Supplement for

Robust Procurement Design

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This online supplement contains additional results for the article Mishra et al. (2025) (hereafter referred to as *the main text*). The notation is the same as in the original article. All sections, definitions, and results specific to this document have the suffix "S" to avoid confusion with the corresponding parts in the main text. Section S.1 establishes existence of a robustly optimal mechanism. Section S.2 shows that every robustly optimal mechanism (q, u) in which q is left-continuous is undominated. Section S.3, considers an extension in which the designer's short list is more permissive than in the main text and establishes robustness of the key qualitative results. Section S.4 extends Lemma 2 (characterizing the short list of quantity mechanisms) and Proposition 2 (providing a partial characterization of robustly optimal quantity mechanisms) in the main text to a setting in which the set $\mathcal{F}$ of technologies from which the seller's cost is draw is a subset of the entire set CDF(Θ) of cds supported on $\Theta = [\underline{\theta}, \bar{\theta}]$. Finally, Section S.5 examines comparative statics of robustly optimal mechanisms with respect to the lowest distribution $\underline{F}$ in $\mathcal{F}$ (in the main text such a distribution is a Dirac measure at $\bar{\theta}$. Appendix S.A contains some of the proofs.

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S.1 Existence of robustly optimal mechanisms

In this section, we show that a robustly optimal mechanism exists. In the main text, Proposition 1 contains necessary and sufficient conditions under which the Baron-Myerson-with-quantity-floor mechanism is robustly optimal. However, when these conditions are not satisfied, Proposition 2 in the main text only contains a partial characterization of the optimal mechanism. The following lemma shows that even in that case, a robustly optimal mechanism always exists.

Lemma S.1 *A robustly optimal mechanism exists.*

Proof: Recall that a mechanism is robustly optimal if and only if it solves the following program:

$$\max_{q:[\underline{\theta},\overline{\theta}]\to[0,\bar{\mathrm{q}}]} \int_{\underline{\theta}}^{\overline{\theta}} \Big[V^{\star}(q(\theta)) - z^{\star}(\theta)q(\theta)\Big] f^{\star}(\theta)(d\theta)$$

$$\text{subject to}$$

$$q \text{ weakly decreasing}$$

$$\underline{V}(q(\theta)) - \theta q(\theta) - \int_{\theta}^{\overline{\theta}} q(y)dy \geq G^{*} \qquad \forall\ \theta \in [\underline{\theta}, \overline{\theta}].$$

Hereafter, we refer to the set of schedules q satisfying the restrictions in the above problem as the "feasible set".

Since each q in the feasible set is uniformly bounded, i.e., $0 \leq q(\theta) \leq \bar{\mathrm{q}}$ for all $\theta \in [\underline{\theta}, \overline{\theta}]$, by Helly's selection theorem, the set of weakly decreasing schedules q is sequentially compact under the point-wise convergence topology. Since $\underline{V}$ is continuous and q is uniformly bounded, by the dominated convergence theorem, the function $\underline{V}(q(\theta)) - \theta q(\theta) - \int_{\theta}^{\overline{\theta}} q(y)dy$ is thus sequentially continuous. Hence, the feasible set is sequentially compact under the point-wise convergence topology. That this set is non-empty is immediate.

Next, observe that the objective function is continuous in q. To see this, define,

$$\phi(\theta, \mathrm{q}) = \Big[V^{\star}(\mathrm{q}) - z^{\star}(\theta)\mathrm{q}\Big] f^{\star}(\theta) \qquad \forall\ \theta \in \Theta, \forall\ \mathrm{q} \in [0, \bar{\mathrm{q}}]$$

Note that for a feasible function q, the value of the objective function equals

$$\int_{\underline{\theta}}^{\overline{\theta}} \phi(\theta, q(\theta))d\theta$$

Clearly, $\phi(\theta, \mathrm{q})$ is continuous in q. Furthermore, for every $\theta \in [\underline{\theta}, \overline{\theta}]$ and $\mathrm{q} \in [0, \bar{\mathrm{q}}]$,

$$\phi(\theta, \mathrm{q}) \leq \Big[V^{\star}(D^{*}(z^{\star}(\theta))) - z^{\star}(\theta)D^{*}(z^{\star}(\theta))\Big] f^{\star}(\theta) \equiv \bar{\phi}(\theta).$$

Because $\bar{\phi}$ is continuous, ϕ is uniformly bounded.

Now take a sequence (q_n) of feasible schedules converging to q under the point-wise convergence topology. Then, for every $\theta \in [\underline{\theta}, \overline{\theta}]$, we have that

$$\lim_{n\to\infty} \phi(\theta, q_n(\theta)) = \phi(\theta, q(\theta))$$

by continuity of ϕ in the second argument. Furthermore, for each q_n in the sequence, we have that

$$\phi(\theta, q_n(\theta)) \leq \bar{\phi}(\theta) \qquad \forall\ \theta \in [\underline{\theta}, \overline{\theta}].$$

Then, by the dominated convergence theorem,

$$\lim_{n\to\infty} \int_{\underline{\theta}}^{\overline{\theta}} \phi(\theta, q_n(\theta))d\theta = \int_{\underline{\theta}}^{\overline{\theta}} \phi(\theta, q(\theta))d\theta.$$

This establishes the sequential continuity of the objective function under the point-wise convergence topology. Since the range of the objective function is a subset of $\mathbb{R}$, from the extreme value theorem, we conclude that the above optimization program has a solution, i.e., a robustly optimal mechanism exists. ■

S.2 Undomination

In this section, we formally define what it means for a mechanism to be undominated, and then establish that robustly optimal mechanisms are undominated.

Recall that $\mathcal{M}$ is the set of all IC and IR mechanisms.

Definition S.1 *For any pair of mechanisms $M = (q, u)$ and $\widehat{M} = (\hat{q}, \hat{u})$, M* ***dominates*** *$\widehat{M}$ if, for every $(V, F) \in \mathcal{V} \times \mathcal{F}$,*

$$\int_{\underline{\theta}}^{\overline{\theta}} \Big[V(q(\theta)) - \theta q(\theta) - u(\theta)\Big] F(d\theta) \geq \int_{\underline{\theta}}^{\overline{\theta}} \Big[V(\hat{q}(\theta)) - \theta \hat{q}(\theta) - \hat{u}(\theta)\Big] F(d\theta),$$

with the inequality strict for some (V, F).

A mechanism $\widehat{M} \in \mathcal{M}$ is ***undominated*** *if there does not exist a mechanism $M \in \mathcal{M}$ that* ***dominates*** *it.*

The following lemma points to an internal consistency property of the set of robustly optimal mechanisms: each robustly optimal mechanism is either undominated, or it is dominated by another robustly optimal mechanism.

Lemma S.2 *Suppose $M^{\mathrm{OPT}} = (q^{\mathrm{OPT}}, u^{\mathrm{OPT}})$ is a robustly optimal mechanism and $M = (q, u) \in \mathcal{M}$ dominates M^{OPT}. Then M is a robustly optimal mechanism.*

Proof: Since $M^{\mathrm{OPT}} = (q^{\mathrm{OPT}}, u^{\mathrm{OPT}})$ is a robustly optimal mechanism, we know that, for all $\theta \in [\underline{\theta}, \overline{\theta}]$,

$$\underline{V}(q^{\mathrm{OPT}}(\theta)) - \theta q^{\mathrm{OPT}}(\theta) - u^{\mathrm{OPT}}(\theta) \geq G^{*}.$$

Now pick any $\theta \in [\underline{\theta}, \overline{\theta}]$. Since $M = (q, u)$ dominates M^{OPT}, under $V = \underline{V}$ and $F = \delta_\theta$ (where δ_θ is the Dirac distribution that puts unit point mass at θ), we have that

$$\underline{V}(q(\theta)) - \theta q(\theta) - u(\theta) \geq \underline{V}(q^{\mathrm{OPT}}(\theta)) - \theta q^{\mathrm{OPT}}(\theta) - u^{\mathrm{OPT}}(\theta).$$

Combining the two inequalities, we have that $\underline{V}(q(\theta)) - \theta q(\theta) - u(\theta) \geq G^{*}$. Since this holds for all θ, by Lemma 2 in the main text, we have that $M \in \mathcal{M}^{\mathrm{SL}}$.

Next, pick any $\theta \in [\underline{\theta}, \overline{\theta}]$. Since $M = (q, u)$ dominates M^{OPT}, by considering $V = V^{\star}$ and $F = \delta_\theta$, we have that

$$V^{\star}(q(\theta)) - \theta q(\theta) - u(\theta) \geq V^{\star}(q^{\mathrm{OPT}}(\theta)) - \theta q^{\mathrm{OPT}}(\theta) - u^{\mathrm{OPT}}(\theta),$$

which implies that, for any $F \in \mathcal{F}$,

$$\int_{\underline{\theta}}^{\overline{\theta}} \Big[V^\star(q(\theta)) - \theta q(\theta) - u(\theta)\Big] F^\star(d\theta) \geq \int_{\underline{\theta}}^{\overline{\theta}} \Big[V^\star(q^{\text{OPT}}(\theta)) - \theta q^{\text{OPT}}(\theta) - u^{\text{OPT}}(\theta)\Big] F^\star(d\theta).$$

Since $M \in \mathcal{M}^{\text{SL}}$ and M^{OPT} is robustly optimal, the above inequality implies that M is also robustly optimal. In fact, this implies that the above inequality is an equality. Therefore, for almost all $\theta \in \Theta$,

$$V^\star(q^{\text{OPT}}(\theta)) - \theta q^{\text{OPT}}(\theta) - u^{\text{OPT}}(\theta) = V^\star(q(\theta)) - \theta q(\theta) - u(\theta).$$

■

We then have the following result:

Lemma S.3 *If the Baron-Myerson-with-quantity-floor mechanism $M^\star = (q^\star, u^\star)$ is robustly optimal, it is undominated.*

Proof: Suppose $M^\star$ is robustly optimal and $M = (q, u)$ dominates it. By Lemma S.2, M is also robustly optimal. By Corollary 1 in the main text, $q(\theta) = q^\star(\theta)$ for all $\theta > \underline{\theta}$. This implies that $u(\theta) = u^\star(\theta)$ for all θ.

Consider the pair $(V^\star, \delta_{\underline{\theta}})$, where $\delta_{\underline{\theta}}$ is the Dirac distribution that puts unit mass at $\underline{\theta}$. Then

$$V^\star(q^\star(\underline{\theta})) - \underline{\theta} q^\star(\underline{\theta}) - u^\star(\underline{\theta}) > V^\star(q(\underline{\theta})) - \underline{\theta} q(\underline{\theta}) - u(\underline{\theta}).$$

The inequality holds because $u(\underline{\theta}) = u^\star(\underline{\theta})$, and $q^\star(\underline{\theta}) \equiv q^{\text{BM}}(\underline{\theta})$ uniquely maximizes surplus $V^\star(\text{q}) - \underline{\theta}\text{q}$. This inequality, however, contradicts to the fact that $M = (q, u)$ dominates $(q^\star, u^\star)$. ■

A consequence of the last Lemma S.2 is that, when the conditions in Proposition 1 in the main text are satisfied, the Baron-Myerson-with-quantity-floor mechanism is not only robustly optimal but also undominated. This result generalizes, albeit under a mild technical condition (which is satisfied by $(q^\star, u^\star)$).

Proposition S.1 *Suppose $M = (q, u)$ is a robustly optimal mechanism and q is left-continuous. Then, (q, u) is undominated.*

S.3 More permissive short list

In this section, we consider a short-list that contains also mechanisms that are not worst-case optimal but for which the guarantee is not too small relative to the maximal one. Formally, let

$$\mathcal{M}_\gamma^{\mathrm{SL}} \equiv \{M \in \mathcal{M} : G(M) \geq \gamma G(M') \ \forall\ M' \in \mathcal{M}\},$$

where $\gamma \in (0, 1]$. The analysis in the main text corresponds to the case $\gamma = 1$. Here, we extend the results to $\gamma \in (0, 1)$.

Lemma 1 in the main text remains unchanged. The short list is now characterized as follows:[1]

Lemma S.4 (Short-list characterization) *Take any IC and IR mechanism $M \equiv (q, u) \in \mathcal{M}$. Then $M \in \mathcal{M}_\gamma^{\mathrm{SL}}$ if and only if*

$$\underline{V}(q(\theta)) - \theta q(\theta) - u(\theta) \geq \gamma G^* \qquad \forall\ \theta \in \Theta. \tag{1}$$

Two important distinctions from the analogue of the same result for $\gamma = 1$ (Lemma 2 in the main text) is that when $\gamma < 1$, (i) the rent $u(\overline{\theta})$ is not necessarily pinned down to zero, and (ii) $q(\overline{\theta})$ need not be equal to q_ℓ. Moreover, short list for $\gamma' > \gamma$ is a subset of the shortlist associated with γ. Moving on, we focus on mechanisms in the $\mathcal{M}_\gamma^{\mathrm{SL}}$ that has $u(\overline{\theta}) = 0$, which will follow from the robust-optimality, and state a generalized version of Lemma 4 in the main text.

Lemma S.5 *Take any weakly decreasing function $q : \Theta \to \mathbb{R}_+$. The following statements are equivalent:*

1. *for all $\theta \in \Theta$,*
$$\underline{V}(q(\theta)) - \theta q(\theta) - \int_\theta^{\overline{\theta}} q(y)dy \geq \gamma G^*; \tag{2}$$

2. *for all $\theta \in \Theta$,*
$$\int_\theta^{\overline{\theta}} q(y)dy \leq \int_\theta^{\overline{\theta}} \underline{D}(y)dy - \int_\theta^{\underline{P}(q(\theta))} \left(\underline{D}(y) - q(\theta)\right) dy + (1-\gamma)G^*; \tag{3}$$

[1] The proofs of all results in this section are available upon request.

3. Condition (3) holds for $\theta \in \{\underline{\theta}, \overline{\theta}\}$ and, for all $\theta \in (\underline{\theta}, \overline{\theta})$,

$$\int_{\theta}^{\overline{\theta}} q(y)dy \leq \int_{\theta}^{\overline{\theta}} \underline{D}(y)dy + (1-\gamma)G^*. \tag{4}$$

We now modify Baron-Myerson-with-quantity-floor to account for the pessimism parameter γ. This is reflected by setting a floor which now depends on γ. To define the new floor (dependent on γ), notice that since $\underline{V}(\mathrm{q}) - \overline{\theta}\mathrm{q}$ is strictly concave in q, there are precisely two solutions to the equation: $\underline{V}(\mathrm{q}) - \overline{\theta}\mathrm{q} = \gamma G^*$. Denote these two solutions as $\bar{\mathrm{q}}_\ell^\gamma$ and $\underline{\mathrm{q}}_\ell^\gamma$, where $\bar{\mathrm{q}}_\ell^\gamma > \underline{\mathrm{q}}_\ell^\gamma$. The generalization of Baron-Myerson-with-quantity-floor uses the lower of the two floors: $\underline{\mathrm{q}}_\ell^\gamma$. In this way, the generalized Baron-Myerson-with-quantity-floor depends on γ, but to have notational simplicity, we supress its dependence on γ.

Definition S.2 *The **Baron-Myerson-with-quantity-floor** is the mechanism $M^\star \equiv (q_\gamma^\star, u_\gamma^\star)$, where $q_\gamma^\star$ is the quantity schedule defined, for all θ, by*

$$q_\gamma^\star(\theta) \equiv \max\{q^{\mathrm{BM}}(\theta), \underline{\mathrm{q}}_\ell^\gamma\} \tag{5}$$

and where $u^\star$ is given by $u_\gamma^\star(\theta) = \int_\theta^{\overline{\theta}} q_\gamma^\star(y)dy$ for all θ.

Proposition S.2 (Optimality of Baron-Myerson-with-quantity-floor) *Baron-Myerson-with-quantity-floor is robustly optimal if and only if*

$$\int_{\underline{\theta}}^{\overline{\theta}} q_\gamma^\star(y)dy \leq \int_{\underline{\theta}}^{\overline{\theta}} \underline{D}(y)dy - \int_{\underline{\theta}}^{\underline{P}(q_\gamma^\star(\underline{\theta}))} \left[\underline{D}(y) - q_\gamma^\star(\underline{\theta})\right]dy + (1-\gamma)G^*, \tag{6}$$

and

$$\int_{\theta}^{\overline{\theta}} q_\gamma^\star(y)dy \leq \int_{\theta}^{\overline{\theta}} \underline{D}(y)dy + (1-\gamma)G^* \qquad \forall \theta \in \Theta. \tag{7}$$

Let θ_γ^m be as defined in the main text. Let $\theta_\gamma^\star$ be the threshold defined as follows. If $q^{\mathrm{BM}}(\overline{\theta}) \leq \underline{\mathrm{q}}_\ell^\gamma$, by continuity of q^{BM} along with the fact that $q^{\mathrm{BM}}(\underline{\theta}) > \underline{\mathrm{q}}_\ell^\gamma$, let $\theta_\gamma^\star$ be the unique solution to $q^{\mathrm{BM}}(\theta_\gamma^\star) = \underline{\mathrm{q}}_\ell^\gamma$. If, instead, $q^{\mathrm{BM}}(\overline{\theta}) > \underline{\mathrm{q}}_\ell^\gamma$ (i.e., if q^{BM} never crosses $\underline{\mathrm{q}}_\ell^\gamma$), then let $\theta_\gamma^\star \equiv \overline{\theta}$. Note that both θ_γ^m and $\theta_\gamma^\star$ depend on γ.

The extended version of Proposition 2 is as follows.

Proposition S.3 *Suppose Baron-Myerson-with-quantity-floor is not robustly optimal. Then* $\theta_\gamma^m < \theta_\gamma^\star$*, and every robustly optimal mechanism* $M^{\mathrm{OPT}} = (q^{\mathrm{OPT}}, u^{\mathrm{OPT}})$ *satisfies the following properties:*

(a) $q^{\mathrm{OPT}}(\theta) = \underline{\mathrm{q}}_\ell^\gamma$ *for all* $\theta \in (\theta_\gamma^\star, \overline{\theta}]$ *when* $\theta_\gamma^\star < \overline{\theta}$.

(b) $q^{\mathrm{OPT}}(\theta) \leq q^{\mathrm{BM}}(\theta)$ *for almost all* $\theta \leq \theta_\gamma^\star$*, with the inequality strict over a Lebesgue positive measure set of types* $I \subseteq [\theta_\gamma^m, \theta_\gamma^\star]$.

(c) $q^{\mathrm{OPT}}(\theta) = q^{\mathrm{BM}}(\theta)$ *for all* $\theta \in (\underline{\theta}, \theta_\gamma^m)$.

Note that the three intervals identified in Proposition S.3 depend on γ, and hence, are different from the analogous proposition for $\gamma = 1$ (Proposition 2 in the main text). More importantly, the floor used in part (a) of Proposition S.3 is different from that in Proposition 2 of main text since the worst-case optimality does not uniquely pin down $q(\overline{\theta})$ to be $\underline{\mathrm{q}}_\ell^\gamma$, and only restricts it to be in the interval $[\underline{q}_\ell^\gamma, \bar{q}_\ell^\gamma]$.

Finally, when it comes to the price regulation, Baron-Myerson-with-price-cap still remains robustly-optimal with a higher price cap of $\underline{P}(\underline{q}_\ell^\gamma) > \overline{\theta}$. Consequently, Proposition 4 in the main text, which compared price and quantity regulation, extends with quantity regulation dominating price regulation less often because of increased price-cap.

S.4 More general forms of technological uncertainty

In this section, we relax the fact that $\mathcal{F}$ contains all distributions over Θ. We show how our results in the main text change under this relaxation. We begin by defining $\mathcal{F}$, the set of plausible technologies, as a subset of the set $\mathrm{CDF}(\Theta)$ with some structure. In particular, we assume there exists a cdf $\underline{F} \in \mathrm{CDF}(\Theta)$ such that $\mathcal{F}$ is the set of all cdfs $F \in \mathrm{CDF}(\Theta)$ such that $F(\theta) \geq \underline{F}(\theta)$ for all $\theta \in \Theta$. In the main text, we assumed $\underline{F}(\theta)$ is the Dirac distribution that puts unit mass at $\overline{\theta}$. This amounted to $\mathcal{F} = \mathrm{CDF}(\Theta)$. Now, we allow $\underline{F}$ to have support $[\theta_s, \overline{\theta}]$, where $\theta_s \in (\underline{\theta}, \overline{\theta})$. In particular, we assume the following:

Definition S.3 *The cdf* $\underline{F}$ *is regular with respect to* θ_s *if it is absolutely continuous over* $\mathbb{R}$ *with density* $f(\theta) > 0$ *if only if* $\theta \in [\theta_s, \overline{\theta}]$ *and with* $\underline{z}(\theta) \equiv \theta + \underline{F}(\theta)/\underline{f}(\theta)$ *continuous and increasing over* $[\theta_s, \overline{\theta}]$.

Let $\underline{M}^{\text{BM}} \equiv (\underline{q}^{\text{BM}}, \underline{u}^{\text{BM}}) \in \mathcal{M}$ denote an arbitrary IC and IR mechanism that is optimal under the model $(\underline{V}, \underline{F})$. Note that such a mechanism is not unique, but in any such a mechanism $\underline{q}^{\text{BM}}$ is non-increasing, $\underline{u}^{\text{BM}}(\bar{\theta}) = 0$, and $\underline{u}^{\text{BM}}(\theta) = \int_{\theta}^{\bar{\theta}} \underline{q}^{\text{BM}}(y)dy$ for all θ. Then let

$$G_s^* \equiv \int_{\theta_s}^{\bar{\theta}} \underline{W}(\theta, \underline{q}^{\text{BM}}) \underline{F}(\mathrm{d}\theta). \tag{8}$$

be the buyer's expected welfare under the mechanism $\underline{M}^{\text{BM}}$ when the gross value function is $\underline{V}$, and the technology is $\underline{F}$. Note that, when $\theta_s = \bar{\theta}$, as in the previous sections, $G_s^* = G^*$. Finally, for any IC and IR mechanism $M = (q, u) \in \mathcal{M}$, let $\underline{w}_q \equiv \inf_{\theta \leq \theta_s} \underline{W}(\theta, q)$. The next proposition generalizes Lemma 2 in the main text by providing a complete characterization of the short-list when $\underline{F}$ is regular with respect to θ_s.

Proposition S.4 *Suppose $\underline{F}$ is regular with respect to θ_s. The following are then true:*

1. *For any $M \in \mathcal{M}^{\text{SL}}$, $G(M) = G_s^*$;*
2. *A mechanism $M \equiv (q, u) \in \mathcal{M}^{\text{SL}}$ if and only if the following conditions jointly hold*

   (a) q is non-increasing,

   (b) for all θ, $u(\theta) = \int_{\theta}^{\bar{\theta}} q(y)dy$,

   (c) $q(\theta) = \underline{q}^{\text{BM}}(\theta)$ for all $\theta \in (\theta_s, \overline{\theta})$,

   (d) $\underline{V}(q(\theta)) - \theta q(\theta) - \int_{\theta}^{\bar{\theta}} q(y)dy \geq G_s^*$ for all $\theta \leq \theta_s$,

   (e) $\underline{w}_q \underline{F}(\theta) + \int_{\theta}^{\bar{\theta}} \underline{W}(y, \underline{q}^{\text{BM}}) \underline{F}(dy) \geq G_s^*$ for all $\theta \in [\theta_s, \overline{\theta}]$.

Part (1) follows from the fact that Nature can always pick $(\underline{V}, \underline{F})$, which implies that the guarantee of any IC and IR mechanism is bounded above by the maximal welfare attainable under the lowest gross value function $\underline{V}$ and the worst technology $\underline{F}$. This upper bound on guarantee can be achieved by offering a mechanism $M \equiv (q, u)$ in which $q(\theta) = \underline{q}^{\text{BM}}(\theta_s)$ for all $\theta \leq \theta_s$, $q(\theta) = \underline{q}^{\text{BM}}(\theta)$ for all $\theta > \theta_s$, and $u(\theta) = \int_{\theta}^{\bar{\theta}} q(y)dy$ for all θ. Against such a

A: Function $\underline{W}(\cdot, q)$. B: Technology F_1 generates welfare below G_s^*.

Figure S.1: Graphical illustration of new robustness constraint in Part (e) of Proposition S.4.

mechanism an adversarial Nature cannot do better than selecting $(V, F) = (\underline{V}, \underline{F})$, yielding the buyer a payoff of G_s^*.

Conditions (a)-(d) in Part (2) are generalizations of robustness constraints in Lemma 2 in the main text. Condition (c) follows from the fact that Nature can always choose the model $(\underline{V}, \underline{F})$ and the worst-case optimality uniquely pins down the buyer's response to be $\underline{M}^{\mathrm{BM}} \equiv (\underline{q}^{\mathrm{BM}}, \underline{u}^{\mathrm{BM}})$. As for condition (d), it now applies only to $\theta \leq \theta_s$ as Nature can now select a Dirac distribution only on $\theta \leq \theta_s$. General technological uncertainty adds a novel constraint given in part (e). It stems from the fact that Nature's best response can be a non-Dirac distribution. To understand this constraint, consider Figure S.1. Let $\theta_1 < \theta_s$ be a cost level at which the function $\underline{W}(\cdot, q)$ reaches the minimum over $[\underline{\theta}, \theta_s]$, i.e., $\underline{W}(\theta_1, q) = \underline{w}_q$, and let $\theta_2 > \theta_s$ be a cost level such that $\underline{W}(\theta_2, q) = \underline{W}(\theta_2, \underline{q}^{\mathrm{BM}}) = \underline{w}_q$. Suppose Nature picks a distribution F_1 with an atom at θ_1 equal to $\underline{F}(\theta_2)$ and which agrees with $\underline{F}$ on all $\theta \geq \theta_2$. Because $\underline{W}(\cdot, \underline{q}^{\mathrm{BM}})$ is decreasing, even if $\underline{w}_q \geq G_s^*$ (as implied by the constraint in Part (c)), in the absence of the new constraint in Part (e) it may well be the case that

$$\underline{w}_q F_1(\theta_1) + \int_{\theta_2}^{\overline{\theta}} \underline{W}(\theta, \underline{q}^{\mathrm{BM}}) F_1(\mathrm{d}\theta) = \underline{w}_q \underline{F}(\theta_2) + \int_{\theta_2}^{\overline{\theta}} \underline{W}(\theta, \underline{q}^{\mathrm{BM}}) \underline{F}(\mathrm{d}\theta) < G_s^*,$$

meaning that the mechanism's guarantee is below G_s^*. Therefore, to guard against such worst-case possibilities, we need constraint in part (e).

Continuing further, we now establish an analog of Proposition 2, which provided a partial characterization of robustly optimal mechanism when $\mathcal{F} = \mathrm{CDF}(\Theta)$. To do so, we generalize the definitions of q_ℓ, $q^\star$, $\theta^\star$ and θ^m as follows. Let

$$\mathrm{q}_\ell^s \equiv \underline{D}(\theta_s)$$

denote the efficient output when the inverse demand is $\underline{P}$ and the cost is θ_s. Then let $q_s^\star$ be the quantity schedule defined by

$$q_s^\star(\theta) \equiv \begin{cases} \max\{q^{\mathrm{BM}}(\theta), \mathrm{q}_\ell^s\} & \theta < \theta_s \\ \underline{q}^{\mathrm{BM}}(\theta) & \theta \geq \theta_s, \end{cases} \tag{9}$$

where q^{BM} continues to denote the optimal quantity schedule of Baron and Myerson (1982) when the model is $(V^\star, F^\star)$, with $F^\star$ regular, whereas $\underline{q}^{\mathrm{BM}}$ is the optimal quantity schedule of Baron and Myerson (1982) when the model is $(\underline{V}, \underline{F})$. The following mechanism is a natural generalization of Baron-Myerson-with-quantity-floor in the main text.

Definition S.4 *The* ***Baron-Myerson-with-quantity-bridge*** *is the mechanism $M_s^\star = (q_s^\star, u_s^\star)$ where $q_s^\star$ is the quantity schedule in (9) and $u_s^\star$ is the function given by $u_s^\star(\theta) = \int_\theta^{\overline{\theta}} q_s^\star(y)dy$ for all θ.*

Finally, let $\theta_s^\star$ be the threshold cost defined as follows. If $q^{\mathrm{BM}}(\theta_s) \leq \mathrm{q}_\ell^s$, by continuity of q^{BM} along with the fact that $q^{\mathrm{BM}}(\underline{\theta}) > \mathrm{q}_\ell^s$ (assured by the regularity of $F^\star$), $\theta_s^\star$ is the unique solution to $q^{\mathrm{BM}}(\theta_s^\star) = \mathrm{q}_\ell^s$. If, instead, $q^{\mathrm{BM}}(\theta_s) > \mathrm{q}_\ell^s$ (i.e., if q^{BM} never crosses q_ℓ^s over the interval $[\underline{\theta}, \theta_s^\star]$), then $\theta_s^\star \equiv \theta_s$. In either case, $\theta_s^\star \leq \theta_s$. Similarly, let

$$\theta_s^m \equiv \max\{\theta : \theta \in \arg\min_{y \in [\underline{\theta}, \theta_s]} \underline{W}(y, q_s^\star)\}.$$

Thus, $\underline{w}_{q_s^\star} = \underline{W}(\theta_s^m, q_s^\star)$. Finally, let

$$G_s^{**} \equiv \sup_{\theta \in (\theta_s, \overline{\theta}]} \frac{1}{\underline{F}(\theta)} \int_{\theta_s}^{\theta} \underline{W}(y, \underline{q}^{\mathrm{BM}}) \underline{F}(\mathrm{d}y). \tag{10}$$

We are now ready to state the generalization of Proposition 2 in the main text, extending the partial characterization of robustly optimal mechanism.

Proposition S.5 *Suppose $F^\star$ is regular and $\underline{F}$ is regular with respect to θ_s. Then, the following are true.*

1. *The Baron-Myerson-with-quantity-bridge mechanism is robustly optimal if and only if $\underline{W}(\theta_s^m, q_s^\star) \geq \max\{G_s^*, G_s^{**}\}$.*

2. *If $\underline{W}(\theta_s^m, q_s^\star) < \max\{G_s^*, G_s^{**}\}$, then $\theta_s^m < \theta_s^\star$ and every robustly optimal mechanism $M^{\mathrm{OPT}} = (q^{\mathrm{OPT}}, u^{\mathrm{OPT}})$ satisfies the following properties:*

A: Baron-Myerson-with-quantity-bridge. B: Robustly optimal mechanism when $V^\star \neq \underline{V}$.

Figure S.2: Graphical illustration of Proposition S.5.

(a) $q^{\mathrm{OPT}}(\theta) = \underline{q}^{\mathrm{BM}}(\theta)$ *for all* $\theta \in (\theta_s, \overline{\theta})$,

(b) $q^{\mathrm{OPT}}(\theta) = \mathrm{q}_\ell^s$ *for all* $\theta \in (\theta_s^\star, \theta_s)$,

(c) $q^{\mathrm{OPT}}(\theta) \leq q^{\mathrm{BM}}(\theta)$ *for almost all* $\theta < \theta_s^\star$*, with the inequality strict over a Lebesgue positive measure set of types* $I \subseteq [\theta_s^m, \theta_s^\star]$*, and*

(d) $q^{\mathrm{OPT}}(\theta) = q^{\mathrm{BM}}(\theta)$ *for all* $\theta \in (\underline{\theta}, \theta_s^m)$.

We end this section with three remarks about Proposition S.5.

Remark S.1 *To be precise, part (2) of Proposition S.5 generalizes Proposition 2 in the main text. Part (1) of Proposition S.5 provides necessary and sufficient conditions for the Baron-Myerson-with-quantity-bridge mechanism to be robustly optimal. This is analogous to Proposition 1 in the main text which provided necessary and sufficient conditions for the Baron-Myerson-with-quantity-floor mechanism to be robustly optimal when* $\mathcal{F} = \mathrm{CDF}(\Theta)$.

Remark S.2 *If* $V^\star = \underline{V}$ *(as when the only uncertainty is over the cost technology), then* $\theta_s^m = \theta_s$*, and the condition in Part 1 of Proposition S.5 is satisfied. In this case, the Baron-Myerson-with-quantity-bridge mechanism, illustrated in Panel A of Figure S.2, is a robustly optimal mechanism. Panel B illustrates the features of robustly optimal mechanism when, instead,* $V^\star \neq \underline{V}$*. A comparison of these figures with Figure 1 and Figure 2 in the main text illustrates that the key features of the robustly optimal mechanism continue to hold under general technological uncertainty.*

Remark S.3 *Proposition S.5 further highlights that the key forces identified in Sections 3 and 4 of the main text continue to determine the shape of robustly optimal mechanisms to the left of* θ_s*. Most importantly, the level of the plateau in robustly optimal mechanisms is determined by* $\underline{D}(\theta_s)$*, the efficient quantity at* θ_s *under the lowest possible inverse demand. Moreover, this level is robust to even more general specifications of technological uncertainty where* $\mathcal{F} \subset \mathrm{CDF}(\Theta)$ *is an arbitrary set with the property that there exists* $\underline{F} \in \mathcal{F}$ *that first-order stochastically dominates all other cost distributions in* $\underline{F}$.

S.5 Non-monotonicity of the procured quantity in $\underline{F}$

As Proposition S.5 shows the robustly optimal mechanism depends on $\mathcal{F}$ only through $F^\star$ and $\underline{F}$. In this section, we show that as $\underline{F}$ changes the robustly optimal mechanism may change in a non-monotonic way. To simplify exposition, we focus on the case $V^\star = \underline{V}$. As is clear from Proposition S.5 (see Remark S.2), the robustly optimal mechanism is the Baron-Myerson-with-quantity-bridge. We formalize the changes in the Baron-Myerson-with-quantity-bridge mechanism as $\underline{F}$ changes.

Consider a sequence $(\underline{F}_n)$ of cdfs corresponding to the lowest elements of the set $\mathcal{F}$ while fixing $F^\star$ with the following properties:

(a) for every n there exists $\underline{\theta}_n \in \Theta$ and $\delta_n \geq 0$ such that,

  (1) $\underline{F}_n$ is absolutely continuous over $(-\infty, \overline{\theta})$, with density $\underline{f}_n(\theta) > 0$ for all $\theta \in [\underline{\theta}_n, \overline{\theta})$,

  (2) $\underline{F}_n(\theta) = 0$ for all $\theta < \underline{\theta}_n$, $\underline{F}_n(\theta) = 1$ for all $\theta \geq \overline{\theta}$,

  (3) $\lim_{\theta \uparrow \bar{\theta}} \underline{F}_n(\theta) = 1 - \delta_n$,

(b) for every n, $\underline{\theta}_{n+1} \geq \underline{\theta}_n$, and for every $\theta \in (\underline{\theta}, \overline{\theta})$, there exists n such that $\theta < \underline{\theta}_n < \overline{\theta}$,

(c) for every n, $\delta_{n+1} \geq \delta_n$,

(d) there exists $\overline{n}, \overline{\overline{n}} \in \mathbb{N} \cup \{+\infty\}$ with $\overline{\overline{n}} > \overline{n}$ such that $\underline{\theta}_n = \underline{\theta}$ if, and only if, $n \leq \overline{n}$, and $\delta_n > 0$ if, and only if, $n \geq \overline{\overline{n}}$.

(e) for every n, the function $\underline{z}_n : [\underline{\theta}_n, \overline{\theta}] \to \mathbb{R}$ defined by

$$\underline{z}_n(\theta) \equiv \begin{cases} \theta + \underline{F}_n(\theta)/\underline{f}_n(\theta) & \text{if } \theta \in [\underline{\theta}_n, \overline{\theta}) \\ \overline{\theta} + 1/\underline{f}_n(\overline{\theta}) & \text{if } \theta = \overline{\theta} \text{ and } \delta_n = 0 \\ \overline{\theta} + (1-\delta_n)/\delta_n & \text{if } \theta = \overline{\theta} \text{ and } \delta_n > 0 \end{cases}$$

is increasing over $[\underline{\theta}_n, \overline{\theta}]$ and continuous over $[\underline{\theta}_n, \overline{\theta})$.

(f) for all $\theta \in [\underline{\theta}_{n+1}, \overline{\theta}]$,

$$\frac{\underline{F}_{n+1}(\theta)}{\underline{f}_{n+1}(\theta)} \leq \frac{\underline{F}_n(\theta)}{\underline{f}_n(\theta)}, \tag{11}$$

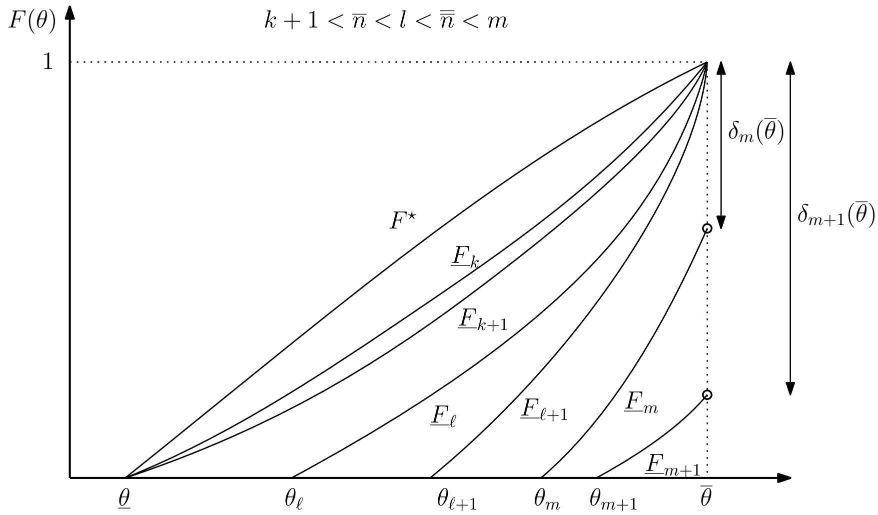

Figure S.3: Pictorial depiction of the sequence $(\underline{F}_n)$.

and for every n and every $\theta \in [\underline{\theta}_n, \overline{\theta}]$,

$$\underline{z}_n(\theta) < z^\star(\theta), \tag{12}$$

which is the case when $\frac{\underline{F}_n(\theta)}{\underline{f}_n(\theta)} < \frac{F^\star(\theta)}{f^\star(\theta)}$.

Figure S.3 provides an illustration of the sequence $(\underline{F}_n)$. Note that property (c) above means that the technologies are ranked in the reverse-hazard-rate order. The sequence can thus be interpreted as capturing an increase in the severity of the buyer's uncertainty over the technology that determines the seller's cost.

Let q_n^{OPT} be a robustly optimal quantity schedule when the lowest technology in $\mathcal{F}$ is $\underline{F}_n$. The following proposition establishes that the quantity procured under a robustly optimal mechanism is not monotone in the lowest distribution $\underline{F}_n$. This property holds despite the fact that, as is well known, the Baron-Myerson quantity schedule $\underline{q}_n^{\text{BM}}$ defined, for all $\theta \in [\underline{\theta}_n, \overline{\theta})$, by

$$\underline{q}_n^{\text{BM}}(\theta) \equiv \arg\max_{\text{q} \in [0,\bar{\text{q}}]} \{V^\star(\text{q}) - \underline{z}_n(\theta)\text{q}\}$$

is increasing in the inverse-hazard rare order: for any $n, n' \in \mathbb{N}$, with $n' > n$ and any $\theta \geq \underline{\theta}_{n'}$, $\underline{q}_{n'}^{\text{BM}}(\theta) \geq \underline{q}_n^{\text{BM}}(\theta)$. That is, when the buyer's model over the technology of the seller's cost coincides with the distribution $\underline{F}_n$, an increase in the distribution (in the inverse-hazard-rate order) leads to an increase in the output procured.

Figure S.4: Illustration of Proposition S.6.

Proposition S.6 (Non-monotonicity of output in severity of cost uncertainty) *Suppose $V^\star = \underline{V}$ and $F^\star$ is regular. Let $(\underline{F}_n)$ be any sequence of cdfs satisfying properties (a)-(f) above and let (M_n^{OPT}) be any sequence of mechanisms such that, for each n, $M_n^{\text{OPT}} \equiv (q_n^{\text{OPT}}, u_n^{\text{OPT}})$ is a robustly optimal mechanism when the lowest distribution in $\mathcal{F}$ is $\underline{F}_n$. Then, for every $\theta \in (\underline{\theta}, \overline{\theta})$,*

1. *there exists $n(\theta) \in \mathbb{N}$ such that $q_n^{\text{OPT}}(\theta)$ is non-decreasing in n (respectively, non-increasing in n) over $n \leq n(\theta) - 1$ (respectively, over $n > n(\theta)$).*

2. *there exists $j, k \in \mathbb{N}$ with $j < k$ such that $q_j^{\text{OPT}}(\theta) > q_k^{\text{OPT}}(\theta)$.*

Figure S.4 illustrates the result in Proposition S.6. For any $\theta \in [\underline{\theta}, \theta^\dagger]$, as the lowest technology changes from $\underline{F}_1$ to $\underline{F}_2$, the quantity procured increases. In fact, the robustly optimal quantity schedule changes from the dash-dotted line to the dash-double-dotted line. Note that both $\underline{F}_1$ to $\underline{F}_2$ have support Θ; a reduction in the inverse of reverse hazard rate then implies a reduction in the value of reducing the rents paid to the most efficient types and hence an increase in the output procured under the optimal mechanism. When the lowest technology changes from $\underline{F}_2$ to $\underline{F}_3$, the robustly optimal quantity schedule changes from the dash-double-dotted line to the solid line and the quantity procured from types in the range $[\underline{\theta}, \theta^\dagger]$ goes down. This is because the support of $\underline{F}_3$ no longer contains low-cost types. The buyer can then afford to procure less output from these types without jeopardizing robustness. Thus, the quantity procured from types in the range $[\underline{\theta}, \theta^\dagger]$ is not monotone in n, equivalently, in the worst possible technology. The formal proof is in Appendix S.A.

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S.A Proofs of Propositions S.1, S.4, S.5, and S.6

Proof of Proposition S.1: Let $M \equiv (q, u)$ be a robustly optimal mechanism in which q is left-continuous. By contradiction, assume M is dominated by another mechanism $\widehat{M} \equiv (\hat{q}, \hat{u})$. Lemma S.2 implies that $\widehat{M}$ is also robustly optimal, and therefore, by Proposition 2 in the main text, $\hat{q}$ differs from q only on the interval $(\theta^m, \theta^\star)$. Let

$$\Theta_q := \{\theta \in (\theta^m, \theta^\star) : \hat{q}(\theta) < q(\theta)\}.$$

By Lemma S.6 (which is stated and proved below after this proof), Θ_q has a positive Lebesgue measure. By Proposition 2 in Mishra et al. (2025), we know that

$$\hat{q}(\theta) < q(\theta) \leq q^{\mathrm{BM}}(\theta) < D^\star(\theta) \qquad \forall\ \theta \in \Theta_q. \tag{13}$$

Next, define

$$\theta_h := \sup\ \{\theta : \theta \in \Theta_q\}.$$

Since Θ_q is non-empty and bounded θ_h is a finite real number. Clearly, $\theta_h > \theta^m$. The proof below establishes a contradiction by showing that there exists θ, in a left neighborhood of θ_h, at which the buyer gets a higher welfare under the mechanism M when $(V, F) = (V^\star, \delta_\theta)$.

By left-continuity of q and since q and $\tilde{q}$ are weakly decreasing, there exists a non-empty left-neighborhood $\mathcal{N} := [\theta_\ell, \theta_h)$ such that $\mathcal{N} \subseteq \Theta_q$ and

$$q(\theta) < D^\star(\theta_h) \qquad \forall\ \theta \in \mathcal{N}.$$

The last inequality follows from the fact that $q(\theta_h) < D^\star(\theta_h)$ (by (13)) and q is left-continuous. Let

$$\Delta = \sup_{y \in \mathcal{N}} \left[q(y) - \hat{q}(y)\right]$$

By definition of $\mathcal{N}$, $\Delta > 0$. Thus, there exists $\theta \in \mathcal{N}$ such that $q(\theta) - \hat{q}(\theta) = \Delta - \epsilon$, where $\epsilon < \Delta \frac{P^\star(q(\theta_\ell)) - \theta_h}{P^\star(q(\theta_\ell)) - \theta^m}$. Such $\epsilon > 0$ can be chosen because, by definition of $\mathcal{N}$, we have

$q(\theta_\ell) < D^\star(\theta_h)$ and this implies $P^\star(q(\theta_\ell)) > \theta_h > \theta^m$. Observe that

$$W(M; V^\star, \delta_\theta) - W(\widehat{M}; V^\star, \delta_\theta) = \int_{\hat{q}(\theta)}^{q(\theta)} \big(P^\star(z) - \theta\big)dz - \int_{\theta}^{\bar{\theta}} \big(q(y) - \hat{q}(y)\big)dy.$$

To reach a contradiction and complete the proof, it suffices to show that the right-hand side of the above condition is positive.

$$\begin{aligned}
&W(M; V^\star, \delta_\theta) - W(\widehat{M}; V^\star, \delta_\theta)\\
&= \int_{\hat{q}(\theta)}^{q(\theta)} \big(P^\star(z) - \theta\big)dz - \Big(\int_{\theta}^{\theta_h} \big(q(y) - \hat{q}(y)\big)dy\Big)\\
&\qquad - \Big(\int_{\theta_h}^{\bar{\theta}} \big(q(y) - \hat{q}(y)\big)dy\Big)\\
&\geq \int_{\hat{q}(\theta)}^{q(\theta)} \big(P^\star(z) - \theta\big)dz - \Big(\int_{\theta}^{\theta_h} \big(q(y) - \hat{q}(y)\big)dy\Big) && \text{(since } q(y) \leq \hat{q}(y) \text{ for all } y > \theta_h\text{)}\\
&\geq \big(P^\star(q(\theta_\ell)) - \theta\big)(q(\theta) - \hat{q}(\theta)) - \Delta(\theta_h - \theta) && \text{(since } q(\theta_\ell) \geq q(\theta) \text{ and by definition of } \Delta\text{)}\\
&= \big(P^\star(q(\theta_\ell)) - \theta\big)(\Delta - \epsilon) - \Delta(\theta_h - \theta)\\
&= \Delta\big(P^\star(q(\theta_\ell)) - \theta_h\big) - \epsilon\big(P^\star(q(\theta_\ell)) - \theta\big)\\
&\geq \Delta\big(P^\star(q(\theta_\ell)) - \theta_h\big) - \epsilon\big(P^\star(q(\theta_\ell)) - \theta^m\big)\\
&> 0,
\end{aligned}$$

where the last inequality follows from the fact that $\epsilon < \Delta \frac{P^\star(q(\theta_\ell)) - \theta_h}{P^\star(q(\theta_\ell)) - \theta^m}$. ■

Lemma S.6 *The set Θ_q has positive Lebesgue measure.*

Proof: Assume, toward a contradiction, that Θ_q has zero Lebesgue measure. Then, either (1) $q(\theta) \leq \hat{q}(\theta)$ for all $\theta \in (\theta^m, \theta^\star)$, or (2) $q(\theta) > \hat{q}(\theta)$ only on countably many $\theta \in (\theta^m, \theta^\star)$. Below, we obtain a contradiction in each of these two cases.

Case 1: Suppose $q(\theta) \leq \hat{q}(\theta)$ for all $\theta \in (\theta^m, \theta^\star)$. If this inequality holds with equality for all $\theta \in (\theta^m, \theta^\star)$, then $\hat{q}$ coincides with q at all θ, a contradiction. Thus, $q(\theta) < \hat{q}(\theta)$ for some $\theta \in (\theta^m, \theta^\star)$. Moreover, because q is left-continuous, and both q and $\hat{q}$ are weakly-decreasing, it must be that $q(\theta) < \hat{q}(\theta)$ on a positive Lebesgue measure of $\theta \in (\theta^m, \theta^\star)$. Thus, $u(\theta^m) < \hat{q}(\theta^m)$. Moreover, any robustly optimal mechanism $q(\theta^m) = \hat{q}(\theta^m) = \underline{D}(\theta^m)$ (Lemma 10 in the main text). Therefore,

$$V^\star(q(\theta^m)) - \theta^m q(\theta^m) - \int_{\theta^m}^{\overline{\theta}} q(y)dy > V^\star(\hat{q}(\theta^m)) - \theta^m \hat{q}(\theta^m) - \int_{\theta^m}^{\overline{\theta}} \hat{q}(y)dy,$$

That is, $W(M; V^\star, \delta_{\theta^m}) > W(\widehat{M}; V^\star, \delta_{\theta^m})$, a contradiction to the fact that $\widehat{M}$ dominates M.

Case 2: Now suppose, $q(\theta) > \hat{q}(\theta)$ only on countably many $\theta \in (\theta^m, \theta^\star)$. Because q is left-continuous, and both q and $\hat{q}$ are weakly-decreasing, these points must be the ones where q is discontinuous. Let θ' be one such point of discontinuity of q in $(\theta^m, \theta^\star)$. Because $q(\theta) = \hat{q}(\theta)$ for almost all $\theta \in \Theta$, we have $u(\theta') = \hat{q}(\theta')$. Furthermore, because $q(\theta) < D^\star(\theta)$ for all $\theta \in (\theta^m, \theta^\star)$, we have that $\hat{q}(\theta') < q(\theta') < D^\star(\theta')$. Combining these facts with the quasi-concavity of $V^\star(\mathrm{q}) - \theta'\mathrm{q}$, we have that

$$V^\star(q(\theta')) - \theta' q(\theta') - \int_{\theta'}^{\overline{\theta}} q(y)dy > V^\star(\hat{q}(\theta')) - \theta' \hat{q}(\theta') - \int_{\theta'}^{\overline{\theta}} \hat{q}(y)dy.$$

That is, $W(M; V^\star, \delta_{\theta'}) > W(\widehat{M}; V^\star, \delta_{\theta'})$, a contradiction to the fact that $\widehat{M}$ dominates M. Consequently, Θ_q has positive Lebesgue measure. ■

Proof of Proposition S.4. **Part 1**. For any IC and IR mechanism $M = (q, u) \in \mathcal{M}$,

$$G(M) = \inf_{(V,F)\in\mathcal{V}\times\mathcal{F}} W(M; V, F) \leq W(M; \underline{V}, \underline{F}) \leq_{(a)} W(\underline{M}^{\mathrm{BM}}; \underline{V}, \underline{F}) = G_s^*. \tag{14}$$

Inequality (a) holds because $\underline{M}^{\mathrm{BM}}$ maximizes $W(\cdot; \underline{V}, \underline{F})$ over $\mathcal{M}$. We now show that there exists an IC and IR mechanism $\underline{M} \in \mathcal{M}$ such that $G(\underline{M}) = G_s^*$. Let $\underline{M} \equiv (\underline{q}, \underline{u})$ be the mechanism in which

$$\underline{q}(\theta) = \begin{cases} \underline{q}^{\mathrm{BM}}(\theta_s) & \text{if } \theta < \theta_s \\ \underline{q}^{\mathrm{BM}}(\theta) & \text{if } \theta \geq \theta_s, \end{cases}$$

and $\underline{u}(\theta) = \int_\theta^{\overline{\theta}} \underline{q}(y)dy$ for all θ. Clearly, $\underline{M} \in \mathcal{M}$. Further, since $\underline{F}$ has support $[\theta_s, \overline{\theta}]$, it follows that $W(\underline{M}; \underline{V}, \underline{F}) = W(\underline{M}^{\mathrm{BM}}; \underline{V}, \underline{F})$. Now, recall that $\underline{M}^{\mathrm{BM}} = (\underline{q}^{\mathrm{BM}}, \underline{u}^{\mathrm{BM}})$ is the optimal mechanism for the model $(\underline{V}, \underline{F})$. When $\underline{F}$ is regular with respect to θ_s, $\underline{q}^{\mathrm{BM}}$ is such that, for all $\theta \geq \theta_s$,

$$\underline{q}^{\mathrm{BM}}(\theta) = \underline{D}(\underline{z}(\theta)),$$

where, for all $\theta \geq \theta_s$, $\underline{z}(\theta) \equiv \theta + \underline{F}(\theta)/\underline{f}(\theta)$. Thus, $\underline{q}(\theta) \leq \underline{D}(\theta)$ for all θ, with the inequality strict for $\theta \neq \theta_s$.[2] Part A of Lemma 9 in Mishra et al. (2025) then implies that $\underline{W}(\cdot, \underline{q})$ is non-increasing over Θ. Furthermore, because, for all $F \in \mathcal{F}$, $\underline{F} \succ_{FOSD} F$,

$$W(\underline{M}; \underline{V}, F) \geq W(\underline{M}; \underline{V}, \underline{F}).$$

Because, for any $V \in \mathcal{V}$ and any $F \in \mathcal{F}$, $W(\underline{M}; V, F) \geq W(\underline{M}; \underline{V}, F)$, we thus have that $W(\underline{M}; V, F) \geq W(\underline{M}; \underline{V}, \underline{F})$. We conclude that $G(\underline{M}) = W(\underline{M}^{\mathrm{BM}}; \underline{V}, \underline{F}) = G_s^*$.

Part 2: Necessity. If $M = (q, u) \in \mathcal{M}^{\mathrm{SL}}$, then M is IC and IR, and, therefore, q is non-increasing and $u(\theta) = u(\overline{\theta}) + \int_\theta^{\overline{\theta}} q(y)dy$ for all θ. Further, by the result in Part 1, it must be that $G(M) = G_s^*$. Hence, $u(\overline{\theta}) = 0$.

Recall that, for any $\theta \geq \theta_s$, $\underline{q}^{\mathrm{BM}}(\theta) = \arg\max_{\mathrm{q} \in [0, \bar{\mathrm{q}}]} \{\underline{V}(\mathrm{q}) - \underline{z}(\theta)\mathrm{q}\}$. If $q(\theta) \neq \underline{q}^{\mathrm{BM}}(\theta)$ for a positive Lebesgue measure subset of $[\theta_s, \bar{\theta}]$, then inequality (a) in (14) is strict and hence $W(M; \underline{V}, \underline{F}) < W(\underline{M}^{\mathrm{BM}}; \underline{V}, \underline{F}) = G_s^*$. This means that $G(M) < G_s^*$ and hence $M \notin \mathcal{M}^{\mathrm{SL}}$, a contradiction. Because $\underline{q}^{\mathrm{BM}}$ is decreasing and continuous over $[\theta_s, \bar{\theta}]$, we conclude that $q(\theta) = \underline{q}^{\mathrm{BM}}(\theta)$ for all $\theta \in (\theta_s, \bar{\theta})$.

Next, observe that, for any $\theta < \theta_s$, $\mathcal{F}$ contains a distribution F corresponding to a Dirac measure at $\theta < \theta_s$ (indeed, $\underline{F} \succ_{\mathrm{FOSD}} F$). Welfare under the lowest gross value function $\underline{V}$ and such an F is $\underline{V}(q(\theta)) - \theta q(\theta) - \int\limits_\theta^{\bar{\theta}} q(y)dy = \underline{W}(\theta, q)$. Hence, it must be that $\underline{W}(\theta, q) \geq G_s^*$.

Finally, observe that the inequality in the constraint in part (e) is an equality for $\theta = \theta_s$. Suppose there exists $\tilde{\theta} \in (\theta_s, \overline{\theta}]$ such that

$$\underline{w}_q \underline{F}(\tilde{\theta}) + \int\limits_{\tilde{\theta}}^{\bar{\theta}} \underline{W}(y, \underline{q}^{\mathrm{MB}}) \underline{F}(\mathrm{d}y) < G_s^*. \tag{15}$$

[2] This property holds even if $\underline{F}$ is not regular. In fact, any undominated mechanism $M = (q, u)$ is such that $q(\theta) \leq \underline{D}(\theta)$ for all θ (Mishra and Patil, 2025).

By definition of $\underline{w}_q$, there exists $\theta' \leq \theta_s$ such that $\underline{W}(\theta', q)$ is arbitrarily close to $\underline{w}_q$. Let $\widetilde{F}$ be the cdf given by

$$\widetilde{F}(\theta) = \begin{cases} 0 & \text{if } \theta < \theta' \\ \underline{F}(\tilde{\theta}) & \text{if } \theta \in [\theta', \tilde{\theta}) \\ \underline{F}(\theta) & \text{if } \theta \geq \tilde{\theta}. \end{cases}$$

Clearly, $\widetilde{F}(\theta) \geq \underline{F}(\theta)$ for all θ, and hence, $\widetilde{F} \in \mathcal{F}$. Welfare under the mechanism (q, u) when Nature selects the model $(\underline{V}, \widetilde{F})$ is equal to

$$W(M; \underline{V}, \widetilde{F}) = \underline{W}(\theta', q)\underline{F}(\tilde{\theta}) + \int_{\tilde{\theta}}^{\bar{\theta}} \underline{W}(y, \underline{q}^{\mathrm{MB}})\underline{F}(\mathrm{d}y) < G_s^*,$$

where the inequality follows from inequality (15) and the fact that $\underline{W}(\theta', q)$ is arbitrarily close to $\underline{w}_q$. This, however, is a contradiction to $M \in \mathcal{M}^{\mathrm{SL}}$.

We conclude that properties (a)-(e) are jointly necessary for any $M \in \mathcal{M}^{\mathrm{SL}}$.

Part 2: Sufficiency. Take any mechanism M satisfying properties (a)-(e). By virtue of (a) and (b), M is IC and IR. Thus, it suffices to prove that $G(M) = G_s^*$. By inequality (14), it is enough to show $W(M; V, F) \geq G_s^*$ for any model $(V, F) \in \mathcal{V} \times \mathcal{F}$. First, suppose F is a Dirac distribution on some $\theta \leq \theta_s$. Then, condition in part (c) implies

$$W(M; V, F) \geq W(M; \underline{V}, F) = \underline{V}(q(\theta)) - \theta q(\theta) - \int_{\theta}^{\bar{\theta}} q(y)dy \geq G_s^*.$$

Now consider any model $(V, F) \in \mathcal{V} \times \mathcal{F}$, where F puts a positive mass on $\theta > \theta_s$. Then,

$$\begin{aligned} W(M; V, F) &\geq W(M; \underline{V}, F) \\ &\geq \underline{w}_q F(\theta_s) + \int_{\theta_s}^{\bar{\theta}} \underline{W}(\theta, q)F(\mathrm{d}\theta) \qquad \text{(by definition of } \underline{w}_q) \\ &= \underline{w}_q F(\theta_s) + \int_{\theta_s}^{\bar{\theta}} \underline{W}(\theta, \underline{q}^{\mathrm{BM}})F(\mathrm{d}\theta) \qquad \text{(because } q(\theta) = \underline{q}^{\mathrm{MB}}(\theta) \text{ for all } \theta \in (\theta_s, \bar{\theta})). \end{aligned} \tag{16}$$

Now, partition $[\theta_s, \overline{\theta}]$ into $\Theta_1 \equiv \{\theta \in [\theta_s, \overline{\theta}] : \underline{W}(\theta, \underline{q}^{\mathrm{BM}}) \leq \underline{w}_q\}$ and $\Theta_2 \equiv [\theta_s, \overline{\theta}] \setminus \Theta_1$. Note that $\underline{W}(\cdot, \underline{q}^{\mathrm{BM}})$ is decreasing over $[\theta_s, \overline{\theta}]$ and hence Θ_1 is an (possibly empty) interval. If Θ_1

is empty, let $\hat{\theta} \equiv \theta_s$. Else, let $\hat{\theta}$ be the left endpoint of Θ_1. Using (16), we have that

$$
\begin{aligned}
W(M;V,F) &\geq \underline{w}_q F(\theta_s) + \int_{\theta_s}^{\hat{\theta}} \underline{W}(\theta, \underline{q}^{\mathrm{BM}}) F(\mathrm{d}\theta) + \int_{\hat{\theta}}^{\overline{\theta}} \underline{W}(\theta, \underline{q}^{\mathrm{BM}}) F(\mathrm{d}\theta) \\
&\geq \underline{w}_q F(\theta_s) + \underline{w}_q \left(F(\hat{\theta}) - F(\theta_s) \right) + \int_{\hat{\theta}}^{\overline{\theta}} \underline{W}(\theta, \underline{q}^{\mathrm{MB}}) F(\mathrm{d}\theta) \\
&\qquad \text{(because } \underline{W}(\cdot, \underline{q}^{\mathrm{MB}}) \geq \underline{w}_q \text{ on } [\theta_s, \hat{\theta}]) \\
&= \underline{w}_q F(\hat{\theta}) + \int_{\hat{\theta}}^{\overline{\theta}} \underline{W}(\theta, \underline{q}^{\mathrm{MB}}) F(\mathrm{d}\theta) \qquad (17)
\end{aligned}
$$

Now, let $h : \Theta \to \mathbb{R}$ be the non-increasing function defined by

$$
h(\theta) = \begin{cases} \underline{w}_q & \text{if } \theta \leq \hat{\theta} \\ \underline{W}(\theta, \underline{q}^{\mathrm{MB}}) & \text{otherwise.} \end{cases}
$$

From (17), we then have that

$$
\begin{aligned}
W(M;V,F) &\geq \int_{\underline{\theta}}^{\hat{\theta}} h(\theta) F(\mathrm{d}\theta) + \int_{\hat{\theta}}^{\overline{\theta}} h(\theta) F(\mathrm{d}\theta) = \int_{\underline{\theta}}^{\overline{\theta}} h(\theta) F(\mathrm{d}\theta) \\
&\geq \int_{\underline{\theta}}^{\overline{\theta}} h(\theta) \underline{F}(\mathrm{d}\theta) \qquad \text{(since } \underline{F} \succ_{\mathrm{FOSD}} F) \\
&= \underline{w}_q \underline{F}(\hat{\theta}) + \int_{\hat{\theta}}^{\overline{\theta}} \underline{W}(\theta, \underline{q}^{\mathrm{MB}}) \underline{F}(\mathrm{d}\theta) \qquad \text{(by definition of } h) \\
&\geq G_s^* \qquad \text{(by condition (e) in Part 3)}
\end{aligned}
$$

Hence, by Part (1), $G(M) = G_s^*$. ■

Proof of Proposition S.5. The proof is in several parts, each corresponding to a part of the proposition. Before we get to these parts, observe that for a mechanism $(q, u) \in \mathcal{M}^{\mathrm{SL}}$ the constraint (d) in Part 2 of Proposition S.4 is equivalent to $\underline{w}_q \geq G_s^*$. As for constraint (e),

$$\underline{w}_q \underline{F}(\theta) + \int_{\theta}^{\bar{\theta}} \underline{W}(y, \underline{q}^{\mathrm{BM}}) \underline{F}(\mathrm{d}y) \geq G_s^* = \int_{\theta_s}^{\bar{\theta}} \underline{W}(y, \underline{q}^{\mathrm{BM}}) \underline{F}(\mathrm{d}y) \quad \forall \ \ \theta \geq \theta_s$$

is equivalent to

$$\underline{w}_q \geq \sup_{\theta \in (\theta_s, \bar{\theta}]} \frac{1}{\underline{F}(\theta)} \int_{\theta_s}^{\theta} \underline{W}(y, \underline{q}^{\mathrm{BM}}) \underline{F}(\mathrm{d}y) = G_s^{**}.$$

Thus, constraints (d) and (e) can be equivalently written as $\underline{w}_q \geq \max\{G_s^*, G_s^{**}\}$, or alternatively, $\underline{W}(\theta, q) \geq \max\{G_s^*, G_s^{**}\}$ for all $\theta \leq \theta_s$.

Part 1. If $M_s^\star$ is robustly optimal, then $M_s^\star \in \mathcal{M}^{\mathrm{SL}}$. Observe that $\underline{w}_{q_s^\star} = \underline{W}(\theta_s^m, q_s^\star)$. Thus, as argued above, conditions (d) and (e) in Part 2 of Proposition S.4 imply $\underline{W}(\theta_s^m, q_s^\star) \geq \max\{G_s^*, G_s^{**}\}$.

Next, suppose that $M_s^\star$ is such that $\underline{W}(\theta_s^m, q_s^\star) \geq \max\{G_s^*, G_s^{**}\}$. We want to show that $M_s^\star$ is robustly optimal. By definition, $M_s^\star$ satisfies conditions (a)-(c) in Part 2 of Proposition S.4. As we argued above, that $\underline{W}(\theta_s^m, q_s^\star) \geq \max\{G_s^*, G_s^{**}\}$ implies that the two robustness constraints (d) and (e) in Part 2 of Proposition S.4 are satisfied. This means that $M_s^\star \in \mathcal{M}^{\mathrm{SL}}$. To see that $M_s^\star$ maximizes the buyer's payoff under the conjectured model $(V^\star, F^\star)$ over $\mathcal{M}^{\mathrm{SL}}$, first observe that $q_s^\star$ is non-increasing because $F^\star$ is regular. Second note that every mechanism $M = (q, u)$ in the short list has a non-increasing quantity schedule q that agrees with $\underline{q}^{BM}$ over $(\theta_s, \bar{\theta})$ (Proposition S.4). This means that, in any such a mechanism, $q(\theta) \geq \mathrm{q}_\ell^s$ for all $\theta \in [\underline{\theta}, \theta_s)$. Because, for every $\theta \in [\underline{\theta}, \theta_s)$,

$$q_s^\star(\theta) = \arg \max_{\mathrm{q} \in [\mathrm{q}_\ell^s, \bar{\mathrm{q}}]} \{V^\star(\mathrm{q}) - z^\star(\theta)\mathrm{q}\}$$

we conclude that, for any $M = (q, u) \in \mathcal{M}^{\mathrm{SL}}$,

$$\int_{\underline{\theta}}^{\bar{\theta}} \Big[V^\star(q(\theta)) - z^\star(\theta) q(\theta)\Big] F^\star(d\theta) \leq \int_{\underline{\theta}}^{\bar{\theta}} \Big[V^\star(q_s^\star(\theta)) - z^\star(\theta) q_s^\star(\theta)\Big] F^\star(d\theta)$$

implying that indeed $M_s^\star$ maximizes the buyer's payoff (under the conjectured model $(V^\star, F^\star)$) over $\mathcal{M}^{\mathrm{SL}}$.

Part 2. We start with the following lemma:

Lemma S.7 *Suppose* $\underline{w}_{q_s^\star} < \max\{G_s^*, G_s^{**}\}$*. Then the following are true:*

1. $\theta_s^m < \theta_s^\star$,

2. *if* $\theta_s^m > \underline{\theta}$*, then* $q_s^\star(\theta_s^m) = \underline{D}(\theta_s^m)$*.*

Proof: **Part 1 of Lemma S.7**. We consider two cases. First, suppose $q^{\text{BM}}(\theta_s) > \mathrm{q}_\ell^s = \underline{D}(\theta_s)$. Then, $\theta_s^\star = \theta_s$. Since $\underline{D}$ and q^{BM} are decreasing and continuous, there exists a non-empty left neighborhood of θ_s where $q^{\text{BM}}(\theta) > \underline{D}(\theta)$. Part B of Lemma 9 in Mishra et al. (2025) then implies that $\underline{W}(\theta, q)$ is increasing on this interval implying $\theta_s^m < \theta_s = \theta_s^\star$.

Next, suppose $q^{\text{BM}}(\theta_s) \leq \mathrm{q}_\ell^s = \underline{D}(\theta_s)$. Then, $\theta_s^\star \leq \theta_s$, and for every $\theta \in [\theta_s^\star, \theta_s]$, $q_s^\star(\theta) = \mathrm{q}_\ell^s$ and $\underline{W}(\theta, q_s^\star) = \underline{W}(\theta_s, q_s^\star) \geq \max\{G_s^*, G_s^{**}\} > \underline{w}_{q_s^\star}$, where the first inequality follows from (10), and the second inequality follows from our assumption. This implies that $\underline{W}(\theta_s^\star, q_s^\star) > \underline{w}_{q_s^\star}$. Hence, $\theta_s^m < \theta_s^\star$.

Part 2 of Lemma S.7. Since $\theta_s^m < \theta_s^\star$, we have $q_s^\star(\theta_s^m) = q^{\text{BM}}(\theta_s^m)$. Using this and repeating the arguments of Lemma 10 in Mishra et al. (2025), the proof can be completed. ■

Equipped with this result, we now establish parts Parts 2(a) and 2(b) of the proposition.

Parts 2(a) and 2(b). Part 2(a) follows from Proposition S.4 because any robustly optimal mechanism belongs to $\mathcal{M}^{\text{SL}}$. Thus consider Part 2(b). We consider two cases. First, if $q^{\text{BM}}(\theta_s) \geq \mathrm{q}_\ell^s$, then, by the definition of $\theta_s^\star$, we have that $\theta_s^\star = \theta_s$, and hence the interval $(\theta_s^\star, \theta_s)$ is empty and the result applies vacuously. Therefore, suppose that $q^{\text{BM}}(\theta_s) < \mathrm{q}_\ell^s$. Then, $\theta_s^\star < \theta_s$. Now, assume for a contradiction that there exists $\theta' \in (\theta_s^\star, \theta_s)$ such that $q^{\text{OPT}}(\theta') > \mathrm{q}_\ell^s$. Monotonicity of q^{OPT} then implies that $q^{\text{OPT}}(\theta) > \mathrm{q}_\ell^s$ for all $\theta \in [\theta_s^\star, \theta']$. This means that there exists a non-zero Lebesgue measure of types such that $q^{\text{OPT}}(\theta) > \mathrm{q}_\ell^s$. Then, consider the mechanism $\widetilde{M} = (\tilde{q}, \tilde{u})$ where the quantity schedule is given by

$$\tilde{q}(\theta) = \begin{cases} q^{\text{OPT}}(\theta) & \text{if } \ \theta < \theta_s^\star \\ \mathrm{q}_\ell^s & \text{if } \ \theta \in [\theta_s^\star, \theta_s] \\ \underline{q}^{\text{BM}}(\theta) = q^{\text{OPT}}(\theta) & \text{if } \ \theta \geq \theta_s \end{cases}$$

and where the rents $\tilde{u}$ are given by $\tilde{u}(\theta) = \int_{\theta}^{\bar{\theta}} \tilde{q}(y)dy$ for all θ. Because $\tilde{q}$ is non-increasing, this ensures that $\widetilde{M}$ is IC and IR. The buyer's payoff from $\widetilde{M}$ (under the conjectured model) is equal to

$$\int_{\underline{\theta}}^{\overline{\theta}} \Big[V^\star(\tilde{q}(\theta)) - z^\star(\theta)\tilde{q}(\theta)\Big] F^\star(\mathrm{d}\theta)$$

which is strictly higher than under M^{OPT}. This follows from the fact that, for any $\theta \in [\theta^\star_s, \theta_s]$, q^s_ℓ maximizes $V^\star(\mathrm{q}) - z^\star(\theta)\mathrm{q}$ over $\mathrm{q} \geq \mathrm{q}^s_\ell$, along with the fact that $F^\star$ is absolutely continuous. Thus, to produce a contradiction to the robust optimality of M^{OPT}, it suffices to show that $\widetilde{M} \in \mathcal{M}^{\mathrm{SL}}$.

By definition, $\widetilde{M}$ satisfies properties (a)-(c) in Part 2 of Proposition S.4. As for properties (d) and (e), they are equivalent to verifying that $\underline{w}_{\tilde{q}} \geq \max\{G^*_s, G^{**}_s\}$. That $\underline{w}_{\tilde{q}} \geq G^*_s$ follows from the arguments in the proof of Lemma 6 in Mishra et al. (2025), along with the fact that $\tilde{q}(\theta) \leq q^{\mathrm{OPT}}(\theta)$ for all θ. To establish that $\underline{w}_{\tilde{q}} \geq G^{**}_s$, notice that, for all $\theta \in [\theta^\star_s, \theta_s]$, $\underline{W}(\theta, \tilde{q}) = \underline{W}(\theta_s, \underline{q}^{\mathrm{BM}}) \geq G^{**}_s$. Thus, it suffices to focus on $\theta < \theta^\star_s$. Observe that

$$\inf_{\theta < \theta^\star_s} \underline{W}(\theta, \tilde{q}) \geq \inf_{\theta < \theta^\star_s} \underline{W}(\theta, q^{\mathrm{OPT}}) \geq G^{**}_s,$$

where the first inequality follows from the fact that $\tilde{q}(\theta) = q^{\mathrm{OPT}}(\theta)$ for $\theta < \theta^\star_s$, along with the fact that $\tilde{q}(\theta) \leq q^{\mathrm{OPT}}(\theta)$ for all θ, which implies that $\tilde{u}(\theta) \leq u^{\mathrm{OPT}}(\theta)$ for all θ. The second inequality holds because $M^{\mathrm{OPT}} \in \mathcal{M}^{\mathrm{SL}}$.

Part 2(c). From Part 2(b), $q^{\mathrm{OPT}}(\theta) = \mathrm{q}^s_\ell$ for all $\theta \in (\theta^\star_s, \theta_s)$. Now suppose there is a positive-Lebesgue-measure set $I \subseteq [\underline{\theta}, \theta^\star_s)$ such that $q^{\mathrm{OPT}}(\theta) > q^\star_s(\theta) = q^{\mathrm{BM}}(\theta)$. Consider the mechanism $\widetilde{M} = (\tilde{q}, \tilde{u})$ where the quantity schedule is given by

$$\tilde{q}(\theta) = \min\{q^\star_s(\theta), q^{\mathrm{OPT}}(\theta)\} \qquad \forall\ \theta \in \Theta,$$

and where $\tilde{u}(\theta) = \int_\theta^{\overline{\theta}} \tilde{q}(y)dy$ for all θ. Clearly, because $\tilde{q}$ is non-increasing and $\tilde{u}$ satisfies the above properties, the mechanism $\widetilde{M}$ is IC and IR. Notice, by Part 2 (a) and (b), $\tilde{q}(\theta) = q^\star_s(\theta) = q^{\mathrm{OPT}}(\theta)$ for all $\theta \geq \theta^\star_s$.

The buyer's payoff under $\widetilde{M}$ is strictly higher than under M^{OPT} following arguments similar to those in the proof of Lemma 7 in Mishra et al. (2025). Clearly, $\widetilde{M}$ satisfies conditions (a)-(c) of Part 2 of Proposition S.4. The next two claims establish that $\widetilde{M}$ also satisfies the conditions in parts (d) and (e), that is, $\underline{w}_{\tilde{q}} \geq \max\{G^*_s, G^{**}_s\}$ or $\underline{W}(\theta, \tilde{q}) \geq \max\{G^*_s, G^{**}_s\}$ for all $\theta \leq \theta_s$. First, observe for every $\theta \in [\theta^\star_s, \theta_s]$, $\tilde{q}(\theta) = q^\star_s(\theta) = \mathrm{q}^s_\ell$ and

$\underline{W}(\theta, \tilde{q}) = \underline{W}(\theta_s, \tilde{q}) > \max\{G_s^*, G_s^{**}\}$. Now, we establish the desired inequality for $\theta < \theta_s^\star$ using the following two claims.

Claim S.1 *Suppose $\theta < \theta_s^\star$ is such that either $\tilde{q}(\theta) = q^{\mathrm{OPT}}(\theta)$ or $\underline{D}(\theta) \leq \tilde{q}(\theta) = q_s^\star(\theta) < q^{\mathrm{OPT}}(\theta)$. Then $\underline{W}(\theta, \tilde{q}) \geq \max\{G_s^*, G_s^{**}\}$.*

Proof: Pick $\theta < \theta_s^\star$. We establish $\underline{W}(\theta, \tilde{q}) \geq \underline{W}(\theta, q^{\mathrm{OPT}})$, and because $\underline{W}(\theta, q^{\mathrm{OPT}}) \geq \max\{G_s^*, G_s^{**}\}$, the claim follows.

Note that $q_s^\star(\theta) = q^{\mathrm{BM}}(\theta)$. For any θ such that $\tilde{q}(\theta) = q^{\mathrm{OPT}}(\theta)$, since $\tilde{q}(y) \leq q^{\mathrm{OPT}}(y)$ for all $y \geq \theta$, we have that $\underline{W}(\theta, \tilde{q}) \geq \underline{W}(\theta, q^{\mathrm{OPT}})$. Thus, consider a θ for which $\underline{D}(\theta) \leq \tilde{q}(\theta) = q_s^\star(\theta) = q^{\mathrm{BM}}(\theta) < q^{\mathrm{OPT}}(\theta)$. The quasi-concavity of the function $\underline{V}(\mathrm{q}) - \theta\mathrm{q}$ in q implies that

$$\underline{V}(q_s^\star(\theta)) - \theta q_s^\star(\theta) > \underline{V}(q^{\mathrm{OPT}}(\theta)) - \theta q^{\mathrm{OPT}}(\theta).$$

Together with the fact that $\tilde{q}(y) \leq q^{\mathrm{OPT}}(y)$ for all $y \geq \theta$, this means that $\underline{W}(\theta, \tilde{q}) \geq \underline{W}(\theta, q^{\mathrm{OPT}})$. ■

Claim S.2 *Suppose $\theta < \theta_s^\star$ is such that $q_s^\star(\theta) < \min\{\underline{D}(\theta), q^{\mathrm{OPT}}(\theta)\}$. Then, $\underline{W}(\theta, \tilde{q}) \geq \max\{G_s^*, G_s^{**}\}$.*

Proof: The proof considers two cases to establish the existence of $\theta' > \theta$ such that $W(\cdot, \tilde{q})$ is non-increasing on $[\theta, \theta']$ with $W(\theta', \tilde{q}) \geq \max\{G_s^*, G_s^{**}\}$.

Case 1. Suppose $q_s^\star(\theta_s) = \mathrm{q}_\ell^s = \underline{D}(\theta_s)$. Because $q_s^\star$ and $\underline{D}$ are both continuous, there exists $\theta < \theta' \leq \theta_s$ such that $q_s^\star(y) \leq \underline{D}(y)$ for all $y \in [\theta, \theta']$, with $q_s^\star(\theta') = \underline{D}(\theta')$. Thus,

$$\tilde{q}(\theta') = \min\{\underline{D}(\theta'), q^{\mathrm{OPT}}(\theta')\}.$$

Further, for all $y \in [\theta, \theta']$,

$$\tilde{q}(y) = \min\{q^{\mathrm{OPT}}(y), q_s^\star(y)\} \leq \underline{D}(y).$$

Part A of Lemma 9 in Mishra et al. (2025) implies that $\underline{W}(\cdot, \tilde{q})$ is non-increasing over $[\theta, \theta']$ whereas Claim S.1 implies that $\underline{W}(\theta', \tilde{q}) \geq \max\{G_s^*, G_s^{**}\}$. Hence, $\underline{W}(\theta, \tilde{q}) \geq \max\{G_s^*, G_s^{**}\}$.

Case 2. Now suppose $q_s^\star(\theta_s) = q^{\mathrm{BM}}(\theta_s) > \mathrm{q}_\ell^s = \underline{D}(\theta_s)$. Then, because $q_s^\star(\theta) < \underline{D}(\theta)$, and $\underline{D}$ and $q_s^\star$ are continuous (latter due to regularity of $F^\star$), there exists $\theta < \hat{\theta} < \overline{\theta}$ such that $q_s^\star(\hat{\theta}) = \underline{D}(\hat{\theta})$ and $q_s^\star(y) > \underline{D}(y)$ for all $y > \hat{\theta}$. Again, just like we argued in Case 1, there exists $\theta < \theta' \leq \hat{\theta}$ such that $q_s^\star(y) \leq \underline{D}(y)$ for all $y \in [\theta, \theta']$ with $q_s^\star(\theta') = \underline{D}(\theta')$. Repeating the remaining arguments in Case 1 completes the proof. ■

The above two claims establish that $\tilde{q}$ satisfies the constraint $\underline{w}_{\tilde{q}} \geq \max\{G_s^*, G_s^{**}\}$, and by Proposition S.4, $\widetilde{M} = (\tilde{q}, \tilde{u}) \in \mathcal{M}^{\mathrm{SL}}$.

We complete the proof by showing that there must exist a set of types $I \subseteq [\underline{\theta}, \theta_s^\star]$ of positive Lebesgue measure such that $q^{\mathrm{OPT}}(\theta) < q^{\mathrm{BM}}(\theta)$ for all $\theta \in I$. To do that assume for contradiction $q^{\mathrm{OPT}}(\theta) = q^{\mathrm{BM}}(\theta)$ almost everywhere on $[\underline{\theta}, \theta_s^\star]$. Moreover, because $q_s^\star$ is continuous and $q^{\mathrm{BM}}(\theta)$ is the unique maximizer of $V^\star(\mathrm{q}) - z^\star(\theta)\mathrm{q}$, it is without loss to assume that $q^{\mathrm{OPT}}(\theta) = q^{\mathrm{BM}}(\theta)$ for all $\theta < \theta_s^\star$. This however implies that $M_s^\star$ is robustly optimal, a contradiction.

Part 2(d). Assume for contradiction that there exists a $\theta \in (\underline{\theta}, \theta_s^m)$ such that $q^{\mathrm{OPT}}(\theta) \neq q^{\mathrm{BM}}(\theta)$. Because q^{BM} is continuous and decreasing, this means that there exists a positive Lebesgue measure set of types $I \subseteq [\underline{\theta}, \theta_s^m)$ such that $q^{\mathrm{OPT}}(\theta) \neq q^{\mathrm{BM}}(\theta)$ for all $\theta \in I$. By Part (c), we have that $q^{\mathrm{OPT}}(\theta) < q^{\mathrm{BM}}(\theta)$ for all $\theta \in I$ (as q^{BM} is continuous and both q^{BM} and q^{OPT} are non-increasing). Then, let $\widetilde{M} = (\tilde{q}, \tilde{u})$ be the mechanism where the quantity schedule is given by

$$\tilde{q}(\theta) = \begin{cases} q^{\mathrm{BM}}(\theta) & \text{if } \theta \in [\underline{\theta}, \theta_s^m] \\ q^{\mathrm{OPT}}(\theta) & \text{otherwise} \end{cases}$$

and where $\tilde{u}(\theta) = \int_\theta^{\overline{\theta}} \tilde{q}(y)dy$ for all θ. Clearly, $\widetilde{M}$ is IC and IR. Below, we show that $\widetilde{M}$ yields a higher payoff to the buyer than M^{OPT} and $\widetilde{M} \in \mathcal{M}^{\mathrm{SL}}$, contradicting the optimality of M^{OPT}.

Because, for any θ, $q^{\mathrm{BM}}(\theta)$ is the unique maximizer of $V^\star(\mathrm{q}) - z^\star(\theta)\mathrm{q}$, the objective function

$$\int_{\underline{\theta}}^{\overline{\theta}} \Big[V^\star(q(\theta)) - z^\star(\theta)q(\theta)\Big] F^\star(d\theta)$$

is strictly higher under $\widetilde{M}$ than under M^{OPT}.

We now show that $\tilde{q}$ satisfies the robustness constraint $\underline{w}_{\tilde{q}} \geq \max\{G_s^*, G_s^{**}\}$. To do so, it suffice to show $\underline{W}(\theta, \tilde{q}) \geq \max\{G_s^*, G_s^{**}\}$ for $\theta \leq \theta_s^m$ because the inequality holds for any $\theta > \theta_s^m$. Thus consider $\theta \in [\underline{\theta}, \theta_s^m]$. For any $\theta \leq \theta_s^m$, $\tilde{q}(\theta) = q^{\mathrm{BM}}(\theta) = q_s^\star(\theta)$. The latter equality follows from Lemma S.7, which states $\theta_s^m < \theta_s^\star$. Moreover,

$$
\begin{aligned}
\underline{W}(\theta, \tilde{q}) - \underline{W}(\theta, q_s^\star) = &\int_{\theta_s^m}^{\bar{\theta}} q_s^\star(y)dy - \int_{\theta_s^m}^{\bar{\theta}} q^{\mathrm{OPT}}(y)dy \\
\geq_{(a)}& \left[\underline{V}(q^{\mathrm{OPT}}(\theta_s^m)) - \theta_s^m q^{\mathrm{OPT}}(\theta_s^m)\right] - \left[\underline{V}(q^{\mathrm{BM}}(\theta_s^m)) - \theta_s^m q^{\mathrm{BM}}(\theta_s^m)\right] \\
&+ \int_{\theta_s^m}^{\bar{\theta}} q_s^\star(y)dy - \int_{\theta_s^m}^{\bar{\theta}} q^{\mathrm{OPT}}(y)dy \\
=_{(b)}& \ \underline{W}(\theta_s^m, q^{\mathrm{OPT}}) - \underline{W}(\theta_s^m, q_s^\star) \\
\geq_{(c)}& \max\{G^*, G_s^{**}\} - \underline{W}(\theta_s^m, q_s^\star) \\
\geq_{(d)}& \max\{G^*, G_s^{**}\} - \underline{W}(\theta, q_s^\star),
\end{aligned}
$$

Inequality (a) follows from the fact that $\underline{D}(\theta_s^m)$ maximizes $\underline{V}(\mathrm{q}) - \theta_s^m \mathrm{q}$ over all q and $q^{\mathrm{BM}}(\theta_s^m) = \underline{D}(\theta_s^m)$ (Lemma S.7). Equality (b) follows from the fact that $q^{\mathrm{BM}}(\theta) = q_s^\star(\theta)$. Inequality (c) follows from the fact that $M^{\mathrm{OPT}} \in \mathcal{M}^{\mathrm{SL}}$ which implies that $q^{\mathrm{OPT}}(y)$ satisfies the robustness constraint. Inequality (d) follows from the definition of θ_s^m. Hence, $\underline{W}(\theta, \tilde{q}) \geq \max\{G^*, G_s^{**}\}$ also for all $\theta \in [\underline{\theta}, \theta_s^m]$. We conclude that $\widetilde{M} \in \mathcal{M}^{\mathrm{SL}}$ and yields a higher payoff to the buyer than M^{OPT} contradicting the optimality of M^{OPT}. ■

This completes the proof of Proposition S.5. ■

Proof of Proposition S.6. Each part below establishes the corresponding part in the proposition.

Part 1. For any $\theta \in (\underline{\theta}, \bar{\theta})$, let $n(\theta)$ be the largest $n > \overline{n}$ such that $\underline{\theta}_n \leq \theta < \underline{\theta}_{n+1}$. Existence of $n(\theta)$ is guaranteed because of (b) in the definition of $(\underline{F}_n)$, which imply $\underline{\theta}_n \leq \underline{\theta}_{n+1} < \bar{\theta}$ and $\lim_{n\to\infty} \underline{\theta}_n = \bar{\theta}$. For any $n \leq n(\theta) - 1$, we have $\theta \in [\underline{\theta}_n, \bar{\theta}]$ and $\theta \in [\underline{\theta}_{n+1}, \bar{\theta}]$. Thus, by Proposition S.5, $q_n^{\mathrm{OPT}}(\theta) = \underline{q}_n^{\mathrm{BM}}(\theta)$, and $q_{n+1}^{\mathrm{OPT}}(\theta) = \underline{q}_{n+1}^{\mathrm{BM}}(\theta)$. Condition (11) in turn implies that $\underline{q}_n^{\mathrm{BM}}(\theta) \leq \underline{q}_{n+1}^{\mathrm{BM}}(\theta)$, that is, $q_n^{\mathrm{OPT}}(\theta)$ is non-decreasing in n for $n \leq n(\theta) - 1$.

For any $n > n(\theta)$, $\theta < \underline{\theta}_n$, and therefore, $q_n^{\mathrm{OPT}}(\theta) = \max\{q^{\mathrm{BM}}(\theta), \underline{D}(\underline{\theta}_n)\}$. The quantity $\underline{D}(\underline{\theta}_n)$ is non-increasing in n because $\underline{\theta}_n \leq \underline{\theta}_{n+1} < \bar{\theta}$ for every n. Consequently, $q_n^{\mathrm{OPT}}(\theta)$ is also non-increasing in n.

Part 2. To establish the second part of the proposition it suffices to exhibit a pair $j, k \in \mathbb{N}$, with $j < k$, such that $q_j^{\mathrm{OPT}}(\theta) > q_k^{\mathrm{OPT}}(\theta)$. To do so, consider the following two cases.

Case 1. Suppose $q^{\mathrm{BM}}(\theta) \geq \underline{D}(\underline{\theta}_{n(\theta)+1})$. Then let $j = n(\theta)$ and $k = n(\theta) + 1$, and observe that

$$q_j^{\mathrm{OPT}}(\theta) = \underline{q}_j^{\mathrm{BM}}(\theta) = \underline{D}\left(\theta + \frac{\underline{F}_j(\theta)}{\underline{f}_j(\theta)}\right) > \underline{D}\left(\theta + \frac{F^\star(\theta)}{f^\star(\theta)}\right) = q^{\mathrm{BM}}(\theta) = q_k^{\mathrm{OPT}}(\theta),$$

where the inequality follows from (12).

Case 2. Suppose $q^{\mathrm{BM}}(\theta) < \underline{D}(\underline{\theta}_{n(\theta)+1})$. Then let $j = n(\theta)+1$ and let k be such that $\underline{\theta}_k > \underline{\theta}_j$. Existence of such an k is ensured by condition (b) in the definition of $(\underline{F}_n)$. Then

$$q_k^{\mathrm{OPT}}(\theta) = \max\{q^{\mathrm{BM}}(\theta), \underline{D}(\underline{\theta}_k)\} < \underline{D}(\underline{\theta}_j) = q_j^{\mathrm{OPT}}(\theta).$$

To see this, observe that $\underline{\theta}_k > \underline{\theta}_j$ implies that $\underline{D}(\underline{\theta}_k) < \underline{D}(\underline{\theta}_j)$. Hence, if $q_k^{\mathrm{OPT}}(\theta) = \underline{D}(\underline{\theta}_k)$, then $q_k^{\mathrm{OPT}}(\theta) = \underline{D}(\underline{\theta}_k) < \underline{D}(\underline{\theta}_j) = q_j^{\mathrm{OPT}}(\theta)$. If, instead, $q_k^{\mathrm{OPT}}(\theta) = q^{\mathrm{BM}}(\theta)$, the result follows from the fact that, by assumption, $q^{\mathrm{BM}}(\theta) < \underline{D}(\underline{\theta}_j)$. ■