xi(a^T)} > \frac{1}{\lambda}\frac{1}{\Lambda(\tau_\omega(a^*)) + \Delta(a^*) + \Xi(a^*)}$$

which implies that $\Delta(a^*) + \Xi(a^*) > 0$. That $\Delta(a^*) + \Xi(a^*) > 0$ in turn implies that

$$\mathrm{F}^T(a^*) = \frac{1}{\lambda\Lambda(\tau_\omega(a^*))} - \frac{1}{\lambda}\frac{1}{\Lambda(\tau_\omega(a^*)) + \Delta(a^*) + \Xi(a^*)} > 0$$

which implies that $a^* > a^T$. Finally, that $a^* > a^T$ implies that

$$\mathrm{F}(a^T) = \frac{1}{\lambda}\frac{1}{\Lambda(\tau_\omega(a^T)) + \Delta(a^T) + \Xi(a^T)} - \frac{1}{\lambda\Lambda(\tau_\omega(a^T))} < 0,$$

which implies that $\Delta(a^T) + \Xi(a^T) > 0$. Replicating the arguments above for the case in which the inequalities are reversed then permits us to establish that

$$a^* - a^T \overset{sgn}{=} \Xi(a^T) + \Delta(a^T) \overset{sgn}{=} \Xi(a^*) + \Delta(a^*).$$

Next, consider the last two equalities in the proposition. In the proof of Lemma 1, we established that, for any sensitivity a of the efficient trades to private information, the sensitivity of the efficient trades to the endogenous signal z contained in the market-clearing price is given by

$$c = \frac{1}{\beta+\lambda}\left[\left(1-\lambda a-\lambda a\frac{\tau_\theta}{y\tau_\eta}\right)\frac{\tau_\omega(a)}{\tau_\omega(a)+\tau_\theta}-\beta a\right]$$

and coincides with the sensitivity of the equilibrium trades to z when the sensitivity of the equilibrium trades to private information is a. Using the formula for $\tau_\omega(a)$, we then have that $a+c>0$. Now use Condition (5) to observe that

$$\hat{c} = -\frac{c}{\beta(a+c)}. \tag{24}$$

Because $a+c>0$, we conclude that $sgn(\hat{c}) = -sgn(c)$. Combining this property with Condition (14), we conclude that

$$\hat{c} \overset{sgn}{=} \beta a - \left(1-\lambda a-\lambda a\frac{\tau_\theta}{y\tau_\eta}\right)\frac{\tau_\omega(a)}{\tau_\omega(a)+\tau_\theta}.$$

Next observe that

$$\Delta(a)+\Xi(a) = \frac{\beta\tau_\eta}{\lambda\left(y\tau_\eta-\tau_\omega(a)\right)}\left(\frac{\tau_\omega(a)+\tau_\theta}{\tau_e} - \frac{\beta y^3\tau_\eta^2\tau_u\left(1-\lambda a-\lambda a\frac{\tau_\theta}{y\tau_\eta}\right)^2}{\lambda\left(\beta^2a^2\tau_u+y\tau_\eta\right)^2\left(\tau_\omega(a)+\tau_\theta\right)}\right). \tag{25}$$

Because $y\tau_\eta-\tau_\omega(a)>0$,

$$\Delta(a)+\Xi(a) \overset{sgn}{=} \lambda\left(\beta^2a^2\tau_u+y\tau_\eta\right)^2\left(\tau_\omega(a)+\tau_\theta\right)^2 - \tau_e\beta y^3\tau_\eta^2\tau_u\left(1-\lambda a-\lambda a\frac{\tau_\theta}{y\tau_\eta}\right)^2.$$

It is then easy to see that $\Delta(a)+\Xi(a) \overset{sgn}{=} \hat{c}$. The above derivations hold no matter whether a is the sensitivity of the equilibrium schedules (equivalently, trades) to private information, or the sensitivity of the efficient schedules (equivalently, trades) to private information. Hence, $\hat{c}^* \overset{sgn}{=} \Xi(a^*)+\Delta(a^*)$ and $\hat{c}^T \overset{sgn}{=} \Xi(a^T)+\Delta(a^T)$. Because $\Xi(a^*)+\Delta(a^*) \overset{sgn}{=} \Xi(a^T)+\Delta(a^T)$, we then have that

$$\Xi(a^*)+\Delta(a^*) \overset{sgn}{=} \hat{c}^* \overset{sgn}{=} \hat{c}^T.$$

Q.E.D.

Proof of Proposition 4.

Under the proposed policy, each trader's demand schedule must satisfy the optimality condition

$$X_i(p;I_i) = \frac{1}{\lambda+\delta}\left(\mathbb{E}[\theta|I_i,p]-(1+t_p)p+t_0\right).$$

For any vector $(a,\hat{b},\hat{c})$, when all traders submit affine demand schedules $x_i = as_i+\hat{b}-\hat{c}p$, the equilibrium price then continues to satisfy the same representation as in (1) but with $(a^*,\hat{b}^*,\hat{c}^*)$ replaced by $(a,\hat{b},\hat{c})$. This also means that the equilibrium trades can be expressed as a function of the endogenous public signal z, as in the laissez-faire equilibrium with no policy. Letting $x_i = as_i+b+cz$ denote the trades generated by the demand schedules $x_i = as_i+\hat{b}-\hat{c}p$ (with z representing the endogenous public signal contained in the market-clearing price), we then have that the functions that map the coefficients $\hat{c}$ and $\hat{b}$ in the demand schedules into the coefficients c and b in the induced trades continue to be given by (5) and (4). Using the fact that $\mathbb{E}[\theta|s_i, z] = \gamma_1(\tau_\omega(a))s_i + \gamma_2(\tau_\omega(a))z$, with the functions $\gamma_1(\cdot)$ and $\gamma_2(\cdot)$ as defined in (7) and (8), along with the fact that the market-clearing price satisfies $p = \alpha + \beta b + \beta(a+c)z$ as shown in (6), we then have that the equilibrium trades must satisfy

$$\begin{aligned} x_i &= \frac{1}{\lambda+\delta}\left[\gamma_1(\tau_\omega(a))s_i + \gamma_2(\tau_\omega(a))z - (1+t_p)\alpha - (1+t_p)\beta b - (1+t_p)\beta(a+c)z + t_0\right] \\ &= \frac{1}{\lambda+\delta}\left\{\gamma_1(\tau_\omega(a))s_i - (1+t_p)\left(\alpha+\beta b\right) + \left[\gamma_2(\tau_\omega(a)) - (1+t_p)\beta(a+c)\right]z + t_0\right\}. \end{aligned}$$

The sensitivity of the equilibrium trades to private information s_i under the proposed policy thus satisfies $a = \gamma_1(\tau_\omega(a))/(\lambda+\gamma)$. Using the formula for γ_1 in (7), we then have that the equilibrium value of a under the proposed policy is the unique solution to the following equation:

$$a = \frac{1}{\lambda+\delta}\frac{\tau_\epsilon y^2\tau_\eta^2 - \tau_\omega(a)\tau_\epsilon y\tau_\eta}{y^2\tau_\eta^2(\tau_\omega(a)+\tau_\epsilon+\tau_\theta) - \tau_\omega(a)\tau_\epsilon\left(\tau_\theta + 2y\tau_\eta\right)},$$

Using the fact that, for ant τ_ω,

$$\Lambda(\tau_\omega) \equiv \frac{y^2\tau_\eta^2(\tau_\omega+\tau_\epsilon+\tau_\theta) - \tau_\omega\tau_\epsilon\left(\tau_\theta+2y\tau_\eta\right)}{\tau_\epsilon y\tau_\eta(y\tau_\eta - \tau_\omega)},$$

we thus have that the equilibrium value of a is given by

$$a = \frac{1}{\lambda+\delta}\frac{1}{\Lambda(\tau_\omega(a))}.$$

The equilibrium value of b is given by the unique solution to

$$b = \frac{-(1+t_p)\left(\alpha+\beta b\right) + t_0}{\lambda+\delta}$$

which is equal to

$$b = \frac{t_0 - (1+t_p)\alpha}{\lambda+\delta+(1+t_p)\beta}.$$

The equilibrium value of c, instead, is given by the unique solution to

$$c = \frac{1}{\lambda+\delta}\left[\gamma_2(\tau_\omega(a)) - (1+t_p)\beta(a+c)\right]$$

which is equal to

$$c = \frac{\gamma_2(\tau_\omega(a)) - (1+t_p)\beta a}{\lambda+\delta+(1+t_p)\beta}.$$

Now recall that the sensitivity a^T of the efficient trades to private information is given by the unique solution to

$$a = \frac{1}{\lambda}\frac{1}{\Lambda(\tau_\omega(a)) + \Xi(a) + \Delta(a)}.$$

Therefore, the equilibrium value a under the proposed policy coincides with the efficient level a^T if and only if δ satisfies

$$\begin{gathered}(\lambda+\delta)\Lambda(\tau_\omega(a^T)) \\ = \lambda\left[\Lambda(\tau_\omega(a^T)) + \Xi(a^T) + \Delta(a^T)\right],\end{gathered}$$

from which we obtain that

$$\delta \quad = \quad \frac{\lambda\left[\Xi(a^T)+\Delta(a^T)\right]}{\Lambda(\tau_\omega(a^T))}.$$

Now recall that, given a^T, the other two coefficients c^T and b^T describing the efficient trades are given by the functions in (14) and (15), implying that

$$c^T = \frac{1}{\beta+\lambda}\left(\left(1-\lambda a^T-\lambda a^T\frac{\tau_\theta}{y\tau_\eta}\right)\frac{\tau_\omega(a^T)}{\tau_\omega(a^T)+\tau_\theta}-\beta a^T\right)$$

and $b^T = -\alpha/(\beta+\lambda)$. Hence, for the equilibrium levels of c and b under the proposed policy to coincide with the efficient levels it must be that

$$\frac{\gamma_2(\tau_\omega(a^T))-(1+t_p)\beta a^T}{\lambda+\delta+(1+t_p)\beta} = \frac{1}{\beta+\lambda}\left(\left(1-\lambda a^T-\lambda a^T\frac{\tau_\theta}{y\tau_\eta}\right)\frac{\tau_\omega(a^T)}{\tau_\omega(a^T)+\tau_\theta}-\beta a^T\right)$$

and

$$\frac{t_0-(1+t_p)\alpha}{\lambda+\delta+(1+t_p)\beta} = -\frac{\alpha}{\beta+\lambda}.$$

It is easy to see that the above two equations are satisfied when

$$t_p = \frac{\gamma_2(\tau_\omega(a^T))-\frac{\lambda+\delta+\beta}{\beta+\lambda}\left[\left(1-\lambda a-\lambda a\frac{\tau_\theta}{y\tau_\eta}\right)\frac{\tau_\omega(a)}{\tau_\omega(a)+\tau_\theta}-\beta a\right]-\beta a^T}{\beta\left\{\frac{1}{\beta+\lambda}\left[\left(1-\lambda a-\lambda a\frac{\tau_\theta}{y\tau_\eta}\right)\frac{\tau_\omega(a)}{\tau_\omega(a)+\tau_\theta}-\beta a\right]+a^T\right\}}$$

and

$$t_0 = (1+t_p)\alpha-\frac{\alpha\left[\lambda+\delta+(1+t_p)\beta\right]}{\beta+\lambda}.$$

Q.E.D.

Proof of Proposition 5.

Given $I_i = (y_i, s_i)$, trader i's demand schedule maximizes, for each price p, the trader's expected payoff

$$\mathbb{E}\left[\left(\theta-(1+t_p)p\right)x_i-\lambda\frac{x_i^2}{2}|I_i,p\right].$$

The solution to this problem is the demand schedule given by

$$X(p;I_i) = \frac{1}{\lambda}(\mathbb{E}[\theta|I_i,p]-(1+t_p)p), \tag{26}$$

where, as in the laissez-faire equilibrium, $\mathbb{E}[\theta|I_i,p]$ denotes the trader's expectation of θ given I_i and p.

In any symmetric equilibrium in which the price is an affine function of (θ,u,η), the equilibrium trades continue to be given by

$$x_i = as_i + b + cz \tag{27}$$

for some scalars (a,b,c) that may depend on the level of the tax t_p and on the quality $y_i = y$ of the agents' information.

When the individual trades are given by (27), the aggregate trade is equal to

$$\tilde{x} = (a+c)z + \frac{u}{\beta} + b,$$

where we used the fact that $z + u/(\beta a) = \theta + f(y)\eta$. Replacing $\tilde{x}$ into the expression for the inverse aggregate supply function, we then have that the equilibrium price

$$p = \alpha + \beta b + \beta(a+c)z \tag{28}$$

can be expressed as a function of (a, b, c) and the endogenous public signal z, as in the laissez-fare equilibrium. Furthermore,

$$\mathbb{E}[\theta|I_i, p] = \gamma_1(\tau_\omega(a))s_i + \gamma_2(\tau_\omega(a))z, \tag{29}$$

with $\gamma_1(\cdot)$ and $\gamma_2(\cdot)$ given by (7) and (8), respectively. Combining (26) with (28) and (29), we thus have that the equilibrium trades satisfy

$$x_i \;=\; \frac{1}{\lambda}\left[\gamma_1(\tau_\omega(a))s_i - (1+t_p)(\alpha+\beta b) + (\gamma_2(\tau_\omega(a)) - (1+t_p)\beta(a+c))\, z\right]. \tag{30}$$

We conclude that the sensitivity of the equilibrium trades to private information satisfies

$$a = \frac{\gamma_1(\tau_\omega(a))}{\lambda}. \tag{31}$$

That is, no matter the value of t_p, the equilibrium level of a is given by a^*, as in the laissez-fare economy in which $t_p = 0$. Furthermore, combining (30) with (31) and using (8), we have that the equilibrium sensitivity of the trades to the endogenous public signal is given by

$$c = \frac{1}{\beta(1+t_p)+\lambda}\left[\left(1 - \lambda a\frac{\tau_\theta + y\tau_\eta}{y\tau_\eta}\right)\frac{\tau_\omega(a)}{\tau_\omega(a)+\tau_\theta} - (1+t_p)\beta a\right], \tag{32}$$

whereas the constant b in the equilibrium trades is given by

$$b = -(1+t_p)\frac{\alpha}{(1+t_p)\beta+\lambda}. \tag{33}$$

Hence, any ad-valorem tax $t_p \neq 0$ induces the same sensitivity a^* of the equilibrium trades to private information as in the laissez-faire equilibrium in which $t_p = 0$ but different values of b and c. Because, given a^*, the values of b and c (equivalently, of $\hat{b}$ and $\hat{c}$) in the laissez-fare economy maximize welfare, as shown in Lemma 1, we conclude that any policy $t_p \neq 0$ results in strictly lower welfare than $t_p = 0$. Q.E.D.

Proof of Proposition 6.

The proof is in four steps. Step 1 shows that, for any $y \in [0, +\infty)$, when all other agents acquire information of quality y and submit the equilibrium limit orders for information of quality y, each agent's net private marginal benefit $N(y)$ of increasing the quality of his information at $y_i = y$ (and then trade optimally) is a strictly decreasing function of y. Step 2 uses the result in step 1 to show that, when $\mathcal{C}'(0)$ is small enough, there is one, and only one, value of y for which $N(y) = 0$. Step 3 shows that, when the cost of information is sufficiently convex, then if all other agents acquire information of quality y^* (where y^* is the unique solution to $N(y) = 0$) and then submit the equilibrium limit orders for information of quality y^*, the payoff $V^{\#}(y^*, y_i)$ that each agent obtains by acquiring information of quality y_i and then trading optimally is strictly quasi-concave in y_i. Jointly, the above properties establish the claim in the proposition.

Step 1. First observe that, when all other agents acquire information of quality y and then submit the equilibrium limit orders for information of quality y, the maximal payoff that agent i can obtain by acquiring information of quality y_i and then trading optimally is given by

$$V^{\#}(y, y_i) \equiv \sup_{g(\cdot)} \left\{ \mathbb{E}[\pi_i^{\#}(y, y_i; g(\cdot))] - \mathcal{C}(y_i) \right\}$$

with

$$\mathbb{E}[\pi_i^{\#}(y, y_i; g(\cdot))] \equiv \mathbb{E}\left[\theta g(s_i, z) - (\alpha + \beta b + \beta(a+c)z)\, g(s_i, z) - \frac{\lambda}{2}\left(g(s_i, z)\right)^2; y_i\right],$$

where g is an arbitrary (measurable) function of the agent's private signal s_i and the public signal $z \equiv \theta + f(y)\eta - u/(\beta a)$ contained in the equilibrium price, with noise $\omega \equiv f(y)\eta - u/\beta a$ of precision $\tau_\omega(a) \equiv \beta^2(a)^2 y\tau_u\tau_\eta / \left(\beta^2(a)^2\tau_u + y\tau_\eta\right)$, describing the amount of the good traded by agent i under the limit orders he submits. Note that, in writing $\mathbb{E}[\pi_i^{\#}(y, y_i; g(\cdot))]$, we used the fact that the relationship between z and the equilibrium price is given by $p = \alpha + \beta b + \beta(a+c)z$, where (a, b, c) are the coefficients describing the equilibrium trades when the quality of information is y and all agents submit the equilibrium limit orders for information of quality y. Also note that the dependence of $\mathbb{E}[\pi_i^{\#}(y, y_i; g(\cdot))]$ on y_i is through the fact that the agent's private signal is given by $s_i = \theta + f(y_i)(\eta + e_i)$. Using the envelope theorem, we then have that

$$N(y) \equiv \left.\frac{\partial V^{\#}(y, y_i)}{\partial y_i}\right|_{y_i = y} = \frac{(\beta + \lambda)(a+c)\,a}{2\tau_\eta y^2} + \frac{\lambda a^2}{2y^2\tau_e} - \mathcal{C}'(y). \tag{34}$$

Next, use Conditions (3) and (14) to verify that $N(y) = F(a, y) - \mathcal{C}'(y)$, where, for any (a, y),

$$F(a, y) \equiv \frac{1}{2}a^2 \frac{a^2\beta^2\lambda\tau_u\tau_\theta + y\left[\lambda a^2\beta^2\tau_u\tau_\eta + \lambda(\tau_e + \tau_\eta)\tau_\theta + \beta^2\tau_e\tau_u a\right]}{y^2\tau_e\left[y\tau_\theta\tau_\eta + a^2\beta^2\tau_u\left(\tau_\theta + y\tau_\eta\right)\right]}. \tag{35}$$

As shown in the proof of Proposition 1, the equilibrium value of a (given y) is given by the unique real root to the cubic equation in (18). Equivalently, letting $Z \equiv a/y$ and

$$R(Z, y) \equiv Z^3 y\beta^2\lambda\tau_u\left(\tau_\theta + y\tau_\eta\right) + Z\lambda\left(\tau_e\tau_\theta + \tau_\theta\tau_\eta + y\tau_e\tau_\eta\right) - \tau_e\tau_\eta,$$

we have, for any y, the equilibrium level of Z is given by the unique positive real solution to the equation $R(Z, y) = 0$, and is such that $Z < \tau_e/\lambda\tau_y$. Furthermore,

$$\frac{\partial}{\partial y}R(Z, y) = Z\lambda\left(\tau_e\tau_\eta + Z^2\beta^2\tau_u\tau_\theta + 2yZ^2\beta^2\tau_u\tau_\eta\right) > 0.$$

Now let $Z^*(y)$ be the equilibrium value of Z, given y. From the Implicit Function Theorem, we thus have that $Z^*(y)$ is decreasing in y.

Next, let $G(y) \equiv F(Z^*(y)y, y)$, where $F(a, y)$ is the function defined in Condition (35) above, and where we used the fact $a = Z^*(y)y$.

Now use the fact that the equilibrium value of a is given by

$$a = y^2 \tau_e \frac{\tau_\eta}{\lambda \left(y \left(\tau_e \tau_\theta + \tau_\theta \tau_\eta + y \tau_e \tau_\eta \right) + a^2 \beta^2 \tau_u \left(\tau_\theta + y \tau_\eta \right) \right)}$$

or, equivalently,

$$a^3 = \frac{y^2 \tau_e \tau_\eta - a y \lambda \left(\tau_e \tau_\theta + \tau_\theta \tau_\eta + y \tau_e \tau_\eta \right)}{\beta^2 \lambda \tau_u \left(\tau_\theta + y \tau_\eta \right)}$$

to express the function $F(a, y)$ as follows:

$$F(a, y) = -\frac{1}{2} \left(\tau_e + a \lambda \tau_\eta \right) \frac{-y \tau_\eta + a \lambda \tau_\theta + a y \lambda \tau_\eta}{\left(y \tau_\theta \tau_\eta + a^2 \beta^2 \tau_u \left(\tau_\theta + y \tau_\eta \right) \right) \lambda \left(\tau_\theta + y \tau_\eta \right)}.$$

The latter expression can be simplified to

$$F(a, y) = \frac{1}{2} \frac{a \left(\tau_e + a \lambda \tau_\eta \right)}{y \tau_e \left(\tau_\theta + y \tau_\eta \right)}.$$

We thus have that

$$G(y) = \frac{1}{2} Z^*(y) \frac{\tau_e + y Z^*(y) \lambda \tau_\eta}{\tau_e \left(\tau_\theta + y \tau_\eta \right)}.$$

Note that

$$\frac{dG(y)}{dy} = \frac{1}{2} Z^*(y) \tau_\eta \frac{-\tau_e + Z^*(y) \lambda \tau_\theta}{\tau_e \left(\tau_\theta + y \tau_\eta \right)^2} + \frac{1}{2} \frac{\tau_e + 2 y Z^*(y) \lambda \tau_\eta}{\tau_e \left(\tau_\theta + y \tau_\eta \right)} \frac{dZ^*(y)}{dy} < 0,$$

where the inequality follows from the fact that $Z^*(y) < \tau_e / \lambda \tau_y$ and $dZ^*(y)/dy < 0$. Because $N(y) = G(y) - \mathcal{C}'(y)$, we conclude that $N(y)$ is a strictly decreasing function of y.

Step 2. Next, consider the limit properties of $N(y)$. Because

$$\lim_{y \to 0} Z^*(y) = \frac{\tau_e \tau_\eta}{\lambda \tau_\theta \left(\tau_e + \tau_\eta \right)},$$

we have that

$$\lim_{y \to 0} G(y) = \frac{1}{2} \frac{\tau_e \tau_\eta}{\lambda \tau_\theta^2 \left(\tau_e + \tau_\eta \right)},$$

and hence

$$\lim_{y \to 0} N(y) = \frac{1}{2} \frac{\tau_e \tau_\eta}{\lambda \tau_\theta^2 \left(\tau_e + \tau_\eta \right)} - \mathcal{C}'(0).$$

Furthermore,

$$\lim_{y \to \infty} N(y) = \lim_{y \to \infty} G(y) - \lim_{y \to \infty} \mathcal{C}'(y).$$

Because $\lim_{y \to \infty} Z^*(y) = 0$, we have that $\lim_{y \to \infty} G(y) = 0$. Hence,

$$\lim_{y \to \infty} N(y) = -\lim_{y \to \infty} \mathcal{C}'(y) < 0.$$

Letting

$$L \equiv \frac{1}{2}\frac{\tau_e \tau_\eta}{\lambda \tau_\theta^2 \left(\tau_e + \tau_\eta\right)},$$

we conclude that, when $\mathcal{C}'(0) < L$, there exists one, and only one, value of y for which $N(y) = 0$.

Step 3. Assume $\mathcal{C}'(0) < L$ and let y^* be the unique solution to $N(y) = 0$. Suppose that all other agents acquire information of quality y^* and then submit the equilibrium limit orders for information of quality y^*. Let (a^*, b^*, c^*) denote the coefficients describing the equilibrium trades under the equilibrium limit orders for information of quality y^* (these coefficients are given by Conditions (12), (15), and (14), applied to $y = y^*$). Let $\tau_\omega^* = \tau_\omega(a^*)$ denote the precision of the endogenous signal $z \equiv \theta + f(y^*)\eta - u/(\beta a^*)$ contained in the equilibrium price when all other agents acquire information of quality y^* and then submit the equilibrium limit orders for information of quality y^*.

We show that, when $\mathcal{C}$ is sufficiently convex, $V^{\#}(y^*, y_i)$ is strictly quasi-concave in y_i. To see this, first recall that optimality requires that, for any y_i, any (s_i, p), the trades that the agent induces through his limit orders given (s_i, p) are equal to

$$x_i = \frac{1}{\lambda}\left(\mathbb{E}[\theta | s_i, p; y_i] - p\right).$$

Equivalently, for any y_i, the function $g^*(\cdot; y_i)$ that maximizes the agent's payoff $\mathbb{E}[\pi_i^{\#}(y^*, y_i; g(\cdot))] - \mathcal{C}(y_i)$ is such that, for any (s_i, z),

$$g^*(s_i, z; y_i) = \frac{1}{\lambda}\left(\mathbb{E}[\theta | s_i, z; y_i] - (\alpha + \beta b^*) - \beta(a^* + c^*)z\right).$$

Observe that

$$\begin{aligned}
\mathbb{E}[\theta | s_i, z; y_i] &= \begin{bmatrix} Cov(\theta, s_i; y_i) & Cov(\theta, z; y_i) \end{bmatrix} \times \\
&\quad \begin{bmatrix} Var(s_i; y_i) & Cov(s_i, z; y_i) \\ Cov(s_i, z : y_i) & Var(z; y_i) \end{bmatrix}^{-1} \begin{bmatrix} s_i - \mathbb{E}[s_i; y_i] \\ z - \mathbb{E}[z; y_i] \end{bmatrix} \\
&= \begin{bmatrix} \sigma_\theta^2 & \sigma_\theta^2 \end{bmatrix} \begin{bmatrix} \sigma_\theta^2 + \sigma_\epsilon^2(y_i) & \sigma_\theta^2 + f(y)f(y_i)\sigma_\eta^2 \\ \sigma_\theta^2 + f(y)f(y_i)\sigma_\eta^2 & \sigma_\theta^2 + \sigma_\omega^2 \end{bmatrix}^{-1} \begin{bmatrix} s_i - \mathbb{E}[s_i] \\ z - \mathbb{E}[z] \end{bmatrix},
\end{aligned}$$

where $\sigma_\epsilon^2(y_i) \equiv \tau_\epsilon^{-1}(y_i)$. Substituting for the inverse of the variance-covariance matrix, and using the fact that, for any y_i, $\mathbb{E}[s_i; y_i] = \mathbb{E}[z; y_i] = 0$, we have that

$$\begin{aligned}
\mathbb{E}[\theta | s_i, z; y_i] &= \frac{1}{(\sigma_\theta^2 + \sigma_\epsilon^2(y_i))(\sigma_\theta^2 + \sigma_\omega^2) - (\sigma_\theta^2 + f(y^*)f(y_i)\sigma_\eta^2)^2} \times \\
&\quad \begin{bmatrix} \sigma_\theta^2 & \sigma_\theta^2 \end{bmatrix} \begin{bmatrix} \sigma_\theta^2 + \sigma_\omega^2 & -(\sigma_\theta^2 + f(y^*)f(y_i)\sigma_\eta^2) \\ -(\sigma_\theta^2 + f(y^*)f(y_i)\sigma_\eta^2) & \sigma_\theta^2 + \sigma_\epsilon^2(y_i) \end{bmatrix} \begin{bmatrix} s_i \\ z \end{bmatrix}.
\end{aligned}$$

Expanding the quadratic form, we have that

$$\begin{aligned}\mathbb{E}[\theta|s_i,z;y_i] &= \frac{\sigma_\theta^2\left(\sigma_\omega^2 - f(y^*)f(y_i)\sigma_\eta^2\right)}{(\sigma_\theta^2+\sigma_\epsilon^2(y_i))(\sigma_\theta^2+\sigma_\omega^2)-(\sigma_\theta^2+f(y^*)f(y_i)\sigma_\eta^2)^2}s_i \\ &\quad + \frac{\sigma_\theta^2\left(\sigma_\epsilon^2 - f(y^*)f(y_i)\sigma_\eta^2\right)}{(\sigma_\theta^2+\sigma_\epsilon^2(y_i))(\sigma_\theta^2+\sigma_\omega^2)-(\sigma_\theta^2+f(y^*)f(y_i)\sigma_\eta^2)^2}z.\end{aligned}$$

Simplifying, and using the fact that $\sigma_\theta^2 \equiv \tau_\theta^{-1}$, $\sigma_\omega^2 \equiv (\tau_\omega^*)^{-1}$, $\sigma_\eta^2 \equiv \tau_\eta^{-1}$, $(\sigma_\epsilon(y_i))^2 \equiv (\tau_\epsilon(y_i))^{-1}$, we have that

$$\begin{aligned}\mathbb{E}[\theta|s_i,z;y_i] &= \frac{\frac{1}{\tau_\theta}\left(\frac{1}{\tau_\omega^*} - \frac{f(y^*)f(y_i)}{\tau_\eta}\right)}{(\frac{1}{\tau_\theta}+\frac{1}{\tau_\epsilon(y_i)})(\frac{1}{\tau_\theta}+\frac{1}{\tau_\omega^*})-(\frac{1}{\tau_\theta}+\frac{f(y^*)f(y_i)}{\tau_\eta})^2}s_i \\ &\quad + \frac{\frac{1}{\tau_\theta}\left(\frac{1}{\tau_\epsilon(y_i)} - \frac{f(y^*)f(y_i)}{\tau_\eta}\right)}{(\frac{1}{\tau_\theta}+\frac{1}{\tau_\epsilon(y_i)})(\frac{1}{\tau_\theta}+\frac{1}{\tau_\omega^*})-(\frac{1}{\tau_\theta}+\frac{f(y^*)f(y_i)}{\tau_\eta})^2}z,\end{aligned}$$

or, equivalently,

$$\begin{aligned}\mathbb{E}[\theta|s_i,z;y_i] &= \frac{\tau_\epsilon(y_i)\tau_\eta\left(\tau_\eta y^* y_i - \frac{\tau_\omega^*}{f(y^*)f(y_i)}\right)}{\tau_\eta^2 y^* y_i(\tau_\epsilon(y_i)+\tau_\omega^*+\tau_\theta) - \tau_\omega^*\tau_\epsilon(y_i)\left(\frac{2\tau_\eta}{f(y^*)f(y_i)}+\tau_\theta\right)}s_i \\ &\quad + \frac{\tau_\omega^*\left(\tau_\eta^2 y^* y_i - \frac{\tau_\epsilon(y_i)\tau_\eta}{f(y^*)f(y_i)}\right)}{\tau_\eta^2 y^* y_i(\tau_\epsilon(y_i)+\tau_\omega^*+\tau_\theta) - \tau_\omega^*\tau_\epsilon(y_i)\left(\frac{2\tau_\eta}{f(y^*)f(y_i)}+\tau_\theta\right)}z.\end{aligned}$$

Using the fact that $\tau_\epsilon(y_i) \equiv \tau_e\tau_\eta y_i/(\tau_e+\tau_\eta)$, $f(y^*) = 1/\sqrt{y^*}$, and $f(y_i) = 1/\sqrt{y_i}$, we conclude that

$$\mathbb{E}[\theta|s_i,z;y_i] = \tilde{\gamma}_1(y_i)s_i + \tilde{\gamma}_2(y_i)z$$

where

$$\tilde{\gamma}_1(y_i) \equiv \frac{\tau_e\tau_\eta\sqrt{y^*y_i}\left(\tau_\eta\sqrt{y^*y_i} - \tau_\omega^*\right)}{\tau_\eta y^*\left[\tau_e\tau_\eta y_i + (\tau_\omega^*+\tau_\theta)(\tau_e+\tau_\eta)\right] - \tau_\omega^*\tau_e\left(2\tau_\eta\sqrt{y^*y_i}+\tau_\theta\right)} \tag{36}$$

and

$$\tilde{\gamma}_2(y_i) \equiv \frac{\tau_\omega^*\tau_\eta\left[(\tau_e+\tau_\eta)y^* - \tau_e\sqrt{y^*y_i}\right]}{\tau_\eta y^*\left[\tau_e\tau_\eta y_i + (\tau_\omega^*+\tau_\theta)(\tau_e+\tau_\eta)\right] - \tau_\omega^*\tau_e\left(2\tau_\eta\sqrt{y^*y_i}+\tau_\theta\right)}. \tag{37}$$

In other words, for any y_i, the function $g^*(\cdot;y_i)$ is given by $g^*(s_i,z;y_i) = \tilde{a}(y_i)s_i + \tilde{b}(y_i) + \tilde{c}(y_i)z$, with $\tilde{a}(y_i) \equiv \tilde{\gamma}_1(y_i)/\lambda$, $\tilde{b}(y_i) \equiv -(\alpha+\beta b^*)/\lambda$, and $\tilde{c}(y_i) \equiv [\tilde{\gamma}_2(y_i) - \beta(a^*+c^*)]/\lambda$.

Now note that, given any affine strategy $g(s_i,z) = As_i + B + Cz$, where A, B, C are scalars,

$$\begin{aligned}\mathbb{E}[\pi_i^{\#}(y^*,y_i;g(\cdot))] &= \mathbb{E}\left[\theta\left(As_i+B+Cz\right)|y_i\right] \\ &\quad -\mathbb{E}\left[(\alpha+\beta b^*+\beta(a^*+c^*)z)\left(As_i+B+Cz\right)|y_i\right] \\ &\quad -\mathbb{E}\left[\tfrac{\lambda}{2}\left(As_i+B+Cz\right)^2|y_i\right].\end{aligned}$$

Hence, fixing the affine strategy $g(s_i,z) = As_i + B + Cz$, and using the fact that

$$s_i = \theta + f(y_i)(\eta + e_i)$$

and

$$z \equiv \theta + f(y^*)\eta - \frac{u}{\beta a^*}$$

we have that

$$\frac{\partial \mathbb{E}[\pi_i^{\#}(y^*, y_i; g(\cdot))]}{\partial y_i} = -A\left[\beta(a^* + c^*) + \lambda C\right] \frac{\partial}{\partial y_i} \mathbb{E}\left[s_i z | y_i\right] - \frac{\lambda}{2} A^2 \frac{\partial}{\partial y_i} \mathbb{E}\left[s_i^2 | y_i\right],$$

where

$$\frac{\partial}{\partial y_i} \mathbb{E}\left[s_i z | y_i\right] = f'(y_i) f(y^*) \frac{1}{\tau_\eta}$$

and

$$\frac{\partial}{\partial y_i} \mathbb{E}\left[s_i^2 | y_i\right] = 2 f(y_i) f'(y_i) \left(\frac{1}{\tau_\eta} + \frac{1}{\tau_e}\right).$$

Using the Envelope Theorem, we thus have that

$$\frac{\partial V^{\#}(y^*, y_i)}{\partial y_i} = \frac{\partial \mathbb{E}[\pi_i^{\#}(y^*, y_i; g^*(\cdot; y_i))]}{\partial y_i} - \mathcal{C}'(y_i)$$

with

$$\frac{\partial \mathbb{E}[\pi_i^{\#}(y^*, y_i; g(\cdot; y_i))]}{\partial y_i} = -\tilde{a}(y_i)\left[\beta(a^* + c^*) + \lambda \tilde{c}(y_i)\right] f'(y_i) f(y^*) \frac{1}{\tau_\eta} - \lambda \tilde{a}(y_i)^2 f(y_i) f'(y_i) \left(\frac{1}{\tau_\eta} + \frac{1}{\tau_e}\right).$$

Observe that

$$\beta(a^* + c^*) + \lambda \tilde{c}(y_i) = \tilde{\gamma}_2(y_i).$$

It follows that

$$\frac{\partial V^{\#}(y^*, y_i)}{\partial y_i} = -\tilde{a}(y_i)\tilde{\gamma}_2(y_i) f'(y_i) f(y^*) \frac{1}{\tau_\eta} - \lambda \tilde{a}(y_i)^2 f(y_i) f'(y_i) \left(\frac{1}{\tau_\eta} + \frac{1}{\tau_e}\right) - \mathcal{C}'(y_i).$$

Next, observe that

$$\begin{aligned}
\frac{\partial^2 V^{\#}(y^*, y_i)}{\partial y_i^2} = \; & -\tilde{a}'(y_i)\tilde{\gamma}_2(y_i) f'(y_i) f(y^*) \frac{1}{\tau_\eta} \\
& -\tilde{a}(y_i) \frac{d\tilde{\gamma}_2(y_i)}{dy_i} f'(y_i) f(y^*) \frac{1}{\tau_\eta} - \tilde{a}(y_i)\tilde{\gamma}_2(y_i) f''(y_i) f(y^*) \frac{1}{\tau_\eta} \\
& -2\lambda \tilde{a}(y_i)\tilde{a}'(y_i) f(y_i) f'(y_i) \left(\frac{1}{\tau_\eta} + \frac{1}{\tau_e}\right) \\
& -\lambda \tilde{a}(y_i)^2 \left(f'(y_i)\right)^2 \left(\frac{1}{\tau_\eta} + \frac{1}{\tau_e}\right) - \lambda \tilde{a}(y_i)^2 f(y_i) f''(y_i) \left(\frac{1}{\tau_\eta} + \frac{1}{\tau_e}\right) - \mathcal{C}''(y_i).
\end{aligned}$$

We thus have that, at any y_i at which $\partial V^{\#}(y^*, y_i)/\partial y_i = 0$,

$$\begin{aligned}
\frac{\partial^2 V^{\#}(y^*, y_i)}{\partial y_i^2} &= -\tilde{a}'(y_i)\tilde{\gamma}_2(y_i)f'(y_i)f(y)\frac{1}{\tau_\eta} \\
&\quad -\tilde{a}(y_i)\frac{d\tilde{\gamma}_2(y_i)}{dy_i}f'(y_i)f(y)\frac{1}{\tau_\eta} + \frac{f''(y_i)}{f'(y_i)}\left[\mathcal{C}'(y_i)\right] \\
&\quad -2\lambda\tilde{a}(y_i)\tilde{a}'(y_i)f(y_i)f'(y_i)\left(\frac{1}{\tau_\eta} + \frac{1}{\tau_e}\right) \\
&\quad -\lambda\tilde{a}(y_i)^2\left(f'(y_i)\right)^2\left(\frac{1}{\tau_\eta} + \frac{1}{\tau_e}\right) - \mathcal{C}''(y_i).
\end{aligned}$$

The above can be rewritten as

$$\begin{aligned}
\frac{\partial^2 V^{\#}(y^*, y_i)}{\partial y_i^2} &= \left(-f'(y_i)f(y)\frac{1}{\tau_\eta}\right)\frac{\partial}{\partial y_i}\left\{\tilde{a}(y_i)\tilde{\gamma}_2(y_i)\right\} \\
&\quad \left(-f'(y_i)\left(\frac{1}{\tau_\eta} + \frac{1}{\tau_e}\right)\right)\frac{\partial}{\partial y_i}\left\{\lambda\tilde{a}(y_i)^2 f(y_i)\right\} \\
&\quad +\frac{f''(y_i)}{f'(y_i)}\left[\mathcal{C}'(y_i)\right] - \mathcal{C}''(y_i).
\end{aligned}$$

Using the fact that $\tilde{a}(y_i) = \tilde{\gamma}_1(y_i)/\lambda$, we have that, at any point y_i at which $\partial V^{\#}(y^*, y_i)/\partial y_i = 0$,

$$\begin{aligned}
\frac{\partial^2 V^{\#}(y^*, y_i)}{\partial y_i^2} &= \underbrace{\frac{1}{\lambda}\left(-f'(y_i)f(y)\frac{1}{\tau_\eta}\right)}_{>0}\frac{d}{dy_i}\left\{\tilde{\gamma}_1(y_i)\tilde{\gamma}_2(y_i)\right\} \\
&\quad +\underbrace{\frac{1}{\lambda}\left(-f'(y_i)\left(\frac{1}{\tau_\eta} + \frac{1}{\tau_e}\right)\right)}_{>0}\frac{d}{dy_i}\left\{\left(\tilde{\gamma}_1(y_i)\right)^2 f(y_i)\right\} \\
&\quad +\underbrace{\frac{f''(y_i)}{f'(y_i)}\mathcal{C}'(y_i)}_{<0} - \mathcal{C}''(y_i). \qquad (38)
\end{aligned}$$

Using the fact that $f(y) = 1/\sqrt{y}$ and letting $J : \mathbb{R}_+ \to \mathbb{R}$ be the function defined by

$$J(y_i) \equiv \frac{1}{\lambda}\left(\frac{1}{2y_i\sqrt{y_i y^*}}\frac{1}{\tau_\eta}\right)\frac{d}{dy_i}\left\{\tilde{\gamma}_1(y_i)\tilde{\gamma}_2(y_i)\right\} + \frac{1}{\lambda}\left[\frac{1}{2y_i\sqrt{y_i}}\left(\frac{1}{\tau_\eta} + \frac{1}{\tau_e}\right)\right]\frac{d}{dy_i}\left\{\left(\tilde{\gamma}_1(y_i)\right)^2\frac{1}{\sqrt{y_i}}\right\},$$

we thus have that, at any point y_i at which $\partial V^{\#}(y^*, y_i)/\partial y_i = 0$,

$$\frac{\partial^2 V^{\#}(y^*, y_i)}{\partial y_i^2} = J(y_i) - \frac{3}{2y_i}\mathcal{C}'(y_i) - \mathcal{C}''(y_i).$$

Now observe that

$$
\begin{aligned}
J(y_i) &= \frac{1}{\lambda}\left(\frac{1}{2y_i\sqrt{y_i y^*}}\frac{1}{\tau_\eta}\right)\left\{\tilde{\gamma}_1'(y_i)\tilde{\gamma}_2(y_i)+\tilde{\gamma}_1(y_i)\tilde{\gamma}_2'(y_i)\right\} \\
&\quad +\frac{1}{\lambda}\left[\frac{1}{2y_i\sqrt{y_i}}\left(\frac{1}{\tau_\eta}+\frac{1}{\tau_e}\right)\right]\left\{2\tilde{\gamma}_1(y_i)\tilde{\gamma}_1'(y_i)\frac{1}{\sqrt{y_i}}-\left(\tilde{\gamma}_1(y_i)\right)^2\frac{1}{2y_i\sqrt{y_i}}\right\} \\
&= \frac{1}{\lambda}\frac{1}{2y_i\sqrt{y_i}}\frac{1}{\sqrt{y^*}}\frac{1}{\tau_\eta}\left\{\tilde{\gamma}_1'(y_i)\tilde{\gamma}_2(y_i)+\tilde{\gamma}_1(y_i)\tilde{\gamma}_2'(y_i)\right\} \\
&\quad +\frac{1}{\lambda}\frac{1}{2y_i\sqrt{y_i}}\frac{1}{\sqrt{y_i}}\left(\frac{1}{\tau_\eta}+\frac{1}{\tau_e}\right)\left\{2\tilde{\gamma}_1(y_i)\tilde{\gamma}_1'(y_i)-\left(\tilde{\gamma}_1(y_i)\right)^2\frac{1}{2y_i}\right\}.
\end{aligned}
$$

Next observe that

$$
\begin{aligned}
\tilde{\gamma}_1'(y_i) &= \left\{\tau_\eta y^*\left[\tau_e\tau_\eta y_i+\left(\tau_\omega^*+\tau_\theta\right)\left(\tau_e+\tau_\eta\right)\right]-\tau_\omega^*\tau_e\left(2\tau_\eta\sqrt{y^*y_i}+\tau_\theta\right)\right\}^{-2}\times \\
&\quad \times\left\{\left(\tau_\eta y^*\left[\tau_e\tau_\eta y_i+\left(\tau_\omega^*+\tau_\theta\right)\left(\tau_e+\tau_\eta\right)\right]-\tau_\omega^*\tau_e\left(2\tau_\eta\sqrt{y^*y_i}+\tau_\theta\right)\right)\times\right. \\
&\quad \frac{d}{dy_i}\left(\tau_e\tau_\eta\sqrt{y^*y_i}\left(\tau_\eta\sqrt{y^*y_i}-\tau_\omega^*\right)\right) \\
&\quad -\frac{d}{dy_i}\left(\tau_\eta y^*\left[\tau_e\tau_\eta y_i+\left(\tau_\omega^*+\tau_\theta\right)\left(\tau_e+\tau_\eta\right)\right]-\tau_\omega^*\tau_e\left(2\tau_\eta\sqrt{y^*y_i}+\tau_\theta\right)\right)\times \\
&\quad \left.\left(\tau_e\tau_\eta\sqrt{y^*y_i}\left(\tau_\eta\sqrt{y^*y_i}-\tau_\omega^*\right)\right)\right\}.
\end{aligned}
$$

Expanding the derivatives,

$$
\begin{aligned}
\tilde{\gamma}_1'(y_i) &= \left\{\tau_\eta y^*\left[\tau_e\tau_\eta y_i+\left(\tau_\omega^*+\tau_\theta\right)\left(\tau_e+\tau_\eta\right)\right]-\tau_\omega^*\tau_e\left(2\tau_\eta\sqrt{y^*y_i}+\tau_\theta\right)\right\}^{-2}\times \\
&\quad \times\left\{\left(\tau_\eta y^*\left[\tau_e\tau_\eta y_i+\left(\tau_\omega^*+\tau_\theta\right)\left(\tau_e+\tau_\eta\right)\right]-\tau_\omega^*\tau_e\left(2\tau_\eta\sqrt{y^*y_i}+\tau_\theta\right)\right)\times\right. \\
&\quad \left[\tau_e\tau_\eta\sqrt{y^*}\left(\tau_\eta\sqrt{y^*}-\frac{1}{2}\tau_\omega\frac{1}{\sqrt{y_i}}\right)\right] \\
&\quad \left.-\left[\tau_\eta y^*\tau_e\tau_\eta-\tau_\omega^*\tau_e\left(2\tau_\eta\sqrt{y^*}\frac{1}{2\sqrt{y_i}}\right)\right]\left(\tau_e\tau_\eta\sqrt{y^*y_i}\left(\tau_\eta\sqrt{y^*y_i}-\tau_\omega^*\right)\right)\right\}.
\end{aligned}
$$

Simplifying, we have that

$$
\begin{aligned}
\tilde{\gamma}_1'(y_i) &= \frac{\tau_e\tau_\eta\sqrt{y^*}\left(\tau_\eta\sqrt{y^*}-\frac{1}{2}\tau_\omega^*\frac{1}{\sqrt{y_i}}\right)}{\tau_\eta y^*\left[\tau_e\tau_\eta y_i+\left(\tau_\omega^*+\tau_\theta\right)\left(\tau_e+\tau_\eta\right)\right]-\tau_\omega^*\tau_e\left(2\tau_\eta\sqrt{y^*y_i}+\tau_\theta\right)} \\
&\quad -\frac{y^*\tau_\eta-\tau_\omega^*\frac{\sqrt{y^*}}{\sqrt{y_i}}}{y^*\left[\tau_\eta y_i+\left(\tau_\omega^*+\tau_\theta\right)\left(1+\frac{\tau_\eta}{\tau_e}\right)\right]-\tau_\omega^*\left(2\sqrt{y^*y_i}+\frac{\tau_\theta}{\tau_\eta}\right)}\tilde{\gamma}_1(y_i).
\end{aligned}
$$

Simplifying further,

$$
\tilde{\gamma}_1'(y_i) = \frac{\tau_\eta y^*-\tau_\omega^*\frac{\sqrt{y^*}}{2\sqrt{y_i}}}{y^*\left[\tau_\eta y_i+\left(\tau_\omega^*+\tau_\theta\right)\left(1+\frac{\tau_\eta}{\tau_e}\right)\right]-\tau_\omega^*\left(2\sqrt{y^*y_i}+\frac{\tau_\theta}{\tau_\eta}\right)}-\frac{\tilde{\gamma}_1(y_i)^2}{y_i}
$$

Using again the fact that

$$
\frac{\tilde{\gamma}_1(y_i)}{y_i} = \frac{\tau_\eta y^*-\tau_\omega^*\frac{\sqrt{y^*}}{\sqrt{y_i}}}{y^*\left[\tau_\eta y_i+\left(\tau_\omega^*+\tau_\theta\right)\left(1+\frac{\tau_\eta}{\tau_e}\right)\right]-\tau_\omega^*\left(2\sqrt{y^*y_i}+\frac{\tau_\theta}{\tau_\eta}\right)},
$$

we have that

$$\begin{aligned}\tilde{\gamma}_1'(y_i) &= \frac{\tau_\eta y^* - \tau_\omega^* \frac{\sqrt{y^*}}{2\sqrt{y_i}}}{\tau_\eta y^* - \tau_\omega^* \frac{\sqrt{y^*}}{\sqrt{y_i}}} \frac{\gamma_1(y_i)}{y_i} - \frac{\tilde{\gamma}_1(y_i)^2}{y_i} \\ &= \left(\frac{\tau_\eta \sqrt{y^* y_i} - \frac{\tau_\omega^*}{2}}{\tau_\eta \sqrt{y^* y_i} - \tau_\omega^*} - \tilde{\gamma}_1(y_i) \right) \frac{\tilde{\gamma}_1(y_i)}{y_i}\end{aligned}$$

Similarly,

$$\begin{aligned}\tilde{\gamma}_2'(y_i) &= \frac{-\frac{1}{2}\tau_\omega^* \tau_e \tau_\eta \frac{\sqrt{y^*}}{\sqrt{y_i}}}{\tau_\eta y^* \left[\tau_e \tau_\eta y_i + (\tau_\omega^* + \tau_\theta)(\tau_e + \tau_\eta)\right] - \tau_\omega^* \tau_e \left(2\tau_\eta \sqrt{y^* y_i} + \tau_\theta\right)} \\ &\quad - \frac{\tau_\eta^2 y^* \tau_e - \tau_\omega^* \tau_e \tau_\eta \frac{\sqrt{y^*}}{\sqrt{y_i}}}{\tau_\eta y^* \left[\tau_e \tau_\eta y_i + (\tau_\omega^* + \tau_\theta)(\tau_e + \tau_\eta)\right] - \tau_\omega^* \tau_e \left(2\tau_\eta \sqrt{y^* y_i} + \tau_\theta\right)} \tilde{\gamma}_2(y_i).\end{aligned}$$

Simplifying, we have that

$$\begin{aligned}\tilde{\gamma}_2'(y_i) &= \frac{-\frac{1}{2}\tau_\omega^* \frac{\sqrt{y^*}}{\sqrt{y_i}}}{y^* \left[\tau_\eta y_i + (\tau_\omega^* + \tau_\theta)\left(1 + \frac{\tau_\eta}{\tau_e}\right)\right] - \tau_\omega^* \left(2\sqrt{y^* y_i} + \frac{\tau_\theta}{\tau_\eta}\right)} \\ &\quad - \frac{\tau_\eta y^* - \tau_\omega^* \frac{\sqrt{y^*}}{\sqrt{y_i}}}{y^* \left[\tau_\eta y_i + (\tau_\omega^* + \tau_\theta)\left(1 + \frac{\tau_\eta}{\tau_e}\right)\right] - \tau_\omega^* \left(2\sqrt{y^* y_i} + \frac{\tau_\theta}{\tau_\eta}\right)} \tilde{\gamma}_2(y_i).\end{aligned}$$

Using the fact that

$$\frac{\tilde{\gamma}_1(y_i)}{y_i} = \frac{\tau_\eta y^* - \tau_\omega^* \frac{\sqrt{y^*}}{\sqrt{y_i}}}{y^* \left[\tau_\eta y_i + (\tau_\omega^* + \tau_\theta)\left(1 + \frac{\tau_\eta}{\tau_e}\right)\right] - \tau_\omega^* \left(2\sqrt{y^* y_i} + \frac{\tau_\theta}{\tau_\eta}\right)},$$

we have that

$$\tilde{\gamma}_2'(y_i) = \frac{-\frac{1}{2}\tau_\omega^* \frac{\sqrt{y^*}}{\sqrt{y_i}}}{\tau_\eta y^* - \tau_\omega^* \frac{\sqrt{y^*}}{\sqrt{y_i}}} \frac{\tilde{\gamma}_1(y_i)}{y_i} - \frac{\tilde{\gamma}_1(y_i)}{y_i} \tilde{\gamma}_2(y_i)$$

and hence

$$\begin{aligned}\tilde{\gamma}_2'(y_i) &= \frac{-\frac{\tau_\omega^*}{2}}{\tau_\eta \sqrt{y^* y_i} - \tau_\omega^*} \frac{\tilde{\gamma}_1(y_i)}{y_i} - \frac{\tilde{\gamma}_1(y_i)}{y_i} \tilde{\gamma}_2(y_i) \\ &= -\left(\tilde{\gamma}_2(y_i) + \frac{\frac{\tau_\omega^*}{2}}{\tau_\eta \sqrt{y^* y_i} - \tau_\omega} \right) \frac{\tilde{\gamma}_1(y_i)}{y_i}.\end{aligned}$$

Replacing the above derivatives in the expression for J we obtain that

$$\begin{aligned}
J(y_i) &= \frac{1}{\lambda}\frac{1}{2y_i\sqrt{y_i}}\frac{1}{\sqrt{y^*}}\frac{1}{\tau_\eta}\times \\
&\left\{\left(\frac{\tau_\eta\sqrt{y^*y_i}-\frac{\tau_\omega^*}{2}}{\tau_\eta\sqrt{y^*y_i}-\tau_\omega^*}-\tilde{\gamma}_1(y_i)\right)\frac{\tilde{\gamma}_1(y_i)\tilde{\gamma}_2(y_i)}{y_i}-\left(\tilde{\gamma}_2(y_i)+\frac{\frac{\tau_\omega^*}{2}}{\tau_\eta\sqrt{y^*y_i}-\tau_\omega^*}\right)\frac{\tilde{\gamma}_1^2(y_i)}{y_i}\right\} \\
&+\frac{1}{\lambda}\frac{1}{2y_i\sqrt{y_i}}\frac{1}{\sqrt{y_i}}\left(\frac{1}{\tau_\eta}+\frac{1}{\tau_e}\right)\left\{2\left(\frac{\tau_\eta\sqrt{y^*y_i}-\frac{\tau_\omega^*}{2}}{\tau_\eta\sqrt{y^*y_i}-\tau_\omega^*}-\tilde{\gamma}_1(y_i)\right)\frac{\tilde{\gamma}_1^2(y_i)}{y_i}-\frac{\tilde{\gamma}_1^2(y_i)}{2y_i}\right\} \\
&= \frac{1}{\lambda}\frac{1}{2y_i\sqrt{y_i}}\frac{1}{\sqrt{y^*}}\frac{1}{\tau_\eta}\frac{\tilde{\gamma}_1(y_i)}{y_i}\times \\
&\left\{\left(\frac{\tau_\eta\sqrt{y^*y_i}-\frac{\tau_\omega^*}{2}}{\tau_\eta\sqrt{y^*y_i}-\tau_\omega^*}-\tilde{\gamma}_1(y_i)\right)\tilde{\gamma}_2(y_i)-\left(\tilde{\gamma}_2(y_i)+\frac{\frac{\tau_\omega^*}{2}}{\tau_\eta\sqrt{y^*y_i}-\tau_\omega^*}\right)\tilde{\gamma}_1(y_i)\right\} \\
&+\frac{1}{\lambda}\frac{1}{2y_i\sqrt{y_i}}\frac{1}{\sqrt{y_i}}\frac{1}{\tau_\eta}\frac{\tilde{\gamma}_1(y_i)}{y_i}\left(1+\frac{\tau_\eta}{\tau_e}\right)\left\{2\left(\frac{\tau_\eta\sqrt{y^*y_i}-\frac{\tau_\omega^*}{2}}{\tau_\eta\sqrt{y^*y_i}-\tau_\omega^*}-\tilde{\gamma}_1(y_i)\right)\tilde{\gamma}_1(y_i)-\frac{\tilde{\gamma}_1(y_i)}{2}\right\}.
\end{aligned}$$

Hence,

$$\begin{aligned}
J(y_i) &= \frac{1}{\lambda}\frac{1}{2y_i\sqrt{y_i}}\frac{1}{\sqrt{y^*}}\frac{1}{\tau_\eta}\frac{\tilde{\gamma}_1(y_i)}{y_i}\left\{\frac{\tau_\eta\sqrt{y^*y_i}-\frac{\tau_\omega^*}{2}}{\tau_\eta\sqrt{y^*y_i}-\tau_\omega^*}\tilde{\gamma}_2(y_i)-2\tilde{\gamma}_1(y_i)\tilde{\gamma}_2(y_i)-\frac{\frac{\tau_\omega^*}{2}}{\tau_\eta\sqrt{y^*y_i}-\tau_\omega^*}\tilde{\gamma}_1(y_i)\right\} \\
&+\frac{1}{\lambda}\frac{1}{2y_i\sqrt{y_i}}\frac{1}{\sqrt{y^*}}\frac{1}{\tau_\eta}\frac{\tilde{\gamma}_1(y_i)}{y_i}\left(1+\frac{\tau_\eta}{\tau_e}\right)\frac{\sqrt{y^*}}{\sqrt{y_i}}\left\{2\frac{\tau_\eta\sqrt{y^*y_i}-\frac{\tau_\omega^*}{2}}{\tau_\eta\sqrt{y^*y_i}-\tau_\omega^*}\tilde{\gamma}_1(y_i)-2\tilde{\gamma}_1^2(y_i)-\frac{\tilde{\gamma}_1(y_i)}{2}\right\}
\end{aligned}$$

In other terms,

$$J(y_i)=\tilde{H}(y_i)\tilde{R}(y_i)$$

where, for any y_i,

$$\tilde{H}(y_i)\equiv\frac{1}{\lambda}\frac{1}{2y_i\sqrt{y_i}}\frac{1}{\sqrt{y^*}}\frac{1}{\tau_\eta}\frac{\tilde{\gamma}_1(y_i)}{y_i}$$

and

$$\begin{aligned}
\tilde{R}(y_i) &\equiv \frac{\tau_\eta\sqrt{y^*y_i}-\frac{\tau_\omega^*}{2}}{\tau_\eta\sqrt{y^*y_i}-\tau_\omega^*}\tilde{\gamma}_2(y_i)-2\tilde{\gamma}_1(y_i)\tilde{\gamma}_2(y_i)-\frac{\frac{\tau_\omega^*}{2}}{\tau_\eta\sqrt{y^*y_i}-\tau_\omega^*}\tilde{\gamma}_1(y_i) \\
&+\left(1+\frac{\tau_\eta}{\tau_e}\right)\frac{\sqrt{y^*}}{\sqrt{y_i}}\left\{2\frac{\tau_\eta\sqrt{y^*y_i}-\frac{\tau_\omega^*}{2}}{\tau_\eta\sqrt{y^*y_i}-\tau_\omega^*}\tilde{\gamma}_1(y_i)-2\tilde{\gamma}_1^2(y_i)-\frac{\tilde{\gamma}_1(y_i)}{2}\right\}.
\end{aligned}$$

Now observe that

$$\tilde{\gamma}_1(y_i)=\frac{\tau_e\tau_\eta\sqrt{y^*y_i}\left(\tau_\eta\sqrt{y^*y_i}-\tau_\omega^*\right)}{D(y_i)}\text{ and }\tilde{\gamma}_2(y_i)=\frac{\tau_\omega^*\tau_\eta\left[\left(\tau_e+\tau_\eta\right)y^*-\tau_e\sqrt{y^*y_i}\right]}{D(y_i)},$$

where, for any y_i,

$$D(y_i)=\tau_\eta y^*\left[\tau_e\tau_\eta y_i+\left(\tau_\omega^*+\tau_\theta\right)\left(\tau_e+\tau_\eta\right)\right]-\tau_\omega^*\tau_e\left(2\tau_\eta\sqrt{y^*y_i}+\tau_\theta\right).$$

Hence

$$\tilde{\gamma}_2(y_i)=\frac{\tau_\omega^*\tau_\eta\left[\left(\tau_e+\tau_\eta\right)y^*-\tau_e\sqrt{y^*y_i}\right]}{\tau_e\tau_\eta\sqrt{y^*y_i}\left(\tau_\eta\sqrt{y^*y_i}-\tau_\omega^*\right)}\tilde{\gamma}_1(y_i)$$

Replacing these terms in the formula for $\tilde{R}$, we have that

$$\begin{aligned}
\tilde{R}(y_i) &= \left(\tau_\eta\sqrt{y^*y_i} - \frac{\tau_\omega^*}{2}\right)\frac{\tau_\omega^*\tau_\eta\left[\left(\tau_e+\tau_\eta\right)y^* - \tau_e\sqrt{y^*y_i}\right]}{\tau_e\tau_\eta\sqrt{y^*y_i}\left(\tau_\eta\sqrt{y^*y_i}-\tau_\omega^*\right)^2}\tilde{\gamma}_1(y_i) \\
&\quad -2\frac{\tau_\omega^*\tau_\eta\left[\left(\tau_e+\tau_\eta\right)y^* - \tau_e\sqrt{y^*y_i}\right]}{\tau_e\tau_\eta\sqrt{y^*y_i}\left(\tau_\eta\sqrt{y^*y_i}-\tau_\omega^*\right)}\tilde{\gamma}_1^2(y_i) - \frac{\frac{\tau_\omega^*}{2}}{\tau_\eta\sqrt{y^*y_i}-\tau_\omega^*}\tilde{\gamma}_1(y_i) \\
&\quad + \left(1+\frac{\tau_\eta}{\tau_e}\right)\frac{\sqrt{y^*}}{\sqrt{y_i}}\left\{\frac{2\tau_\eta\sqrt{y^*y_i}-\tau_\omega^*}{\tau_\eta\sqrt{y^*y_i}-\tau_\omega^*}\tilde{\gamma}_1(y_i) - 2\tilde{\gamma}_1^2(y_i) - \frac{\tilde{\gamma}_1(y_i)}{2}\right\}.
\end{aligned}$$

This means that

$$J(y_i) = \tilde{H}(y_i)\tilde{\gamma}(y_i)\tilde{W}(y_i)$$

where, for any y_i,

$$\begin{aligned}
\tilde{W}(y_i) &= \left(\tau_\eta\sqrt{y^*y_i} - \frac{\tau_\omega^*}{2}\right)\frac{\tau_\omega^*\tau_\eta\left[\left(\tau_e+\tau_\eta\right)y^* - \tau_e\sqrt{y^*y_i}\right]}{\tau_e\tau_\eta\sqrt{y^*y_i}\left(\tau_\eta\sqrt{y^*y_i}-\tau_\omega^*\right)^2} \\
&\quad -2\frac{\tau_\omega^*\tau_\eta\left[\left(\tau_e+\tau_\eta\right)y^* - \tau_e\sqrt{y^*y_i}\right]}{\tau_e\tau_\eta\sqrt{y^*y_i}\left(\tau_\eta\sqrt{y^*y_i}-\tau_\omega^*\right)}\frac{\tau_e\tau_\eta\sqrt{y^*y_i}\left(\tau_\eta\sqrt{y^*y_i}-\tau_\omega^*\right)}{D(y_i)} - \frac{\frac{\tau_\omega^*}{2}}{\tau_\eta\sqrt{y^*y_i}-\tau_\omega^*} \\
&\quad + \left(\frac{2\tau_\eta\sqrt{y^*y_i}-\tau_\omega^*}{\tau_\eta\sqrt{y^*y_i}-\tau_\omega^*}\right)\left(1+\frac{\tau_\eta}{\tau_e}\right)\frac{\sqrt{y^*}}{\sqrt{y_i}} \\
&\quad -2\frac{\tau_e\tau_\eta\sqrt{y^*y_i}\left(\tau_\eta\sqrt{y^*y_i}-\tau_\omega^*\right)}{D(y_i)}\left(1+\frac{\tau_\eta}{\tau_e}\right)\frac{\sqrt{y^*}}{\sqrt{y_i}} - \frac{1}{2}\left(1+\frac{\tau_\eta}{\tau_e}\right)\frac{\sqrt{y^*}}{\sqrt{y_i}}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\tilde{W}(y_i) &= \left(\tau_\eta\sqrt{y^*y_i} - \tau_\omega^* + \frac{\tau_\omega^*}{2}\right)\frac{\tau_\omega^*\left[\left(\tau_e+\tau_\eta\right)y^* - \tau_e\sqrt{y^*y_i}\right]}{\tau_e\sqrt{y^*y_i}\left(\tau_\eta\sqrt{y^*y_i}-\tau_\omega^*\right)^2} \\
&\quad -2\frac{\tau_\omega^*\tau_\eta\left[\left(\tau_e+\tau_\eta\right)y^* - \tau_e\sqrt{y^*y_i}\right]}{D(y_i)} - \frac{\frac{\tau_\omega^*}{2}}{\tau_\eta\sqrt{y^*y_i}-\tau_\omega^*} \\
&\quad + \left(\frac{2\tau_\eta\sqrt{y^*y_i}-\tau_\omega^*}{\tau_\eta\sqrt{y^*y_i}-\tau_\omega^*}\right)\left(1+\frac{\tau_\eta}{\tau_e}\right)\frac{\sqrt{y^*}}{\sqrt{y_i}} \\
&\quad -2\frac{\tau_\eta y^*\left(\tau_\eta\sqrt{y^*y_i}-\tau_\omega^*\right)}{D(y_i)}\left(\tau_e+\tau_\eta\right) - \frac{1}{2}\left(1+\frac{\tau_\eta}{\tau_e}\right)\frac{\sqrt{y^*}}{\sqrt{y_i}}.
\end{aligned}$$

Equivalently,

$$\begin{aligned}
\tilde{W}(y_i) &= \frac{\tau_\omega^*\left[\left(\tau_e+\tau_\eta\right)\sqrt{y^*} - \tau_e\sqrt{y_i}\right]}{\tau_e\sqrt{y_i}\left(\tau_\eta\sqrt{y^*y_i}-\tau_\omega^*\right)} \\
&\quad + \frac{\left(\tau_\omega^*\right)^2\left[\left(\tau_e+\tau_\eta\right)\sqrt{y^*} - \tau_e\sqrt{y_i}\right]}{2\tau_e\sqrt{y_i}\left(\tau_\eta\sqrt{y^*y_i}-\tau_\omega^*\right)^2} \\
&\quad - \frac{2\tau_\omega^*\tau_\eta\left[\left(\tau_e+\tau_\eta\right)y^* - \tau_e\sqrt{y^*y_i}\right]}{D(y_i)} - \frac{\frac{\tau_\omega^*}{2}}{\tau_\eta\sqrt{y^*y_i}-\tau_\omega^*} \\
&\quad + \left(1+\frac{\tau_\eta\sqrt{y^*y_i}}{\tau_\eta\sqrt{y^*y_i}-\tau_\omega^*}\right)\left(1+\frac{\tau_\eta}{\tau_e}\right)\frac{\sqrt{y^*}}{\sqrt{y_i}} \\
&\quad - \frac{2\tau_\eta y^*\left(\tau_\eta\sqrt{y^*y_i}-\tau_\omega^*\right)}{D(y_i)}\left(\tau_e+\tau_\eta\right) - \frac{1}{2}\left(1+\frac{\tau_\eta}{\tau_e}\right)\frac{\sqrt{y^*}}{\sqrt{y_i}}.
\end{aligned}$$

Simplifying further

$$
\begin{aligned}
\tilde{W}(y_i) \quad = \quad & \frac{\tau_\omega^* \left[(\tau_e+\tau_\eta)\sqrt{y^*}-\tau_e\sqrt{y_i}\right]}{\tau_e\sqrt{y_i}\left(\tau_\eta\sqrt{y^*y_i}-\tau_\omega^*\right)} \\
& + \frac{\left(\tau_\omega^*\right)^2 \left[(\tau_e+\tau_\eta)\sqrt{y^*}-\tau_e\sqrt{y_i}\right]}{2\tau_e\sqrt{y_i}\left(\tau_\eta\sqrt{y^*y_i}-\tau_\omega^*\right)^2} \\
& - \frac{2\tau_\omega^*\tau_\eta\left[(\tau_e+\tau_\eta)y^*-\tau_e\sqrt{y^*y_i}\right]}{D(y_i)} - \frac{\frac{\tau_\omega^*}{2}}{\tau_\eta\sqrt{y^*y_i}-\tau_\omega^*} \\
& + \left(1+\frac{\tau_\eta}{\tau_e}\right)\left(\frac{\tau_\eta y^*}{\tau_\eta\sqrt{y^*y_i}-\tau_\omega^*}\right) \\
& - \frac{2\tau_\eta y^*\left(\tau_\eta\sqrt{y^*y_i}-\tau_\omega^*\right)}{D(y_i)}(\tau_e+\tau_\eta) + \frac{1}{2}\left(1+\frac{\tau_\eta}{\tau_e}\right)\frac{\sqrt{y^*}}{\sqrt{y_i}}.
\end{aligned}
$$

Putting all pieces together we thus have that

$$
\begin{aligned}
J(y_i) \quad = \quad & \frac{\tau_\omega^* \left[(\tau_e+\tau_\eta)\sqrt{y^*}-\tau_e\sqrt{y_i}\right]}{\tau_e\sqrt{y_i}\left(\tau_\eta\sqrt{y^*y_i}-\tau_\omega^*\right)} \left\{\frac{1}{\lambda}\frac{1}{2y_i\sqrt{y_i}}\frac{1}{\sqrt{y^*}}\frac{1}{\tau_\eta}\frac{1}{y_i}\frac{\tau_e^2\tau_\eta^2 y^* y_i\left(\tau_\eta\sqrt{y^*y_i}-\tau_\omega^*\right)^2}{D^2(y_i)}\right\} \\
& + \frac{\left(\tau_\omega^*\right)^2 \left[(\tau_e+\tau_\eta)\sqrt{y^*}-\tau_e\sqrt{y_i}\right]}{2\tau_e\sqrt{y_i}\left(\tau_\eta\sqrt{y^*y_i}-\tau_\omega^*\right)^2} \left\{\frac{1}{\lambda}\frac{1}{2y_i\sqrt{y_i}}\frac{1}{\sqrt{y^*}}\frac{1}{\tau_\eta}\frac{1}{y_i}\frac{\tau_e^2\tau_\eta^2 y^* y_i\left(\tau_\eta\sqrt{y^*y_i}-\tau_\omega^*\right)^2}{D^2(y_i)}\right\} \\
& - \frac{2\tau_\omega^*\tau_\eta\left[(\tau_e+\tau_\eta)y^*-\tau_e\sqrt{y^*y_i}\right]}{D(y_i)} \left\{\frac{1}{\lambda}\frac{1}{2y_i\sqrt{y_i}}\frac{1}{\sqrt{y^*}}\frac{1}{\tau_\eta}\frac{1}{y_i}\frac{\tau_e^2\tau_\eta^2 y^* y_i\left(\tau_\eta\sqrt{y^*y_i}-\tau_\omega^*\right)^2}{D^2(y_i)}\right\} \\
& - \frac{\frac{\tau_\omega^*}{2}}{\tau_\eta\sqrt{y^*y_i}-\tau_\omega^*} \left\{\frac{1}{\lambda}\frac{1}{2y_i\sqrt{y_i}}\frac{1}{\sqrt{y^*}}\frac{1}{\tau_\eta}\frac{1}{y_i}\frac{\tau_e^2\tau_\eta^2 y^* y_i\left(\tau_\eta\sqrt{y^*y_i}-\tau_\omega^*\right)^2}{D^2(y_i)}\right\} \\
& + \left(1+\frac{\tau_\eta}{\tau_e}\right)\left(\frac{\tau_\eta y^*}{\tau_\eta\sqrt{y^*y_i}-\tau_\omega^*}\right) \left\{\frac{1}{\lambda}\frac{1}{2y_i\sqrt{y_i}}\frac{1}{\sqrt{y^*}}\frac{1}{\tau_\eta}\frac{1}{y_i}\frac{\tau_e^2\tau_\eta^2 y^* y_i\left(\tau_\eta\sqrt{y^*y_i}-\tau_\omega^*\right)^2}{D^2(y_i)}\right\} \\
& - \frac{2\tau_\eta y^*\left(\tau_\eta\sqrt{y^*y_i}-\tau_\omega^*\right)}{D(y_i)}(\tau_e+\tau_\eta) \left\{\frac{1}{\lambda}\frac{1}{2y_i\sqrt{y_i}}\frac{1}{\sqrt{y^*}}\frac{1}{\tau_\eta}\frac{1}{y_i}\frac{\tau_e^2\tau_\eta^2 y^* y_i\left(\tau_\eta\sqrt{y^*y_i}-\tau_\omega^*\right)^2}{D^2(y_i)}\right\} \\
& + \frac{1}{2}\left(1+\frac{\tau_\eta}{\tau_e}\right)\frac{\sqrt{y^*}}{\sqrt{y_i}} \left\{\frac{1}{\lambda}\frac{1}{2y_i\sqrt{y_i}}\frac{1}{\sqrt{y^*}}\frac{1}{\tau_\eta}\frac{1}{y_i}\frac{\tau_e^2\tau_\eta^2 y^* y_i\left(\tau_\eta\sqrt{y^*y_i}-\tau_\omega^*\right)^2}{D^2(y_i)}\right\}.
\end{aligned}
$$

Simplifying,

$$
\begin{aligned}
J(y_i) \quad = \quad & \frac{\tau_\omega^* \tau_e \tau_\eta \sqrt{y^*} \left(\tau_\eta \sqrt{y^* y_i} - \tau_\omega^*\right) \left[\left(\tau_e + \tau_\eta\right) \sqrt{y^*} - \tau_e \sqrt{y_i}\right]}{2\lambda y_i^2 D^2(y_i)} \\
& + \frac{\left(\tau_\omega^*\right)^2 \tau_e \tau_\eta \sqrt{y^*} \left[\left(\tau_e + \tau_\eta\right) \sqrt{y^*} - \tau_e \sqrt{y_i}\right]}{4\lambda y_i^2 D^2(y_i)} \\
& - \frac{\tau_e^2 \tau_\eta^2 \tau_\omega^* \sqrt{y^*} \left(\tau_\eta \sqrt{y^* y_i} - \tau_\omega^*\right)^2 \left[\left(\tau_e + \tau_\eta\right) y^* - \tau_e \sqrt{y^* y_i}\right]}{\lambda y_i \sqrt{y_i} D^3(y_i)} \\
& - \frac{\tau_e^2 \tau_\eta \tau_\omega^* \sqrt{y^*} \left(\tau_\eta \sqrt{y^* y_i} - \tau_\omega^*\right)}{4\lambda \sqrt{y_i} y_i D^2(y_i)} \\
& + \frac{\left(\tau_\eta + \tau_e\right) \tau_e \tau_\eta^2 \sqrt{y^*} y^* \left(\tau_\eta \sqrt{y^* y_i} - \tau_\omega^*\right)}{2\lambda y_i \sqrt{y_i} D^2(y_i)} \\
& - \frac{\left(\tau_e + \tau_\eta\right) \tau_e^2 \tau_\eta^2 \sqrt{y^*} y^* \left(\tau_\eta \sqrt{y^* y_i} - \tau_\omega^*\right)^3}{\lambda y_i \sqrt{y_i} D^3(y_i)} \\
& + \frac{\left(\tau_e + \tau_\eta\right) \tau_e \tau_\eta y^* \left(\tau_\eta \sqrt{y^* y_i} - \tau_\omega^*\right)^2}{4\lambda y_i^2 D^2(y_i)}.
\end{aligned}
$$

Simplifying further, we have that

$$
\begin{aligned}
J(y_i) \quad = \quad & \frac{\tau_\omega^* \tau_e \tau_\eta \sqrt{y^*} \left(\tau_\eta \sqrt{y^* y_i} - \tau_\omega^*\right) \left[\left(\tau_e + \tau_\eta\right) \sqrt{y^*} - \tau_e \sqrt{y_i}\right]}{2\lambda y_i^2 D^2(y_i)} \\
& + \frac{\left(\tau_\eta + \tau_e\right) \tau_e \tau_\eta^2 \sqrt{y^*} \sqrt{y_i} y^* \left(\tau_\eta \sqrt{y^* y_i} - \tau_\omega^*\right)}{2\lambda y_i^2 D^2(y_i)} \\
& + \frac{\left(\tau_\omega^*\right)^2 \tau_e \tau_\eta \sqrt{y^*} \left[\left(\tau_e + \tau_\eta\right) \sqrt{y^*} - \tau_e \sqrt{y_i}\right]}{4\lambda y_i^2 D^2(y_i)} \\
& + \frac{\left(\tau_e + \tau_\eta\right) \tau_e \tau_\eta y^* \left(\tau_\eta \sqrt{y^* y_i} - \tau_\omega^*\right)^2 - \tau_e^2 \tau_\eta \tau_\omega^* \sqrt{y^*} \sqrt{y_i} \left(\tau_\eta \sqrt{y^* y_i} - \tau_\omega^*\right)}{4\lambda y_i^2 D^2(y_i)} \\
& - \frac{\tau_e^2 \tau_\eta^2 \tau_\omega^* \sqrt{y^*} \left(\tau_\eta \sqrt{y^* y_i} - \tau_\omega^*\right)^2 \left[\left(\tau_e + \tau_\eta\right) y^* - \tau_e \sqrt{y^* y_i}\right]}{\lambda y_i \sqrt{y_i} D^3(y_i)} \\
& - \frac{\left(\tau_e + \tau_\eta\right) \tau_e^2 \tau_\eta^2 \sqrt{y^*} y^* \left(\tau_\eta \sqrt{y^* y_i} - \tau_\omega^*\right)^3}{\lambda y_i \sqrt{y_i} D^3(y_i)}.
\end{aligned}
$$

Equivalently,

$$
\begin{gathered}
J(y_i) = \frac{2\tau_e \tau_\eta \sqrt{y^*} \left(\tau_\eta \sqrt{y^* y_i} - \tau_\omega^*\right) \left\{\left(\tau_e + \tau_\eta\right) \sqrt{y^*} \left[\tau_\omega^* + \tau_\eta \sqrt{y_i y^*}\right] - \tau_\omega^* \tau_e \sqrt{y_i}\right\}}{4\lambda y_i^2 D^2(y_i)} \\
+ \frac{\tau_e \tau_\eta \sqrt{y^*}}{4\lambda y_i^2 D^2(y_i)} \left[\left(\tau_\omega^*\right)^2 \left(\tau_e + \tau_\eta\right) \sqrt{y^*} - \tau_e \left(\tau_\omega^*\right)^2 \sqrt{y_i}\right] \\
+ \frac{\tau_e \tau_\eta \sqrt{y^*}}{4\lambda y_i^2 D^2(y_i)} \left(\tau_\eta \sqrt{y^* y_i} - \tau_\omega^*\right) \left[\left(\tau_e + \tau_\eta\right) \sqrt{y^*} \left(\tau_\eta \sqrt{y^* y_i} - \tau_\omega^*\right) - \tau_e \tau_\omega^* \sqrt{y_i}\right] \\
- \frac{\tau_e^2 \tau_\eta^2 y^* \left(\tau_\eta \sqrt{y^* y_i} - \tau_\omega^*\right)^2 \left[\left(\tau_e + \tau_\eta\right) y^* \tau_\eta - \tau_\omega^* \tau_e\right]}{\lambda y_i D^3(y_i)}.
\end{gathered}
$$

We conclude that

$$
\begin{aligned}
J(y_i) &= \tfrac{\tau_e \tau_\eta \sqrt{y^* y_i}}{4\lambda y_i^2 D^2(y_i)} \left\{ (\tau_e + \tau_\eta) y^* \tau_\eta \left[3\tau_\eta \sqrt{y_i y^*} - 2\tau_\omega^*\right] - 3\tau_\omega^* \tau_e \tau_\eta \sqrt{y^* y_i} + 2\tau_e \left(\tau_\omega^*\right)^2 \right\} \\
&\quad - \tfrac{\tau_e^2 \tau_\eta^2 y^* \left(\tau_\eta \sqrt{y^* y_i} - \tau_\omega^*\right)^2 \left[(\tau_e + \tau_\eta) y^* \tau_\eta - \tau_\omega^* \tau_e\right]}{\lambda y_i D^3(y_i)}.
\end{aligned}
$$

Next, observe that

$$
\lim_{y_i \to 0} D(y_i) = \tau_\eta y^* \left(\tau_\omega^* + \tau_\theta\right) \left(\tau_e + \tau_\eta\right) - \tau_\omega^* \tau_e \tau_\theta.
$$

Using this limit, we have that

$$
\begin{aligned}
\lim_{y_i \to 0} J(y_i) &= \lim_{y_i \to 0} \left\{ -\frac{\tau_e \tau_\eta \tau_\omega^* \sqrt{y^*} \left[\tau_\eta y^* \left(\tau_e + \tau_\eta\right) - \tau_e \tau_\omega^*\right]}{2\lambda \left[\tau_\eta y^* \left(\tau_e + \tau_\eta\right) \left(\tau_\omega^* + \tau_\theta\right) - \tau_e \tau_\omega^* \tau_\theta\right]^2} \left(\frac{\sqrt{y_i}}{y_i^2}\right) \right\} + \\
&\quad \lim_{y_i \to 0} \left\{ -\frac{\left(\tau_\omega^*\right)^2 \tau_e^2 \tau_\eta^2 y^* \left[\tau_\eta y^* \left(\tau_e + \tau_\eta\right) - \tau_e \tau_\omega^*\right]}{\lambda y_i \left[\tau_\eta y^* \left(\tau_e + \tau_\eta\right) \left(\tau_\omega^* + \tau_\theta\right) - \tau_e \tau_\omega^* \tau_\theta\right]^3} \frac{1}{y_i} \right\} \\
&\equiv \lim_{y_i \to 0} \left\{ -\frac{A_1(y_i)}{2\lambda A_0(y_i)^2} \left(\frac{\sqrt{y_i}}{y_i^2}\right) - \frac{A_2(y_i)}{\lambda y_i A_0(y_i)^3} \frac{1}{y_i} \right\},
\end{aligned}
$$

where, for any y_i,

$$
A_0(y_i) \equiv \left[\tau_\eta y^* \left(\tau_e + \tau_\eta\right) \left(\tau_\omega^* + \tau_\theta\right) - \tau_e \tau_\omega^* \tau_\theta\right],
$$

$$
A_1(y_i) \equiv \tau_e \tau_\eta \tau_\omega^* \sqrt{y^*} \left[\tau_\eta y^* \left(\tau_e + \tau_\eta\right) - \tau_e \tau_\omega^*\right],
$$

and

$$
A_2(y_i) \equiv \left(\tau_\omega^*\right)^2 \tau_e^2 \tau_\eta^2 y^* \left[\tau_\eta y^* \left(\tau_e + \tau_\eta\right) - \tau_e \tau_\omega^*\right].
$$

Using the fact that

$$
\tau_\omega^* = \frac{\beta^2 a^{*2} y^* \tau_u \tau_\eta}{\beta^2 a^{*2} \tau_u + y^* \tau_\eta},
$$

we have that

$$
\begin{aligned}
A_1(y_i) &\overset{sgn}{=} \tau_\eta y^* \left(\tau_e + \tau_\eta\right) - \tau_e \tau_\omega^* \\
&= \tau_e (\tau_\eta y^* - \tau_\omega^*) + \tau_\eta y^* \tau_\eta \\
&= \tau_e \left(\tau_\eta y^* - \frac{\beta^2 a^{*2} y^* \tau_u \tau_\eta}{\beta^2 a^{*2} \tau_u + y^* \tau_\eta}\right) + \tau_\eta y^* \tau_\eta \\
&= \tau_\eta \tau_e y^* \left(1 - \frac{\beta^2 a^{*2} \tau_u}{\beta^2 a^{*2} \tau_u + y^* \tau_\eta}\right) + \tau_\eta y^* \tau_\eta \\
&> 0.
\end{aligned}
$$

Similarly,

$$
A_2(y_i) \overset{sgn}{=} \tau_\eta y^* \left(\tau_e + \tau_\eta\right) - \tau_e \tau_\omega^* > 0.
$$

Observe that

$$\begin{aligned} A_0(y_i) &\propto \tau_\eta y^* (\tau_e + \tau_\eta) (\tau_\omega^* + \tau_\theta) - \tau_e \tau_\omega^* \tau_\theta \\ &= (\tau_\eta y^* - \tau_\omega^*) \tau_e \tau_\theta + \tau_\eta y^* (\tau_e \tau_\omega^* + \tau_\eta \tau_\omega^* + \tau_\eta \tau_\theta) \\ &> 0. \end{aligned}$$

Therefore $\lim_{y_i \to 0} J(y_i) < 0$. It is straightforward to see that $\lim_{y_i \to \infty} J(y_i) = 0$. Finally, we check for asymptotes (namely, for values of y_i for which $D(y_i) = 0$). Suppose that $D(y_i) = 0$ for some y_i. That is, there exists y_i such that

$$\tau_\eta^2 y^* \left[\tau_e \tau_\eta y_i + (\tau_\omega + \tau_\theta)(\tau_e + \tau_\eta)\right] - \tau_\omega \tau_e \tau_\eta \left(2\tau_\eta \sqrt{y^* y_i} + \tau_\theta\right) = 0.$$

The expression in the above equation is quadratic in $\sqrt{y_i}$ so we can calculate the determinant to be:

$$\begin{aligned} 4\tau_\omega^2 \tau_e^2 \tau_\eta^2 y^* - 4\tau_e \tau_\eta^2 y^* \left(\tau_\eta y^* (\tau_\omega + \tau_\theta)(\tau_e + \tau_\eta) - \tau_\omega \tau_e \tau_\theta\right) &\propto \tau_\omega^2 \tau_e - \left(\tau_\eta y^* (\tau_\omega + \tau_\theta)(\tau_e + \tau_\eta) - \tau_\omega \tau_e \tau_\theta\right) \\ &\propto \tau_e (\tau_\omega - \tau_\eta y^*) - \tau_\eta^2 y^*. \end{aligned}$$

Using the definition of τ_ω we then get that the determinant is proportional to

$$\begin{aligned} \frac{\beta^2 a^{*2} y^* \tau_u \tau_\eta}{\beta^2 a^{*2} \tau_u + y^* \tau_\eta} - \tau_\eta y^* &\propto \frac{\beta^2 a^{*2} \tau_u}{\beta^2 a^{*2} \tau_u + y^* \tau_\eta} - 1 \\ &= \frac{-y^* \tau_\eta}{\beta^2 a^{*2} \tau_u + y^* \tau_\eta}, \end{aligned}$$

which is strictly negative. Therefore there is no real value of $\sqrt{y_i}$ for which $D(y_i) = 0$. Because y_i is non-negative, this means that there are no vertical asymptotes for $J(y_i; y^*)$.

Jointly, the properties that (a) $\lim_{y_i \to 0} J(y_i; y^*) = -\infty$, (b) $\lim_{y_i \to +\infty} J(y_i) = 0$, and (c) $J(y_i; y^*)$ does not have vertical asymptotes, imply that $J(y_i; y^*)$ is bounded from above by a constant $M > 0$. Hence, when $\frac{3}{2y_i} \mathcal{C}'(y_i) + \mathcal{C}''(y_i) > M$ for all $y_i \geq 0$, the payoff is quasi-concave. Note that, when $\mathcal{C}(y) = \frac{B}{2} y^2$, the above condition becomes $B > \frac{2}{5} M$, which holds for B large enough, as claimed in the main text.

The above results imply that, under the conditions in the proposition, choosing quality of information $y_i = y^*$ and then submitting the limit orders defined by the coefficients $(a^*, \hat{b}^*, \hat{c}^*)$ in Proposition 1 (for quality of information y^*) is a symmetric equilibrium in the full game. That there are no other symmetric equilibria in affine strategies follows from the uniqueness of the solution to $N(y) = 0$ established in Step 2. Q.E.D.

Proof of Proposition 7.

Let y^T denote the socially optimal quality of private information and $(a^T, \hat{b}^T, \hat{c}^T)$ the coefficients describing the efficient demand schedules when the precision of private information is y^T. Next, for any $\bar{y}$, let $\mathbb{E}[W^T; \bar{y}]$ denote ex-ante gross welfare when all traders acquire information of quality $\bar{y}$ but then submit the efficient demand schedules for information of quality y^T (that is, the schedules corresponding to the coefficients $(a^T, \hat{b}^T, \hat{c}^T)$). Such a welfare function is gross of the costs of information acquisition. Finally, for any $(y_i, \bar{y})$, let $\mathbb{E}[\pi_i^T; y_i, \bar{y}]$ denote the ex-ante gross profit of a trader acquiring information of quality y_i when all other traders acquire information of quality $\bar{y}$, and all traders, including i, submit the efficient demand schedules for information of quality y^T (that is, the schedules corresponding to the coefficients $(a^T, \hat{b}^T, \hat{c}^T)$ mentioned above). The payoff is again gross of the cost of information acquisition. We start by establishing the following result:

Lemma 2. *Let y^T denote the socially optimal quality of private information and suppose that all traders submit the efficient demand schedules for information of quality y^T (parametrized by $(a^T, \hat{b}^T, \hat{c}^T)$). When $\hat{c}^T > 0$ (i.e., when the pecuniary externality dominates over the information externality so that the efficient demand schedules are downward sloping), for any $\bar{y}$,*

$$\frac{\partial}{\partial y_i}\mathbb{E}[\pi_i^T; y_i, \bar{y}]\bigg|_{y_i=\bar{y}} > \frac{d}{d\bar{y}}\mathbb{E}[W^T; \bar{y}]$$

whereas the opposite inequality holds when $\hat{c}^T < 0$ (i.e., when the learning externality dominates over the pecuniary externality and, as a result, the efficient demand schedules are upward sloping).

Proof of Lemma 2. When all traders other than i acquire information of quality $\bar{y}$ and then submit the demand schedules corresponding to $(a^T, \hat{b}^T, \hat{c}^T)$, irrespectively of the information acquired by trader i and of the demand schedule submitted by the latter, the equilibrium price is given by

$$p(\theta, u, \eta; \bar{y}) = \alpha + \beta b^T + \beta(a^T + c^T) z(\theta, u, \eta; \bar{y}),$$

where b^T and c^T are the coefficients obtained from $(a^T, \hat{b}^T, \hat{c}^T)$ using the functions (4) and (5), and where $z(\theta, u, \eta; \bar{y}) \equiv \theta + f(\bar{y})\eta - u/\beta a^T$.[2] Furthermore, the aggregate level of trade is equal

$$\tilde{X}(\theta, u, \eta; \bar{y}) = a^T[\theta + f(\bar{y})\eta] + b^T + c^T z(\theta, u, \eta; \bar{y})$$

whereas the level of trade for agent i when he acquires information of quality y_i and then submits the demand schedule corresponding to the coefficients $(a^T, \hat{b}^T, \hat{c}^T)$ is equal to

$$X_i(\theta, u, \eta, e_i; \bar{y}, y_i) = a^T \underbrace{[\theta + f(y_i)e_i + f(y_i)\eta]}_{s_i} + b^T + c^T z(\theta, u, \eta; \bar{y}).$$

It follows that, when all traders other than i acquire information of quality $\bar{y}$, trader i acquires information of quality y_i and all traders, including trader i, submit the demand schedules corresponding to $(a^T, \hat{b}^T, \hat{c}^T)$, trader i's ex-ante gross payoff is equal to

[2] Observe that the functions (4) and (5) do not depend on y and hence c^T and b^T do not depend on y.

$$\mathbb{E}[\pi_i^T;\bar{y},y_i]=\mathbb{E}\left[\left(\theta-p(\theta,u,\eta;\bar{y})\right)X_i(\theta,u,\eta,e_i;\bar{y},y_i)-\frac{\lambda}{2}X_i^2(\theta,u,\eta,e_i;\bar{y},y_i)\right].$$

Using the fact that the market-clearing price must also be consistent with the inverse-supply function and hence satisfy $p=\alpha-u+\beta\tilde{X}(\theta,u,\eta;\bar{y})$, we then have that

$$\mathbb{E}[\pi_i^T;\bar{y},y_i]=\mathbb{E}\left[\left(\theta-\alpha+u-\beta\tilde{X}(\theta,u,\eta;\bar{y})\right)\mathbb{E}[x_i|\theta,u,\eta;\bar{y},y_i]-\frac{\lambda}{2}\mathbb{E}\left[x_i^2|\theta,u,\eta;\bar{y},y_i\right]\right]$$

or, equivalently,

$$\begin{aligned}\mathbb{E}[\pi_i^T;\bar{y},y_i]=\mathbb{E}\Big[&\left(\theta-\alpha+u-\beta\tilde{X}(\theta,u,\eta;\bar{y})\right)\mathbb{E}[x_i|\theta,u,\eta;\bar{y},y_i]-\tfrac{\lambda}{2}Var[x_i|\theta,\eta,u;\bar{y},y_i]\\&-\tfrac{\lambda}{2}\left(\mathbb{E}[x_i|\theta,\eta,u;\bar{y},y_i]\right)^2\Big],\end{aligned}$$

where

$$\mathbb{E}[x_i|\theta,u,\eta;\bar{y},y_i]\equiv\mathbb{E}[X_i(\theta,u,\eta,e_i;\bar{y},y_i)|\theta,u,\eta;\bar{y},y_i],$$

$$\mathbb{E}[x_i^2|\theta,u,\eta;\bar{y},y_i]\equiv\mathbb{E}\left[\left(X_i(\theta,u,\eta,e_i;\bar{y},y_i)\right)^2|\theta,u,\eta;\bar{y},y_i\right],$$

and

$$Var[x_i|\theta,\eta,u;\bar{y},y_i]\equiv\mathbb{E}[x_i^2|\theta,u,\eta;\bar{y},y_i]-\left(\mathbb{E}[x_i|\theta,u,\eta;\bar{y},y_i]\right)^2.$$

Using the fact that

$$\mathbb{E}[x_i|\theta,u,\eta;\bar{y},y_i]=a^T[\theta+f(y_i)\eta]+b^T+c^Tz(\theta,u,\eta;\bar{y})$$

and

$$Var[x_i|\theta,\eta,u;\bar{y},y_i]=\left(a^Tf(y_i)\right)^2/\tau_e,$$

we have that

$$\begin{aligned}\frac{\partial}{\partial y_i}\mathbb{E}[\pi_i^T;\bar{y},y_i]&=\mathbb{E}\left[\left(\theta-\alpha+u-\beta\tilde{X}(\theta,u,\eta;\bar{y})\right)a^Tf'(y_i)\eta\right]-\lambda\frac{\left(a^T\right)^2}{\tau_e}f(y_i)f'(y_i)\\&\quad-\lambda\mathbb{E}\left[\left(a^T[\theta+f(y_i)\eta]+b^T+c^Tz(\theta,u,\eta;\bar{y})\right)a^Tf'(y_i)\eta\right]\\&=-a^T\beta\mathbb{E}\left[\tilde{X}(\theta,u,\eta;\bar{y})\eta\right]f'(y_i)-\lambda\frac{\left(a^T\right)^2}{\tau_e}f(y_i)f'(y_i)\\&\quad-\lambda\left(a^T\right)^2f(y_i)f'(y_i)\frac{1}{\tau_\eta}-\lambda a^Tc^T\mathbb{E}\left[z(\theta,u,\eta;\bar{y})\eta\right]f'(y_i).\end{aligned}$$

Using the fact that

$$\mathbb{E}\left[\tilde{X}(\theta,u,\eta;\bar{y})\eta\right]=\frac{a^Tf(\bar{y})}{\tau_n}+c^T\mathbb{E}\left[z(\theta,u,\eta;\bar{y})\eta\right]$$

and

$$\mathbb{E}\left[z(\theta,u,\eta;\bar{y})\eta\right]=\frac{f(\bar{y})}{\tau_n},$$

we then have that

$$\begin{aligned}\frac{\partial}{\partial y_i}\mathbb{E}[\pi_i^T;\bar{y},y_i]&=-a^T\beta\left[a^Tf(\bar{y})\frac{1}{\tau_n}+c^Tf(\bar{y})\frac{1}{\tau_n}\right]f'(y_i)-\lambda\frac{\left(a^T\right)^2}{\tau_e}f(y_i)f'(y_i)\\&\quad-\lambda\left(a^T\right)^2f(y_i)f'(y_i)\frac{1}{\tau_\eta}-\lambda a^Tc^Tf(\bar{y})\frac{1}{\tau_n}f'(y_i).\end{aligned}\tag{39}$$

We conclude that

$$\begin{aligned}
\frac{\partial}{\partial y_i}\mathbb{E}[\pi_i^T;\bar{y},y_i]\Big|_{y_i=\bar{y}} &= -a^T\beta\left[a^T f(\bar{y})\frac{1}{\tau_n}+c^T f(\bar{y})\frac{1}{\tau_n}\right]f'(\bar{y})-\lambda\frac{\left(a^T\right)^2}{\tau_e}f(\bar{y})f'(\bar{y}) \\
&\quad -\lambda\left(a^T\right)^2 f(\bar{y})f'(\bar{y})\frac{1}{\tau_\eta}-\lambda a^T c^T f(\bar{y})\frac{1}{\tau_n}f'(\bar{y}) \\
&= -f(\bar{y})f'(\bar{y})a^T\left[\lambda\frac{a^T}{\tau_e}+(\beta+\lambda)(a^T+c^T)\frac{1}{\tau_\eta}\right]. \qquad (40)
\end{aligned}$$

Next, observe that, when trader i also acquires information of quality $\bar{y}$ and all traders submit the demand schedules corresponding to $(a^T,\hat{b}^T,\hat{c}^T)$,

$$\mathbb{E}[\pi_i^T;\bar{y},\bar{y}]=\mathbb{E}\left[\left(\theta-\alpha+u-\beta\tilde{X}(\theta,u,\eta;\bar{y})\right)\tilde{X}(\theta,u,\eta;\bar{y})-\frac{\lambda}{2}\frac{\left(a^T f(\bar{y})\right)^2}{\tau_e}-\frac{\lambda}{2}\left(\tilde{X}(\theta,u,\eta;\bar{y})\right)^2\right].$$

Now observe that, when all traders acquire information of quality $\bar{y}$ and submit the demand schedules corresponding to $(a^T,\hat{b}^T,\hat{c}^T)$, the ex-ante payoff of the representative supplier (which the planner accounts for in the computation of welfare) is equal to

$$\begin{aligned}
\mathbb{E}[\Pi;\bar{y}] &= \mathbb{E}\left[\left(p(\theta,u,\eta;\bar{y})-\alpha+u\right)\tilde{X}(\theta,u,\eta;\bar{y})-\frac{\beta}{2}\left(\tilde{X}(\theta,u,\eta;\bar{y})\right)^2\right] \\
&= \frac{\beta}{2}\mathbb{E}\left[\left(\tilde{X}(\theta,u,\eta;\bar{y})\right)^2\right],
\end{aligned}$$

where we used the fact that $p(\theta,u,\eta;\bar{y})=\alpha-u+\beta\tilde{X}(\theta,u,\eta;\bar{y})$. We thus have that, when all traders acquire information of quality $\bar{y}$ and submit the demand schedules corresponding to $(a^T,\hat{b}^T,\hat{c}^T)$, ex-ante welfare is equal to

$$\begin{aligned}
\mathbb{E}[W^T;\bar{y}] &= \mathbb{E}[\pi_i^T;\bar{y},\bar{y}]+\mathbb{E}[\Pi;\bar{y}] \\
&= \mathbb{E}\left[\left(\theta-\alpha+u\right)\tilde{X}(\theta,u,\eta;\bar{y})-\frac{\lambda}{2}\frac{\left(a^T f(\bar{y})\right)^2}{\tau_e}-\frac{\lambda+\beta}{2}\left(\tilde{X}(\theta,u,\eta;\bar{y})\right)^2\right].
\end{aligned}$$

Hence,

$$\frac{d}{d\bar{y}}\mathbb{E}[W^T;\bar{y}]=\mathbb{E}\left[\begin{array}{c}(\theta-\alpha+u)\frac{\partial\tilde{X}(\theta,u,\eta;\bar{y})}{\partial\bar{y}}-\frac{\lambda\left(a^T\right)^2 f(\bar{y})f'(\bar{y})}{\tau_e}\\ -(\lambda+\beta)\tilde{X}(\theta,u,\eta;\bar{y})\frac{\partial\tilde{X}(\theta,u,\eta;\bar{y})}{\partial\bar{y}}\end{array}\right],$$

where

$$\frac{\partial}{\partial\bar{y}}\tilde{X}(\theta,u,\eta;\bar{y})=(a^T+c^T)f'(\bar{y})\eta.$$

It follows that

$$\frac{d}{d\bar{y}}\mathbb{E}[W^T;\bar{y}]=-\frac{\lambda\left(a^T\right)^2 f(\bar{y})f'(\bar{y})}{\tau_e}-(\lambda+\beta)(a^T+c^T)f'(\bar{y})\mathbb{E}_{\theta,\eta,u}\left[\tilde{X}(\theta,u,\eta;\bar{y})\eta\right].$$

Using the fact that

$$\mathbb{E}\left[\tilde{X}(\theta,u,\eta;\bar{y})\eta\right]=(a^T+c^T)f(\bar{y})\frac{1}{\tau_n},$$

we thus have that

$$\frac{d}{d\bar{y}}\mathbb{E}[W^T;\bar{y}] \quad = \quad -\frac{\lambda\left(a^T\right)^2 f(\bar{y})f'(\bar{y})}{\tau_e} - (\lambda+\beta)\left(a^T+c^T\right)^2 f'(\bar{y})f(\bar{y})\frac{1}{\tau_n}. \tag{41}$$

Comparing (40) with (41), we thus have that, when $c^T < 0$,

$$\left.\frac{\partial}{\partial y_i}\mathbb{E}[\pi_i^T;\bar{y},y_i]\right|_{y_i=\bar{y}} \quad > \quad \frac{d}{d\bar{y}}\mathbb{E}[W^T;\bar{y}],$$

whereas the opposite inequality holds when $c^T > 0$. Finally, use Condition (5) to observe that $\hat{c}^T = -\frac{c^T}{\beta(a^T+c^T)}$ and Condition (14), along with the formula for $\tau_\omega(a)$, to observe that $a^T + c^T > 0$. Jointly, the last two conditions imply that $sgn(\hat{c}^T) = -sgn(c^T)$ thus completing the proof of the lemma.

We now show that the result in Lemma 2 implies the result in the proposition. We start by establishing the (global) concavity of $\mathbb{E}[\pi_i^T;\bar{y},y_i]$ and $\mathbb{E}[W^T;\bar{y}]$ in y_i and $\bar{y}$, respectively. Recall that the coefficients defining the equilibrium trades as a function of the private signals s_i and the endogenous public signal z are kept constant in both cases at (a^T,b^T,c^T), where (a^T,b^T,c^T) is the vector defining the efficient trades when the quality of private information is y^T. Using (39), we have that

$$\begin{aligned}\frac{\partial^2}{\partial y_i^2}\mathbb{E}[\pi_i^T;\bar{y},y_i] \quad &= \quad -a^T\beta f(\bar{y})\frac{1}{\tau_\eta}\left(a^T+c^T\right)f''(y_i) - \lambda\left(a^T\right)^2\left[\frac{1}{\tau_e}+\frac{1}{\tau_\eta}\right]\frac{\partial}{\partial y_i}\left(f(y_i)f'(y_i)\right) \\ &\quad -\lambda a^T c^T f(\bar{y})\frac{1}{\tau_\eta}f''(y_i) \\ &= \quad -a^T f(\bar{y})\frac{1}{\tau_\eta}\left[\beta\left(a^T+c^T\right)+\lambda c^T\right]f''(y_i) - \lambda\left(a^T\right)^2\left[\frac{1}{\tau_e}+\frac{1}{\tau_\eta}\right]\frac{\partial}{\partial y_i}\left(f(y_i)f'(y_i)\right).\end{aligned}$$

Now observe that $f''(y_i) = 3\sqrt{y_i}/4y_i^3 > 0$ and $\frac{\partial}{\partial y_i}\left(f(y_i)f'(y_i)\right) = 1/y_i^3 > 0$. Hence,

$$\frac{\partial^2}{\partial y_i^2}\mathbb{E}[\pi_i^T;\bar{y},y_i] \quad = \quad -\frac{a^T}{y_i^3\tau_\eta}\left[\frac{3\sqrt{y_i}}{4\sqrt{\bar{y}}}\left(\beta a^T+(\beta+\lambda)c^T\right)+\lambda a^T\frac{\tau_\eta+\tau_e}{\tau_e}\right].$$

Recall that, irrespective of the sign of c^T, $a^T > 0$ and $a^T + c^T > 0$, where the last inequality is established in the proof of Lemma 2. Hence, when $c^T \geq 0$, for any $(\bar{y},y_i)$, $\partial^2\mathbb{E}[\pi_i^T;\bar{y},y_i]/\partial y_i^2 < 0$. To see that the same inequality holds when $c^T < 0$, recall that

$$c^T = \frac{1}{\beta+\lambda}\left[\left(1-\lambda a^T-\lambda a^T\frac{\tau_\theta}{y^T\tau_\eta}\right)\frac{\tau_\omega(a^T)}{\tau_\omega(a^T)+\tau_\theta}-\beta a^T\right].$$

Hence,

$$\beta a^T + (\beta+\lambda)c^T = \left(1-\lambda a^T-\lambda a^T\frac{\tau_\theta}{y^T\tau_\eta}\right)\frac{\tau_\omega(a^T)}{\tau_\omega(a^T)+\tau_\theta}.$$

Using

$$\tau_\omega(a^T) = \frac{\beta^2\left(a^T\right)^2 y^T\tau_\eta\tau_u}{\beta^2\left(a^T\right)^2\tau_u + y^T\tau_\eta},$$

we can rewrite the last condition as

$$\beta a^T + (\beta+\lambda)c^T = \left[\left(1-\lambda a^T\right)y^T\tau_\eta - \lambda a^T\tau_\theta\right]\frac{\beta^2\left(a^T\right)^2\tau_u}{\beta^2\left(a^T\right)^2\tau_u\left(y^T\tau_\eta+\tau_\theta\right)+y^T\tau_\eta\tau_\theta}.$$

Hence,
$$\beta a^T + (\beta+\lambda)\, c^T \overset{sgn}{=} \left(1-\lambda a^T\right) y^T\tau_\eta - \lambda a^T\tau_\theta.$$
Now recall that a^T solves
$$a^T = \frac{1}{\lambda}\,\frac{1}{\Lambda(\tau_\omega(a^T)) + \Xi(a^T) + \Delta(a^T)}\,. \tag{42}$$
Using the definition of the Λ function, we have that the latter condition is equal to
$$a^T = \frac{1}{\lambda}\,\frac{\tau_\epsilon y^T\tau_\eta(y^T\tau_\eta - \tau_\omega(a^T))}{\left(y^T\right)^2\tau_\eta^2\left(\tau_\epsilon+\tau_\theta+\tau_\omega(a^T)\right) - \tau_\omega(a^T)\tau_\epsilon\left(\tau_\theta + 2y^T\tau_\eta\right) + \left[\Xi(a^T)+\Delta(a^T)\right]\tau_\epsilon y^T\tau_\eta(y^T\tau_\eta-\tau_\omega(a^T))}\,, \tag{43}$$
with $\tau_\epsilon \equiv \left(y^T\tau_e\tau_\eta\right)/\left(\tau_e+\tau_\eta\right)$ and $\tau_\omega(a) \equiv \beta^2a^2\tau_u y^T\tau_\eta/\left(\beta^2a^2\tau_u + y^T\tau_\eta\right)$. Observe that the numerator in (43) is positive. Because $a^T>0$, as shown above, this means that the denominator in (43) is also positive. Furthermore, using (43), we have that
$$\left(1-\lambda a^T\right) y^T\tau_\eta - \lambda a^T\tau_\theta$$
$$= \frac{y^T\tau_\eta Q}{\left(y^T\right)^2\tau_\eta^2(\tau_\epsilon+\tau_\theta+\tau_\omega(a^T)) - \tau_\omega(a^T)\tau_\epsilon(\tau_\theta+2y^T\tau_\eta) + [\Xi(a^T)+\Delta(a^T)]\tau_\epsilon y^T\tau_\eta(y^T\tau_\eta-\tau_\omega(a^T))},$$
where
$$Q \equiv y^T\tau_\eta\left(y^T\tau_\eta - \tau_\epsilon\right)\left(\tau_\theta+\tau_\omega(a^T)\right) + \left[\Xi(a^T)+\Delta(a^T)\right]\tau_\epsilon y^T\tau_\eta(y^T\tau_\eta - \tau_\omega(a^T)).$$
We thus have that
$$\left(1-\lambda a^T\right) y^T\tau_\eta - \lambda a^T\tau_\theta \overset{sgn}{=} Q.$$
Now, using the fact that $\tau_\epsilon = \left(y^T\tau_e\tau_\eta\right)/\left(\tau_e+\tau_\eta\right)$, we have that Q can be rewritten as
$$Q = \left(y^T\tau_\eta\right)^2\frac{\tau_\eta}{\tau_e+\tau_\eta}\left(\tau_\theta+\tau_\omega(a^T)\right) + \left[\Xi(a^T)+\Delta(a^T)\right]\frac{\tau_e}{\tau_e+\tau_\eta}\left(y^T\tau_\eta\right)^2\left(y^T\tau_\eta - \tau_\omega(a^T)\right).$$
Because $y^T\tau_\eta - \tau_\omega(a^T) > 0$, we conclude that $sgn\,(Q) > 0$ if $\Xi(a^T)+\Delta(a^T) > 0$. The latter property holds because, as explained in the main text, when $c^T<0$, then $\hat{c}^T>0$ in which case $\Xi(a^T)+\Delta(a^T)>0$. We conclude that, no matter the sign of c^T, for any $\bar{y}$, $\mathbb{E}[\pi_i^T;\bar{y},y_i]$ is strictly concave in y_i.

Next, consider the concavity of $\mathbb{E}[W^T;\bar{y}]$ in $\bar{y}$. Using (41), we have that
$$\begin{aligned}\frac{d^2}{d\bar{y}^2}\mathbb{E}[W^T;\bar{y}] &= -\left[\frac{\lambda\left(a^T\right)^2}{\tau_e} + (\lambda+\beta)\left(a^T+c^T\right)^2\frac{1}{\tau_n}\right]\frac{\partial}{\partial\bar{y}}\left(f(\bar{y})f'(\bar{y})\right)\\ &< 0,\end{aligned}$$
where again the inequality follows from the fact that $\frac{\partial}{\partial\bar{y}}\left(f(\bar{y})f'(\bar{y})\right)>0$. Hence $\mathbb{E}[W^T;\bar{y}]$ is strictly concave in $\bar{y}$. Because $\mathbb{E}[\pi_i^T;\bar{y},y_i]$ is strictly concave in y_i, in equilibrium, all traders acquire information of quality y^* such that
$$\left.\frac{\partial}{\partial y_i}\mathbb{E}[\pi_i^T;\bar{y},y_i]\right|_{y_i=\bar{y}=y^*} = \mathcal{C}'(y^*).$$

Now recall that the socially-optimal quality of information satisfies

$$\frac{d}{d\bar{y}}\mathbb{E}[W^T;\bar{y}]\bigg|_{\bar{y}=y^T} = \mathcal{C}'(y^T).$$

Because $\mathbb{E}[W^T;\bar{y}]$ is strictly concave in $\bar{y}$, the result in Lemma 2 then implies that, when $\hat{c}^T < 0$, $y^T > y^*$, whereas, when $\hat{c}^T > 0$, $y^T < y^*$. Q.E.D.

Proof of Proposition 8.

Under the proposed policy, each trader i's ex-ante gross expected payoff when all traders other than i collect information of quality $\bar{y}$, trader i collects information of quality y_i, and all traders (including i) submit the efficient demand schedules (parametrized by $(a^T, \hat{b}^T, \hat{c}^T)$) is equal to

$$\begin{aligned}
\mathbb{E}[\pi_i^T(\bar{y}, y_i); \hat{t}_p] &= \mathbb{E}\left[\theta x_i - (1+\hat{t}_p)px_i - \frac{\lambda}{2}x_i^2\right] \\
&= \mathbb{E}\left[\theta x_i - (1+\hat{t}_p)\left(\alpha - u + \beta\tilde{x}\right)x_i - \frac{\lambda}{2}x_i^2\right]
\end{aligned}$$

with

$$x_i = \mathbf{X}_i(\theta, u, \eta, e_i; \bar{y}, y_i) = a^T \underbrace{\left[\theta + f(y_i)e_i + f(y_i)\eta\right]}_{s_i} + b^T + c^T\left(\theta + f(\bar{y})\eta - \frac{u}{\beta a^T}\right),$$

$$p = P(\theta, u, \eta; \bar{y}) = \alpha - u + \beta\mathbf{X}(\theta, u, \eta; \bar{y}),$$

and

$$\tilde{x} = \mathbf{X}(\theta, u, \eta; \bar{y}) = a^T(\theta + f(\bar{y})\eta) + b^T + c^T\left(\theta + f(\bar{y})\eta - \frac{u}{\beta a^T}\right),$$

and where b^T and c^T are the coefficients describing the equilibrium trades obtained from $\hat{b}^T$ and $\hat{c}^T$ using (4) and (5). Hence,

$$\mathbb{E}[\pi_i^T(\bar{y}, y_i); \hat{t}_p] = N - \beta(a^T + c^T)a^T\frac{1+\hat{t}_p}{\sqrt{\bar{y}}\sqrt{y_i}\tau_\eta} - \frac{\lambda c^T a^T}{\sqrt{\bar{y}}\sqrt{y_i}\tau_\eta} - \frac{\lambda}{2}\frac{\left(a^T\right)^2}{y_i\tau_\eta} - \frac{\lambda}{2}\frac{\left(a^T\right)^2}{y_i\tau_e},$$

where N is a function of all variables that do not interact with y_i. It follows that

$$\frac{\partial}{\partial y_i}\mathbb{E}[\pi_i^T(\bar{y}, y_i); \hat{t}_p] = \frac{\beta(1+\hat{t}_p)(a^T + c^T)a^T}{2\tau_\eta y_i\sqrt{\bar{y}y_i}} + \frac{\lambda a^T}{2\tau_\eta y_i\sqrt{y_i}}\left(\frac{a^T}{\sqrt{y_i}} + \frac{c^T}{\sqrt{\bar{y}}}\right) + \frac{\lambda\left(a^T\right)^2}{2y_i^2\tau_e}.$$

Because $\mathbb{E}[\pi_i^T(\bar{y}, y_i); \hat{t}_p] - \mathcal{C}(y_i)$ is concave in y_i, for $y_i = \bar{y} = y^T$ to be sustained in equilibrium it is both necessary and sufficient that

$$\frac{\partial}{\partial y_i}\mathbb{E}[\pi_i^T(y^T, y^T); \hat{t}_p] = \mathcal{C}'(y^T)$$

which is equivalent to

$$\frac{\left[\beta(1+\hat{t}_p) + \lambda\right](a^T + c^T)a^T}{2\tau_\eta} + \frac{\lambda\left(a^T\right)^2}{2\tau_e} = \mathcal{C}'(y^T)\left(y^T\right)^2.$$

Using the fact that y^T satisfies

$$\frac{(\beta+\lambda)(a^T+c^T)^2}{2\tau_\eta}+\frac{\lambda\left(a^T\right)^2}{2\tau_e}=\mathcal{C}'(y^T)\left(y^T\right)^2,$$

we have that the proposed policy implements the efficient acquisition of private information when

$$\hat{t}_p=\frac{(\beta+\lambda)c^T}{\beta a^T}.$$

Using the fact that

$$c^T=\frac{1}{\beta+\lambda}\left(\gamma_2\left(\tau_\omega(a^T)\right)-\beta a^T\right)$$

we then have that the optimal $\hat{t}_p$ is equal to

$$\hat{t}_p=\frac{\gamma_2\left(\tau_\omega(a^T)\right)-\beta a^T}{\beta a^T}$$

where γ_2 is the function defined in the proof of Proposition 1. Q.E.D.

Proof of Proposition 9.

Assume that all traders other than i acquire information of quality y^T and then submit the efficient demand schedules (that is, those corresponding to the coefficients $(a^T,\hat{b}^T,\hat{c}^T)$ for quality of information y^T). Given any policy $T(x_i,p)$, the expected net payoff for trader i when he chooses information of quality y_i and then selects his demand schedule optimally is equal to

$$V(y^T,y_i)\equiv\sup_{g(\cdot)}\left\{\mathbb{E}[\tilde{\pi}_i(y^T,y_i);g(\cdot)]-\mathcal{C}(y_i)\right\}$$

where $g:\mathbb{R}^2\to\mathbb{R}$ is a generic function specifying the amount of shares $x_i=g(s_i,z)$ that the trader purchases as a function of his private signal s_i and the endogenous signal $z\equiv\theta+f(y^T)\eta-u/(\beta a^T)$ contained in the market-clearing price, and where

$$\begin{aligned}\mathbb{E}[\tilde{\pi}_i(y^T,y_i);g(\cdot)] \;\equiv\; & \mathbb{E}\left[\theta g(s_i,z)-(\alpha-u+\beta\tilde{x})g(s_i,z)-\tfrac{\lambda}{2}\left(g(s_i,z)\right)^2\right]\\ & -\mathbb{E}\left[T\left(g(s_i,z),\alpha-u+\beta\tilde{x}\right)\right].\end{aligned}$$

Note that the definition of $\mathbb{E}[\tilde{\pi}_i(y^T,y_i);g(\cdot)]$ uses the fact that the market-clearing price is given by $p=\alpha-u+\beta\tilde{x}$ with $\tilde{x}=a^T(\theta+f(y^T)\eta)+b^T+c^Tz$, where b^T and c^T are the coefficients describing the equilibrium trades obtained from $\hat{b}^T$ and $\hat{c}^T$ using (4) and (5). It also uses the fact that, when all other traders submit the efficient demand schedules, any demand schedule for trader i (that is, any mapping from (s_i,p) into x_i) can be expressed as a function $g(s_i,z)$ of (s_i,z).[3]

For the policy $T(x_i,p)$ to implement the efficient acquisition and usage of information, it must be that, when $y_i=y^T$, the function $g(\cdot)$ that maximizes the trader's payoff is equal to $g(s_i,z)=a^Ts_i+b^T+c^Tz$. Using the fact that the equilibrium price can be expressed as $p=\alpha+\beta b^T+\beta(a^T+c^T)z$, and the fact that $\mathbb{E}\left[\theta|s_i,z\right]=\gamma_1(\tau_\omega(a^T))s_i+\gamma_2(\tau_\omega(a^T))z$ where γ_1 and γ_2 are the functions defined

[3] It suffices to use (6) to observe that $p=\alpha+\beta b^T+\beta(a^T+c^T)z$.

in the proof of Proposition 1, we thus have that, for the policy T to implement the efficient trades, it must be that T is differentiable in x_i and satisfies

$$\gamma_1(\tau_\omega(a^T))s_i + \gamma_2(\tau_\omega(a^T))z - \left[\alpha + \beta b^T + \beta(a^T + c^T)z\right] - \lambda\left(a^T s_i + b^T + c^T z\right)$$

$$-\tfrac{\partial}{\partial x}T\left(a^T s_i + b^T + c^T z, \alpha + \beta b^T + \beta(a^T + c^T)z\right) = 0$$

for all (s_i, z). Next, observe that, when trader i trades efficiently, the quantity that he purchases is given by $x_i = a^T s_i + b^T + c^T z$. Expressing s_i as a function of x_i using the last expression, and using the relationship $p = \alpha + \beta b^T + \beta(a^T + c^T)z$ to express z as a function of p, we have that

$$\begin{aligned}
&\gamma_1(\tau_\omega(a^T))s_i + \gamma_2(\tau_\omega(a^T))z - \left[\alpha + \beta b^T + \beta(a^T + c^T)z\right] - \lambda\left(a^T s_i + b^T + c^T z\right)\\
&= \left[\gamma_1(\tau_\omega(a^T)) - \lambda a^T\right]\tfrac{x_i - b^T - c^T z}{a^T} + \left[\gamma_2(\tau_\omega(a^T)) - \beta(a^T + c^T) - \lambda c^T\right]\tfrac{p - \alpha - \beta b^T}{\beta(a^T + c^T)}\\
&- \left(\alpha + \beta b^T + \lambda b^T\right) = \left[\gamma_1(\tau_\omega(a^T)) - \lambda a^T\right]\tfrac{x - b^T}{a^T}\\
&+ \left[\gamma_2(\tau_\omega(a^T)) - \beta(a^T + c^T) - \lambda c^T - \left(\gamma_1(\tau_\omega(a^T)) - \lambda a^T\right)\tfrac{c^T}{a^T}\right]\tfrac{p - \alpha - \beta b^T}{\beta(a^T + c^T)} - \left(\alpha + \beta b^T + \lambda b^T\right).
\end{aligned}$$

Note that the term above is the discrepancy between the trader's marginal benefit and marginal cost of expanding his demand evaluated at the efficient trade. But this means that, for the policy $T(x_i, p)$ to implement the efficient use of information, it must be that $T(x_i, p)$ is a polynomial of second order of the form

$$T(x_i, p) = \frac{\delta}{2}x_i^2 + (t_p p - t_0)\,x_i + \tilde{K}(p), \tag{44}$$

for some vector (δ, t_p, t_0) and some function $\tilde{K}(p)$ which plays no role for incentives and which therefore we can disregard. In the proof of Proposition 4, we showed that there exists a unique vector (δ, t_p, t_0) that induces the traders to submit the efficient demand schedules when the precision of their private information is y^T (the vector in Proposition 4 applied to $y = y^T$). Thus, if a policy T induces efficiency in both information acquisition and information usage, it must be of the form in (44) with (δ, t_p, t_0) as in Proposition 4 applied to $y = y^T$. When the policy takes this form, for any y_i, the optimal choice of $g(\cdot)$ is affine and hence can be written as $g(s_i, z) = as_i + b + cz$, for some (a, b, c), implying that

$$\begin{aligned}
\mathbb{E}[\tilde{\pi}_i(y^T, y_i); g(\cdot)] = \mathbb{E}\Bigg[&(\theta + t_0)\,(as_i + b + cz) - \tfrac{\lambda+\delta}{2}\,(as_i + b + cz)^2\\
&-(1 + t_p)\left(\alpha - u + \beta\left[a^T(\theta + f(y^T)\eta) + b^T + c^T z\right]\right)(as_i + b + cz)\Bigg].
\end{aligned}$$

Letting $\tilde{M}$ be a function of all variables that do not interact with y_i, we then have that, when $g(s_i, z) = as_i + b + cz$, for some (a, b, c),

$$\begin{aligned}
\mathbb{E}[\tilde{\pi}_i(y^T, y_i); g(\cdot)] &= \tilde{M} - \beta(1 + t_p)(a^T + c^T)a\tfrac{1}{\sqrt{y^T}\sqrt{y_i}\tau_\eta}\\
&+\tfrac{(\lambda+\delta)ca}{\sqrt{y^T}\sqrt{y_i}\tau_\eta} - \tfrac{\lambda+\delta}{2}\tfrac{a^2}{y_i\tau_\eta} - \tfrac{\lambda+\delta}{2}\tfrac{a^2}{y_i\tau_e}.
\end{aligned}$$

The optimal g when $y_i = y^T$ is $g(x_i, z) = a^T s_i + b^T + c^T z$. Hence, using the envelope theorem, we then have that

$$\frac{\partial}{\partial y_i} V(y^T, y_i)\Big|_{y_i=y^T} = \frac{\left[\beta(1+t_p)+\lambda+\delta\right](a^T+c^T)a^T}{2\tau_\eta \left(y^T\right)^2} + \frac{(\lambda+\delta)\left(a^T\right)^2}{2\tau_e \left(y^T\right)^2} - \mathcal{C}'(y^T).$$

Recall that the efficient y^T is given by the solution to the following equation

$$\frac{(\beta+\lambda)(a^T+c^T)^2}{2\tau_\eta \left(y^T\right)^2} + \frac{\lambda\left(a^T\right)^2}{2\tau_e \left(y^T\right)^2} = \mathcal{C}'(y^T).$$

Hence, for the policy of Proposition 4 (applied to $\bar{y} = y^T$) to implement the efficient acquisition of private information, it must be that

$$\frac{(\beta+\lambda)(a^T+c^T)^2}{\tau_\eta} + \frac{\lambda\left(a^T\right)^2}{\tau_e} = \frac{\left[\beta(1+t_p)+\lambda+\delta\right](a^T+c^T)a^T}{\tau_\eta} + \frac{(\lambda+\delta)\left(a^T\right)^2}{\tau_e}$$

or, equivalently, $(a^T+c^T)\tau_e\left[(\beta+\lambda)c^T-(\beta t_p+\delta)a^T\right] = \delta\left(a^T\right)^2\tau_\eta$. One can verify that the values of δ and t_p from Proposition 4 do not solve the above equation except for a non-generic set of parameters. Q.E.D.

Proof of Proposition 10.

Paralleling the derivations in the proof of Proposition 9, we have that, when the policy takes the proposed form and all traders other than i acquire information of quality y^T and then submit the efficient demand schedules (that is, the affine orders corresponding to the coefficients $(a^T, \hat{b}^T, \hat{c}^T)$ for quality of information y^T), the expected net payoff for trader i when he chooses information of quality y_i is maximized by submitting an affine demand schedule $x_i = as_i + \hat{b} - \hat{c}p$ which induces trades $x_i = as_i + b + cz$ that are affine in (s_i, z), where $z = \theta + f(y^T)\eta - u/\beta a^T$ is the endogenous signal contained in the market-clearing price.

Using this result, let

$$\hat{V}(y^T, y_i) \equiv \sup_{a,b,c}\left\{\mathbb{E}[\tilde{\pi}_i(y^T, y_i); a, b, c] - \mathcal{C}(y_i) + Ay_i\right\}$$

denote the maximal payoff that trader i can obtain by acquiring information of precision y_i when all other traders acquire information of precision y^T and then submit the efficient demand schedules for information of quality y^T. As shown in the proof of Proposition 9, the expected gross payoff that trader i obtains by inducing the affine trades $x_i = as_i + b + cz$ when he chooses information of quality y_i is equal to

$$\mathbb{E}[\tilde{\pi}_i(y^T, y_i); a, b, c] = \bar{M} - \beta(1+t_p)(a+c)a\frac{1}{\sqrt{y^T}\sqrt{y_i}\tau_\eta} - \frac{(\lambda+\delta)ca}{\sqrt{y^T}\sqrt{y_i}\tau_\eta} - \frac{\lambda+\delta}{2}\frac{a^2}{y_i\tau_\eta} - \frac{\lambda+\delta}{2}\frac{a^2}{y_i\tau_e},$$

where $\bar{M}$ is a term collecting all variables that do not interact with y_i. Using the envelope theorem, we have that

$$\frac{\partial}{\partial y_i}\hat{V}(y^T, y_i)\bigg|_{y_i=y^T} = \frac{[\beta(1+t_p)+\lambda+\delta]\,(a^T+c^T)a^T}{2\tau_\eta\,(y^T)^2} + \frac{(\lambda+\delta)\,\left(a^T\right)^2}{2\tau_e\,(y^T)^2} - \mathcal{C}'(y^T)+A.$$

Again, in writing the above derivative we used the fact that, when $y_i = y^T$, the optimal demand schedule for trader i induces trades equal to $a^T s_i + b^T + c^T z$. Using the fact that y^T satisfies

$$\frac{(\beta+\lambda)(a^T+c^T)^2}{2\tau_\eta\,(y^T)^2} + \frac{\lambda\left(a^T\right)^2}{2\tau_e\,(y^T)^2} = \mathcal{C}'(y^T),$$

we thus have that the proposed policy induces the efficient acquisition of private information only if the following condition holds

$$\frac{(\beta+\lambda)(a^T+c^T)^2}{2\tau_\eta} + \frac{\lambda\left(a^T\right)^2}{2\tau_e} = \frac{(\beta(1+t_p)+\lambda+\delta)\,(a^T+c^T)a^T}{2\tau_\eta} + \frac{(\lambda+\delta)\,\left(a^T\right)^2}{2\tau_e} + A\left(y^T\right)^2$$

from which we obtain that

$$A = \frac{a^T+c^T}{2\tau_\eta\,(y^T)^2}\left[(\beta+\lambda)c^T - (\beta t_p+\delta)a^T\right] - \frac{\delta\left(a^T\right)^2}{2\tau_e\,(y^T)^2}.$$

Next, use Condition (5) to express c^T as a function of $\hat{c}^T$ and rewrite A as follows

$$A = -\frac{\left(a^T\right)^2}{2\tau_\eta\,(y^T)^2}\left[\frac{\beta(\beta+\lambda)\hat{c}^T}{(1+\beta\hat{c}^T)^2} + \frac{\beta t_p+\delta}{1+\beta\hat{c}^T}\right] - \frac{\delta\left(a^T\right)^2}{2\tau_e\,(y^T)^2}.$$

That the function $\hat{V}(y^T, y_i)$ is globally quasi-concave in y_i under the conditions in the proposition follows from arguments similar to those in the proof of Proposition 6. We conclude that the proposed policy implements the efficient acquisition and usage of information. Q.E.D.

Proof of Proposition 11.

As in the proof of the last two propositions, assume that all traders other than i acquire information of quality y^T and then submit the efficient demand schedules (that is, those corresponding to the coefficients $(a^T, \hat{b}^T, \hat{c}^T)$ for quality of information y^T). Given any policy $T(x_i, \tilde{x}, p)$, the expected net payoff for trader i when he chooses information of quality y_i and then selects his demand schedule optimally is equal to

$$\tilde{V}(y^T, y_i) \equiv \sup_{g(\cdot)}\left\{\mathbb{E}[\tilde{\pi}_i(y^T, y_i); g(\cdot)] - \mathcal{C}(y_i)\right\}$$

where $g : \mathbb{R}^2 \to \mathbb{R}$ is a generic function specifying the amount of shares $x_i = g(s_i, z)$ that the trader purchases as a function of s_i and z, with $z \equiv \theta + f(y^T)\eta - u/(\beta a^T)$, and

$$\mathbb{E}[\tilde{\pi}_i(y^T, y_i); g(\cdot)] \equiv \mathbb{E}\left[\theta g(s_i, z) - (\alpha - u + \beta\tilde{x})g(s_i, z) - \tfrac{\lambda}{2}\left(g(s_i, z)\right)^2\right] \\ -\mathbb{E}\left[T\left(g(s_i, z), \tilde{x}, \alpha - u + \beta\tilde{x}\right)\right].$$

Note that, in writing $\mathbb{E}[\tilde{\pi}_i(y^T, y_i); g(\cdot)]$, we use the fact that the market-clearing price is given by $p = \alpha - u + \beta\tilde{x}$ with $\tilde{x} = a^T(\theta + f(y^T)\eta) + b^T + c^T z$, where b^T and c^T are the coefficients describing the equilibrium trades obtained from $\hat{b}^T$ and $\hat{c}^T$ using (4) and (5). We also use the fact that, when all other traders submit the efficient demand schedules, any demand schedule for trader i (that is, any mapping from (s_i, p) into x_i) can be expressed as a function $g(s_i, z)$ of (s_i, z) by using (6) to express $p = \alpha + \beta b^T + \beta(a^T + c^T)z$ as an affine transformation of z.

For the policy $T(x_i, \tilde{x}, p)$ to implement efficiency in both information acquisition and usage, it must be that, when $y_i = y^T$, the function $g(\cdot)$ that maximizes the trader's payoff is equal to $g(s_i, z) = a^T s_i + b^T + c^T z$. Using the expression for the equilibrium price $p = \alpha + \beta b^T + \beta(a^T + c^T)z$ and the fact that

$$\mathbb{E}\left[\theta | s_i, z; y_i, y^T\right]\big|_{y_i = y^T} = \gamma_1(\tau_\omega(a^T))s_i + \gamma_2(\tau_\omega(a^T))z,$$

where γ_1 and γ_2 are the functions defined in the proof of Proposition 1, we thus have that, for the policy T to implement the efficient trades, it must be that T is differentiable in x_i and, for all (s_i, z), satisfy

$$\gamma_1(\tau_\omega(a^T))s_i + \gamma_2(\tau_\omega(a^T))z - \left[\alpha + \beta b^T + \beta(a^T + c^T)z\right] - \lambda\left(a^T s_i + b^T + c^T z\right) \\ - \tfrac{\partial}{\partial x_i}\mathbb{E}\left[T\left(a^T s_i + b^T + c^T z, \tilde{x}, \alpha - u + \beta\tilde{x}\right)|s_i, z; y_i, y^T\right]\Big|_{y_i = y^T} = 0,$$

where $\tilde{x} = a^T(\theta + f(y^T)\eta) + b^T + c^T z$, with $z \equiv \theta + f(y^T)\eta - u/(\beta a^T)$.

Next recall from the proof of Proposition 7 that, when the individual trades efficiently,

$$\gamma_1(\tau_\omega(a^T))s_i + \gamma_2(\tau_\omega(a^T))z - \left[\alpha + \beta b^T + \beta(a^T + c^T)z\right] - \lambda\left(a^T s_i + b^T + c^T z\right) \\ = \left[\gamma_1(\tau_\omega(a^T)) - \lambda a^T\right]\tfrac{x - b^T}{a^T} + \left[\gamma_2(\tau_\omega(a^T)) - \beta(a^T + c^T) - \lambda c^T - \left(\gamma_1(\tau_\omega(a^T)) - \lambda a^T\right)\tfrac{c^T}{a^T}\right]\tfrac{p - \alpha - \beta b^T}{\beta(a^T + c^T)} \\ - \left(\alpha + \beta b^T + \lambda b^T\right).$$

This means that, for the policy T to implement the efficient use of information, it must be that $T(x_i, \tilde{x}, p)$ is a polynomial of second order of the form

$$T(x_i, \tilde{x}, p) = \frac{\delta'}{2}x_i^2 + \left(pt_p' - t_0' + t_{\tilde{x}}\tilde{x}\right)x_i + K'(\tilde{x}, p), \tag{45}$$

for some vector $(\delta', t_p', t_0', t_{\tilde{x}})$, where $K'(\tilde{x}, p)$ is a function that does not depend on x_i, plays no role for incentives, and hence can be disregarded. Furthermore, under any such a policy,

$$\tfrac{\partial}{\partial x_i}\mathbb{E}\left[T\left(x_i, \tilde{x}, p\right)|s_i, p; y_i, y^T\right] = \delta' x_i + pt_p' - t_0' + t_{\tilde{x}}\mathbb{E}\left[\tilde{x}|s_i, p; y_i, y^T\right] \\ = \delta' x_i + pt_p' - t_0' + t_{\tilde{x}}\mathbb{E}\left[\tfrac{p - \alpha + u}{\beta}|s_i, p; y_i, y^T\right] = \delta' x_i + pt_p' - t_0' + \tfrac{t_{\tilde{x}}}{\beta}(p - \alpha) + \tfrac{t_{\tilde{x}}}{\beta}\mathbb{E}\left[u|s_i, p; y_i, y^T\right] \\ = \delta' x_i + pt_p' - t_0' + \tfrac{t_{\tilde{x}}}{\beta}(p - \alpha) + \tfrac{t_{\tilde{x}}}{\beta}A^{\#}(y_i, y^T)s_i + \tfrac{t_{\tilde{x}}}{\beta}B^{\#}(y_i, y^T)p + \tfrac{t_{\tilde{x}}}{\beta}C^{\#}(y_i, y^T),$$

where we used the fact that $p = \alpha - u + \beta\tilde{x}$ and the fact that

$$\mathbb{E}\left[u|s_i, p; y_i, y^T\right] = A^{\#}(y_i, y^T)s_i + B^{\#}(y_i, y^T)p + C^{\#}(y_i, y^T)$$

where $A^{\#}(y_i, y^T)$, $B^{\#}(y_i, y^T)$, and $C^{\#}(y_i, y^T)$ are the coefficients of the projection of u on (s_i, p) when all agents other than i acquire information of quality y^T (and trade efficiently) whereas trader i acquires information of quality y_i.

When trader i too acquires information of quality $y_i = y^T$ and trades efficiently, $x_i = a^T s_i + b^T + c^T z$, with $z = \left(p - \alpha - \beta b^T\right) / \left(\beta(a^T + c^T)\right)$. Using the last two conditions to express s_i as a function of x_i and p, we then have that

$$\mathbb{E}\left[u|s_i, p; y_i, y^T\right] = A^{\#}(y^T, y^T)\frac{x_i - b^T - c^T\left(\frac{p-\alpha-\beta b^T}{\beta(a^T+c^T)}\right)}{a^T} + B^{\#}(y^T, y^T)p + C^{\#}(y^T, y^T)$$
$$= \frac{A^{\#}(y^T,y^T)}{a^T}x_i + \left[B^{\#}(y^T, y^T) - \frac{A^{\#}(y^T,y^T)c^T}{a^T\beta(a^T+c^T)}\right]p + C^{\#}(y^T, y^T) - \frac{A^{\#}(y^T,y^T)b^T}{a^T} + \frac{A^{\#}(y^T,y^T)c^T(\alpha+\beta b^T)}{a^T\beta(a^T+c^T)}.$$

Then let

$$\hat{A}^{\#} \equiv \frac{A^{\#}(y^T, y^T)}{a^T},$$

$$\hat{B}^{\#} \equiv \left[B^{\#}(y^T, y^T) - \frac{A^{\#}(y^T, y^T)c^T}{a^T\beta(a^T + c^T)}\right],$$

and

$$\hat{C}^{\#} \equiv C^{\#}(y^T, y^T) - \frac{A^{\#}(y^T, y^T)b^T}{a^T} + \frac{A^{\#}(y^T, y^T)c^T(\alpha + \beta b^T)}{a^T\beta(a^T + c^T)}.$$

We thus have that, when trader i acquires information of quality $y_i = y^T$ and trades efficiently,

$$\frac{\partial}{\partial x_i}\mathbb{E}\left[T\left(x_i, \tilde{x}, p\right)|s_i, p; y^T, y^T\right] = \delta x_i + t_p p - t_0$$

where

$$\delta = \delta' + \frac{t_{\tilde{x}}}{\beta}\hat{A}^{\#}, \tag{46}$$

$$t_p = t'_p + t_{\tilde{x}}\frac{1 + \hat{B}^{\#}}{\beta}, \tag{47}$$

and

$$t_0 = t'_0 + t_{\tilde{x}}\frac{\alpha}{\beta} - \frac{t_{\tilde{x}}}{\beta}\hat{C}^{\#}. \tag{48}$$

In the proof of Proposition 4, we showed that, when agents acquire information of quality y^T, for them to trade efficiently, the values of (δ, t_p, t_0) must coincide with those in Proposition 4 (applied to $y = y^T$). Thus, for the above policy to induce efficiency in both information acquisition and information usage, it must be that the vector $(\delta', t'_p, t'_0, t_{\tilde{x}})$ satisfies Conditions (46)-(48) with (δ, t_p, t_0) given by the values determined in Proposition 4 applied to $y = y^T$. Note that, for any $t_{\tilde{x}}$, there exists unique values of (δ', t'_p, t'_0) that solve the above three conditions. Abusing notation, denote these values by $(\delta'(t_{\tilde{x}}), t'_p(t_{\tilde{x}}), t'_0(t_{\tilde{x}}))$.

Next, note that, when the policy takes the form in (45), for any y_i, the optimal choice of $g(\cdot)$ is affine and hence can be written as $g(s_i, z) = as_i + b + cz$, for some (a, b, c). This implies that

$$\mathbb{E}[\tilde{\pi}_i(y^T, y_i); g(\cdot)] = \mathbb{E}\Bigg[(\theta + t_0'(t_{\tilde{x}}) - t_{\tilde{x}}\tilde{x})(as_i + b + cz) - \tfrac{\lambda+\delta}{2}(as_i + b + cz)^2$$
$$-(1 + t_p'(t_{\tilde{x}}))\left(\alpha - u + \beta\left[a^T(\theta + f(y^T)\eta) + b^T + c^T z\right]\right)(as_i + b + cz)\Bigg].$$

Letting $\hat{M}$ be a function of all variables that do not interact with y_i, we then have that, when $g(s_i, z) = as_i + b + cz$, for some (a, b, c),

$$\mathbb{E}[\tilde{\pi}_i(y^T, y_i); g(\cdot)] = \hat{M} - \left[t_{\tilde{x}} + \beta(1 + t_p'(t_{\tilde{x}}))\right]\tfrac{a(a^T + c^T)}{\sqrt{y^T}\sqrt{y_i}\tau_\eta} - \tfrac{(\lambda+\delta)ca}{\sqrt{y^T}\sqrt{y_i}\tau_\eta} - \tfrac{\lambda+\delta}{2}\tfrac{a^2}{y_i\tau_\eta} - \tfrac{\lambda+\delta}{2}\tfrac{a^2}{y_i\tau_e}.$$

Using the envelope theorem, we then have that

$$\frac{\partial}{\partial y_i}\tilde{V}(y^T, y_i)\bigg|_{y_i = y^T} = \frac{\left[t_{\tilde{x}} + \beta(1 + t_p'(t_{\tilde{x}})) + \lambda + \delta\right](a^T + c^T)a^T}{2\tau_\eta\left(y^T\right)^2} + \frac{(\lambda+\delta)\left(a^T\right)^2}{2\tau_e\left(y^T\right)^2} - \mathcal{C}'(y^T).$$

Once again, in writing the above derivative, we used the fact that, when $y_i = y^T$, the optimal demand schedule for trader i induces trades equal to the efficient trades $a^T s_i + b^T + c^T z$. Finally, recall that the efficient y^T is given by the solution to the following equation

$$\frac{(\beta+\lambda)(a^T + c^T)^2}{2\tau_\eta\left(y^T\right)^2} + \frac{\lambda\left(a^T\right)^2}{2\tau_e\left(y^T\right)^2} = \mathcal{C}'(y^T).$$

Hence, for the above policy to induce efficiency in information acquisition, it must be that

$$\tfrac{(\beta+\lambda)(a^T + c^T)^2}{\tau_\eta} + \tfrac{\lambda\left(a^T\right)^2}{\tau_e} = \tfrac{\left[t_{\tilde{x}} + \beta(1 + t_p'(t_{\tilde{x}})) + \lambda + \delta\right](a^T + c^T)a^T}{\tau_\eta} + \tfrac{(\lambda+\delta)\left(a^T\right)^2}{\tau_e}. \tag{49}$$

Using (47), we have that

$$t_p'(t_{\tilde{x}}) = t_p - t_{\tilde{x}}\frac{1 + \hat{B}^\#}{\beta}$$

with t_p given by the unique value determined in Proposition 4 applied to $y = y^T$. Because the function $\tilde{H} : \mathbb{R} \to \mathbb{R}$ given by $\tilde{H}(t_{\tilde{x}}) \equiv t_{\tilde{x}} + \beta t_p'(t_{\tilde{x}}) = \beta t_p - t_{\tilde{x}}\hat{B}^\#$ is linear, there exists a (unique) value of $t_{\tilde{x}}$ that solves (49).

Following steps similar to those in the proof of Proposition 6, one can show that there exist scalars $\hat{K}, \hat{M} \in \mathbb{R}_{++}$ such that, when the cost of information satisfies the properties in the proposition, the function $\tilde{V}(y^T, y_i)$ is globally quasi-concave in y_i. We conclude that, under the conditions in the proposition, the policy in (45), with $t_{\tilde{x}}$ given by the unique solution to (49) and with (δ', t_p', t_0') given by the unique solution $(\delta'(t_{\tilde{x}}), t_p'(t_{\tilde{x}}), t_0'(t_{\tilde{x}}))$ to Conditions (46)-(48), induces efficiency in both information acquisition and information usage. Q.E.D.

Proof of Proposition 12.

We establish the result by showing that the precision of private information y acquired in equilibrium is invariant in t_p. Once this property is established, the proposition follows from what established in the proof of Proposition 5. Namely, any $t_p \neq 0$ results in an equilibrium in which the precision of private information is $y = y^*$ and the sensitivity of the trades to the private signals is $a = a^*$, where y^* and a^* are as in the laissez-faire economy in which $t_p = 0$. On the other hand, for any $t_p \neq 0$, the sensitivity c of the equilibrium trades to the endogenous public signal z contained in the equilibrium price, and the constant b in the equilibrium trades are different from the corresponding levels in the laissez-faire economy. Because, given y^* and a^*, the sensitivity c^* of the equilibrium trades to the endogenous public signal z and the constant b^* in the equilibrium trades in the laissez-faire economy are welfare maximizing (by virtue of Lemma 1), we thus have that any $t_p \neq 0$ results in strictly lower welfare than $t_p = 0$, as in the case of exogenous private information (Proposition 5).
Hence, based on the arguments above, it suffices to show that any such a policy fails to change the quality of information acquired in equilibrium. To see this, fix t_p, and denote by y and (a, b, c) the precision of private information acquired in equilibrium and the parameters defining the equilibrium trades in the economy with ad-valorem tax equal to t_p.

For any y_i, let

$$V^{\#}(y, y_i) \equiv \sup_{g(\cdot)} \left\{ \mathbb{E}[\pi_i^{\#}; y, y_i, g(\cdot))] - \mathcal{C}(y_i) \right\}$$

denote the maximal payoff that trader i can obtain by selecting private information of quality y_i when all other traders acquire information of quality y and then submit the limit orders corresponding to the parameters (a, b, c), where $g : \mathbb{R}^2 \to \mathbb{R}$ is a generic function specifying the amount of shares $x_i = g(s_i, z)$ the trader purchases as a function of s_i and the endogenous public signal z contained in the equilibrium price. Let (a, b, c) be the parameters defining the equilibrium trades when information is of quality y and the tax rate is t_p. Note that[4]

$$\mathbb{E}[\pi_i^{\#}; y, y_i, g(\cdot))] \equiv \mathbb{E}\left[\theta g(s_i, z) - (1 + t_p)(\alpha + \beta b + \beta(a + c)z)\, g(s_i, z) - \tfrac{\lambda}{2}(g(s_i, z))^2 \, | y_i \right].$$

is the trader's expected payoff, gross of the information cost, when following the rule $g(\cdot)$ after acquiring information of quality y_i. In writing $\mathbb{E}[\pi_i^{\#}(y_i; g(\cdot)]$, we used the fact that the equilibrium price is given by $p = \alpha + \beta b + \beta(a + c)z$ with $z = \theta + f(y)\eta - u/(\beta a)$.

By the definition of equilibrium, if agent i acquires information of quality $y_i = y$, the limit order that maximizes his payoff must be the equilibrium ones (that is, the one corresponding to the coefficients (a, b, c)). The envelope theorem then implies that

$$N(y) \equiv \left. \frac{\partial V^{\#}(y, y_i)}{\partial y_i} \right|_{y_i = y} = \frac{\beta(1 + t_p)(a + c)a}{2\tau_\eta y^2} + \frac{\lambda a(a + c)}{2\tau_\eta y^2} + \frac{\lambda (a)^2}{2y^2 \tau_e} - \mathcal{C}'(y). \tag{50}$$

Hence, the equilibrium value of y must satisfy $N(y) = 0$. Let $M^{\#}(t_p, a, c, y)$ denote the function defined by the right-hand-side of (50). Next, use the derivations in the proof of Proposition 5 to observe that, given (t_p, y), the equilibrium values of (a, b, c) are given by (31), (32), and (33). From

[4] As above, given (a, b, c), the sensitivity of the equilibrium limit orders $\hat{c}$ to the price and the constant $\hat{b}$ in the equilibrium limit orders are obtained from (a, b, c) using (4) and (5).

the implicit function theorem, we then have that

$$\frac{dy}{dt_p} = -\frac{\frac{\partial M^{\#}(t_p,a,c,y)}{\partial t_p} + \frac{\partial M^{\#}(t_p,a,c,y)}{\partial c}\frac{\partial c}{\partial t_p}}{\frac{\partial M^{\#}(t_p,a,c,y)}{\partial y} + \frac{\partial M^{\#}(t_p,a,c,y)}{\partial a}\frac{\partial a}{\partial y} + \frac{\partial M^{\#}(t_p,a,c,y)}{\partial c}\frac{\partial c}{\partial y}},$$

where we used the fact that, given y, the equilibrium level of a is invariant in t_p. Note that $\partial c/\partial t_p$ is the partial derivative of the equilibrium level of c with respect to t_p, holding y constant, whereas $\partial a/\partial y$ and $\partial c/\partial y$ are the partial derivatives of the equilibrium levels of a and c with respect to y, holding t_p fixed.

Because

$$\frac{\partial}{\partial t_p}M^{\#}(t_p, a, c, y) = \frac{\beta(a+c)a}{2\tau_\eta y^2},$$

$$\frac{\partial}{\partial c}M^{\#}(t_p, a, c, y) = \frac{[\beta(1+t_p)+\lambda]\,a}{2\tau_\eta y^2},$$

and

$$\frac{\partial c}{\partial t_p} = \frac{-\beta(a+c)}{\beta(1+t_p)+\lambda},$$

we conclude that $dy/dt_p = 0$, as claimed. Q.E.D.