Online Appendix for
Signaling in a Global Game: Coordination and Policy Traps

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In this Appendix, we provide two auxiliary results for the paper. The first one is Lemma 2, which was used without proof in Proposition 4 (unbounded policy noise). The second is an example in which the payoff that the policy maker enjoys from maintaining the status quo is negatively correlated with its strenght; this case was briefly referred to in Section 5 of the paper.

A1. Proof of Lemma 2

For convenience, we first restate the lemma.

Lemma 2. *For any $r^* \in (\underline{r}, \tilde{r})$ and $\varepsilon > 0$, there exist $\bar{\eta} > 0$ and $\bar{\rho} < \underline{r}/r^*$ such that for any $(\eta, \rho) < (\bar{\eta}, \bar{\rho})$, conditions (3)-(2) below admit a solution $(x', \hat{x}, \theta', \theta'')$ that satisfies $\theta' \leq \theta''$, $|x' - x^*| < \varepsilon$, $\left|\theta' - \theta^*\right| < \varepsilon$, $\left|\theta'' - \theta^{**}\right| < \varepsilon$, and $\hat{x} < -1/\varepsilon$.*

$$1 - \Psi(\tfrac{x'-\theta'}{\sigma}) = \underline{r} - (\underline{r} - \rho r^*)[\Psi(\tfrac{x'-\theta'}{\sigma}) - \Psi(\tfrac{x'-\theta''}{\sigma})] \tag{1}$$

$$1 - \Psi(\tfrac{\hat{x}-\theta'}{\sigma}) = \underline{r} + [r^*\rho + r^*(1-\rho)\exp(\tfrac{r^*-\underline{r}}{\eta}) - \underline{r}][\Psi(\tfrac{\hat{x}-\theta'}{\sigma}) - \Psi(\tfrac{\hat{x}-\theta''}{\sigma})] \tag{2}$$

$$\theta' = \rho[1 - \exp(-\tfrac{r^*-\underline{r}}{\eta})]\Psi(\tfrac{x'-\theta'}{\sigma}) + [1 - \rho + \rho\exp(-\tfrac{r^*-\underline{r}}{\eta})]\Psi(\tfrac{\hat{x}-\theta'}{\sigma}) + C(r^*) \tag{3}$$

$$C(r^*) = (1-\rho)[1 - \exp(-\tfrac{r^*-\underline{r}}{\eta})][\Psi(\tfrac{x'-\theta''}{\sigma}) - \Psi(\tfrac{\hat{x}-\theta''}{\sigma})] \tag{4}$$

$$C(r^*) \leq (1-\rho)[1 - \exp(-\tfrac{r^*-\underline{r}}{\eta})][\Psi(\tfrac{x'-\theta'}{\sigma}) - \Psi(\tfrac{\hat{x}-\theta'}{\sigma})] \tag{5}$$

Proof. It is useful to change variables as follows. Let

$$W \equiv \Psi(\tfrac{\hat{x}-\theta'}{\sigma}), \qquad Z \equiv \Psi(\tfrac{\hat{x}-\theta''}{\sigma}), \qquad Y \equiv \Psi(\tfrac{x'-\theta''}{\sigma}). \tag{6}$$

Conditions (1)-(4) can then be restated as follows:

$$\delta - \gamma Y = \Psi\left(\Psi^{-1}(Y) - \Psi^{-1}(Z) + \Psi^{-1}(W)\right) \tag{7}$$

$$W = \alpha + \beta Z \tag{8}$$

$$\theta' = \rho[1 - \exp(-\tfrac{r^* - \underline{r}}{\eta})]\left[\Psi\left(\Psi^{-1}(Y) - \Psi^{-1}(Z) + \Psi^{-1}(W)\right) - W\right] + W + C(r^*) \tag{9}$$

$$Y = Z + \frac{C(r^*)}{(1-\rho)[1 - \exp(-\frac{r^* - \underline{r}}{\eta})]} \tag{10}$$

where $\alpha, \beta, \delta \in (0,1)$ and $\gamma > 0$ are given by

$$\alpha \equiv \frac{1-\underline{r}}{1-\underline{r}+r^*[\rho+(1-\rho)\exp(\frac{r^*-\underline{r}}{\eta})]}, \quad \beta \equiv \frac{r^*[\rho+(1-\rho)\exp(\frac{r^*-\underline{r}}{\eta})]-\underline{r}}{1-\underline{r}+r^*[\rho+(1-\rho)\exp(\frac{r^*-\underline{r}}{\eta})]}, \quad \gamma \equiv \frac{\underline{r}-\rho r^*}{1-\underline{r}+\rho r^*}, \quad \delta \equiv \frac{1-\underline{r}}{1-\underline{r}+\rho r^*}.$$

For any given r^*, (7)-(10) is a system of four equations in four unknowns, (Y, Z, W, θ'). We first seek a solution to this system, proceeding as follows. Substituting (8) into (7) gives

$$\Psi^{-1}(\delta - \gamma Y) - \Psi^{-1}(Y) = \Psi^{-1}(\alpha + \beta Z) - \Psi^{-1}(Z). \tag{11}$$

Together with (10), this defines a (sub)system of two equations in two unknowns, (Y, Z). In Step 1, we establish the existence of a solution (Y^*, Z^*) to this (sub)system. Condition (8) then gives W^*, whereas (9) gives θ'. From (6) we can then back out $(x', \hat{x}, \theta'')$. In Step 2, we check that the thresholds $(x', \hat{x}, \theta', \theta'')$ satisfy the inequality in (5). Finally, Step 3 shows convergence to the corresponding thresholds of the benchmark game.

Step 1. We first prove that (10) and (11) admit a solution for (Y, Z). Let $LHS(Y)$ and $RHS(Z)$ denote, respectively, the left-hand and the right-hand side of (11). Note that $LHS(Y)$ and $RHS(Z)$ are defined for $Y \in (0, \min\{1, \delta/\gamma\})$ and $Z \in (0,1)$ and are continuous in Y and Z. Moreover, LHS is decreasing in Y, with $\lim_{Y\to 0} LHS(Y) = \infty$, $\lim_{Y\to\min\{1,\delta/\gamma\}} LHS(Y) = -\infty$ and $LHS(Y) \gtreqless 0$ if and only if $Y \lesseqgtr 1-\underline{r}$, whereas $\lim_{Z\to 0} RHS(Z) = \infty$, $\lim_{Z\to 1} RHS(Z) = -\infty$, and $RHS(Z) \gtreqless 0$ if and only if $Z \lesseqgtr 1-\underline{r}$. It follows that (11) defines implicitly a continuous function $Y = g(Z; \eta, \rho)$, with $g : (0,1) \times \mathbb{R}^2 \to (0, \min\{1, \delta/\gamma\})$; note that $\lim_{Z\to 0} g(Z) = 0$, $\lim_{Z\to 1} g(Z) = \min\{1, \delta/\gamma\}$, and $g(Z) \lesseqgtr 1 - \underline{r}$ if and only if $Z \lesseqgtr 1 - \underline{r}$. Condition (10), on the other hand, defines explicitly a function $Y = f(Z; \eta, \rho)$. We thus seek a solution (Y^*, Z^*) to $Y = f(Z) = g(Z)$.

Note that $f(Z; \eta, \rho)$ is continuous and increasing in (Z, η, ρ) with $f(0; \eta, \rho) \to C(r^*) \in (0, 1-\underline{r})$ as $(\eta, r) \to (0,0)$. Then, take any (η_0, ρ_0, Z_0) such that $f(Z_0; \eta_0, \rho_0) < 1-\underline{r}$, and note that $g(Z; \eta, \rho)$ is also continuous in (Z, η, ρ) with $g(Z_0; \eta, \rho) \to 1 - \underline{r}$ as $\eta \to 0$ and $g(Z; \eta, \rho) \to 0$ for any (η, ρ)

as $Z \to 0$. It follows that there exist $\tilde{\eta} \in (0, \eta_0)$, $\tilde{\rho} < \min\{\rho_0, \underline{r}/r^*\}$ and $Z_1 < Z_0$ such that for any $(\eta, \rho) < (\tilde{\eta}, \tilde{\rho})$, $g(Z_0; \eta, \rho) > f(Z_0; \eta, \rho)$ and $g(Z_1; \eta, \rho) < f(Z_1; \eta, \rho)$. The graphs of g and f thus intersect at least twice for (η, ρ) sufficiently small, implying that the system $Y = f(Z) = g(Z)$ admits at least two solutions, as illustrated in Figure A1.

Consider the lowest solution (Z^*, Y^*), let $W^* = \alpha + \beta Z^*$ and note that (Z^*, Y^*, W^*) are continuous in (η, ρ) and satisfy $Z^* \in (0, 1 - \underline{r})$, $Y^* \in (Z^*, 1 - \underline{r})$ and $W^* \in (Z^*, 1 - \underline{r})$. The thresholds $(x', \hat{x}, \theta', \theta'')$ are then the unique solutions to (6) and (9). That $W^* > Z^*$ and $Y^* > Z^*$ imply that $0 < \theta' < \theta''$.

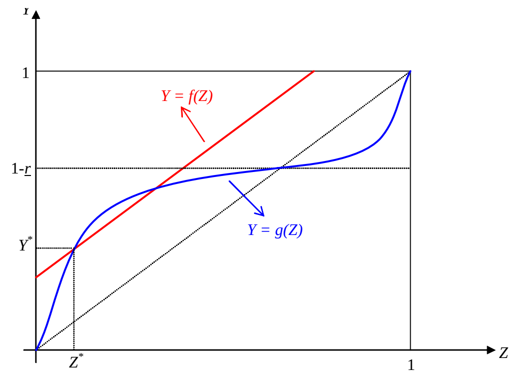

Figure A1

Step 2. We now show that if (η, ρ) are sufficiently small, then (5) holds. Using $\Psi(\frac{x'-\theta'}{\sigma}) = \Psi(\Psi^{-1}(Y^*) - \Psi^{-1}(Z^*) + \Psi^{-1}(W^*))$ and (7), we have that $\Psi(\frac{x'-\theta'}{\sigma}) = \delta - \gamma Y^*$ and hence $\Psi(\frac{x'-\theta'}{\sigma}) - \Psi(\frac{\hat{x}-\theta'}{\sigma}) = \delta - \gamma Y^* - W^*$. As $(\eta, \rho) \to (0, 0)$, $Z^* \to 0$, $Y^* \to C(r^*)$, $W^* \to 0$, $\delta \to 1$, $\gamma \to \frac{\underline{r}}{1-\underline{r}}$ and $\exp(-\frac{r^*-\underline{r}}{\eta}) \to 0$, implying that

$$(1-\rho)[1 - \exp(-\tfrac{r^*-\underline{r}}{\eta})][\Psi(\tfrac{x'-\theta'}{\sigma}) - \Psi(\tfrac{\hat{x}-\theta'}{\sigma})] \to 1 - \tfrac{\underline{r}}{1-\underline{r}}C(r^*).$$

Since $C(r^*) < 1 - \underline{r}$, necessarily $1 - \frac{\underline{r}}{1-\underline{r}}C(r^*) > 1 - \underline{r} > C(r^*)$, which implies that there exist $\eta' \in (0, \tilde{\eta})$ and $\rho' \in (0, \tilde{\rho})$ such that $(x', \hat{x}, \theta', \theta'')$ satisfies (5) for all $(\eta, \rho) < (\eta', \rho')$.

Step 3. We conclude by showing convergence. As $(\eta, \rho) \to (0, 0)$, $Y^* \to C(r^*)$, $W^* \to 0$, and $Z^* \to 0$. Using (6), (9) and (1), we then have that $\theta' \to C(r^*)$, $\hat{x} = \theta' + \sigma\Psi^{-1}(W) \to -\infty$, $x' \to x^*$, and $\theta'' \to \theta^{**}$. Hence, for any $\varepsilon > 0$, there exist $\hat{\eta} \in (0, \eta')$ and $\hat{\rho} \in (0, \rho')$ such that $(\eta, \rho) < (\hat{\eta}, \hat{\rho})$ suffices for $|x' - x^*| < \varepsilon$, $|\theta' - \theta^*| < \varepsilon$, $|\theta'' - \theta^{**}| < \varepsilon$ and $\hat{x} < -1/\varepsilon$, where $(x^*, \theta^*, \theta^{**})$ are as in Proposition 2. *QED*

A2. Alternative payoff structures (continued)

Proposition 5 in the paper assumes that the payoff the policy maker enjoys from maintaining the status quo is positively correlated with (or independent of) its strength. The following example shows that such a positive correlation is not essential.

Proposition 6. *Suppose that $R(\theta, A, r) = \theta - A$ and $U(\theta, A, r) = v(\theta) - A - C(r)$, where v is not necessarily monotonic, but satisfies $v(\theta) > 1 - \underline{r}$ for all $\theta \in [0,1)$. There exists $\hat{r} > \underline{r}$ such that, for any $r^* \in [\underline{r}, \hat{r})$, there is an equilibrium in which the status quo is abandoned if and only if $\theta < 0$ and the policy maker sets r^* for $\theta \in [0, \theta^{**}]$ and $\underline{r}$ otherwise.*

Proof. Let $\hat{r} \in (\underline{r}, \tilde{r})$ be the unique solution to

$$C(\hat{r}) = \sigma \left[\Psi^{-1}\left(1 - \tfrac{\underline{r}}{1-\underline{r}} C(\hat{r})\right) - \Psi^{-1}(C(\hat{r})) \right]$$

and note that $C(\hat{r}) < 1 - \underline{r}$. For any $r^* \in (\underline{r}, \hat{r})$, let $x^* = \sigma \Psi^{-1}(1 - \frac{\underline{r}}{1-\underline{r}} C(r^*))$ and

$$\theta^{**} = x^* - \sigma \Psi^{-1}(C(r^*)) = \sigma \left[\Psi^{-1}\left(1 - \tfrac{\underline{r}}{1-\underline{r}} C(r^*)\right) - \Psi^{-1}(C(r^*)) \right],$$

and note that $\theta^{**} \geq 0$ for any $r^* < \hat{r}$ and solves $\Psi(\frac{x^* - \theta^{**}}{\sigma}) = C(r^*)$. Finally, let $\hat{\theta} \in (0,1)$ be the unique solution to $\Psi(\frac{x^* - \hat{\theta}}{\sigma}) = \hat{\theta}$ and observe that $\theta^{**} > \hat{\theta}$ since $\Psi(\frac{x^* - \theta^{**}}{\sigma}) < \theta^{**}$ when $r^* < \hat{r}$.

We next prove that the following is part of an equilibrium: the policy maker sets $r(\theta) = r^*$ if $\theta \in [0, \theta^{**}]$ and $r(\theta) = \underline{r}$ otherwise; agents attack if and only if $(x, r) < (x^*, r^*)$, or $x < \underline{x}$; and the status quo is abandoned if and only if $\theta < 0$.

Consider the agents. For $r = \underline{r}$, beliefs are pinned down by Bayes' rule (this is immediate when noise is unbounded; with bounded noise, it follows from the fact that $\theta^{**} < 2\sigma$) and satisfy $\mu(0|x, \underline{r}) > \underline{r}$ if and only if $x < x^*$, where x^* solves

$$\frac{1 - \Psi(\frac{x^*}{\sigma})}{1 - \Psi(\frac{x^*}{\sigma}) + \Psi(\frac{x^* - \theta^{**}}{\sigma})} = \underline{r}.$$

For any (x, r) such that $r = r^*$ and $\Theta(x) \cap [0, \theta^{**}] \neq \varnothing$, μ is also determined by Bayes' rule and satisfies $\mu(0|x, \underline{r}) = 0$. For any (x, r) such that either $r = r^*$ and $\Theta(x) \cap [0, \theta^{**}] = \varnothing$, or $r > r^*$, $\Theta(x) \subseteq \Theta(r)$, take any beliefs such that $\mu(0|x, r^*) = 1$ if $x < \underline{x}$ and $\mu(0|x, r^*) = 0$ otherwise. Finally, for any $r \in (\underline{r}, r^*)$, note that $[0, \theta^{**}] \cap \Theta(r) = \varnothing$. Then take any beliefs such that $\mu(\hat{\theta}|x, r) > \underline{r}$ if and only if $x < x^*$ and $\mu(\{\theta \in \Theta(x) \cap \Theta(r)\}|x, r) = 0$ if $\Theta(x) \nsubseteq \Theta(r)$. Given these beliefs, the strategy of the agents is sequentially rational for any (x, r).

Consider the policy maker. Given the agents' strategy, it is optimal to set either $\underline{r}$ or r^*. The payoff from setting $\underline{r}$ is zero for $\theta \leq \hat{\theta}$ and $v(\theta) - \Psi(\frac{x^* - \theta}{\sigma})$ for $\theta > \hat{\theta}$, whereas the payoff from setting r^* is negative for $\theta < 0$ and $v(\theta) - C(r^*)$ for $\theta \geq 0$. Since $\Psi(\frac{x^* - \theta^{**}}{\sigma}) = C(r^*) \leq C(\hat{r}) < 1 - \underline{r} < v(\theta)$, it follows that r^* is optimal if and only if $\theta \in [0, \theta^{**}]$. *QED*

The above result assumes that v is sufficiently high. Multiplicity, however, survives even if v is negative for all θ : there exists a continuum of equilibria in which an intermediate set of θ who would maintain the status quo even by setting $\underline{r}$, prefer to raise the policy at r^*, because the cost of the policy is lower than that of the attack at $\underline{r}$ (i.e., $C(r^*) \leq A(\theta, \underline{r})$). These equilibria differ with respect to both the level of the policy and the regime outcome.