Robust Procurement Design*

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PRELIMINARY DRAFT

Abstract

We study the design of procurement contracts in environments where the buyer faces uncertainty over the product's demand and the seller's cost. The buyer has a belief but does not fully trust it. They first identify all worst-case optimal mechanisms, which deliver the largest payoff guarantee over a set of plausible demand and cost functions. They then select the mechanism that maximizes their expected payoff (under their beliefs) over such a restricted set. We show that robustness calls for an increase in the quantity procured from the least efficient sellers and a decrease in the quantity procured from the sellers with an intermediate cost (relative to the optimal mechanisms in the absence of any uncertainty). The analysis also identifies conditions under which price regulation is superior to quantity regulation, and draws a few policy implications.

KEYWORDS: Robustness, ambiguity, uncertainty, mechanism design, regulation.

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1 Introduction

Hoping for the best, prepared for the worst, and unsurprised by anything in between.

– Maya Angelou, *I Know Why the Caged Bird Sings*

Procurement plays a central role in economics. Governments rely on sophisticated contracts for the purchase of products and services for their citizens, and use regulation to influence the terms offered by firms with market power to consumers. Likewise, in bilateral trade, consumers and firms rely on customized contracts for the purchase of major assets, ranging from real estate to various inputs used in the production of final goods.

A major difficulty in the design of procurement contracts is the information held privately by the providers of the relevant goods and services. A second difficulty is that the social or private value of procuring the good may not be known at the contracting stage. The typical approach to the design of contracts in these situations is based on a subjective expected utility model whereby the buyer, say the government, has a belief over the seller's cost and the value of the good to consumers (equivalently, the product's aggregate inverse demand) and designs a mechanism that maximizes its expected payoff (under such a belief) over all mechanisms that are individually rational and incentive compatible for the seller.

In many situations of interest, though, the buyer may not trust their beliefs and seek for a more robust approach. In this paper, we investigate properties of robustly optimal contracts. For concreteness, we focus on the case of public procurement but the results equally apply to private procurement. The government (i.e., the buyer) first seeks to protect itself against the possibility that its belief is wrong. It does so by identifying all mechanisms that are *worst-case optimal*, namely for which the "welfare guarantee" is maximal. A mechanism's guarantee is the lowest welfare (a combination of consumer and producer's surplus) across all combinations of consumers' demand and seller's cost. The government then maximizes expected welfare under its original belief over the short list of all worst-case-optimal mechanisms. This lexicographic approach either originates in the government's attitude towards uncertainty or in hierarchical constraints that force the relevant agency in charge of designing the mechanism to obtain approval from a supervising entity whose criteria are conservative in that it authorizes only mechanisms for which the guarantee is the highest.

Our first result shows that, when uncertainty is only over the seller's cost, and the government's belief satisfies familiar regularity conditions, the optimal mechanism is *Baron-Myerson-with-quantity-floor.* In a highly influential paper that paved the way to an entire literature, Baron and Myerson (1982) characterize the optimal procurement mechanism when the government fully trusts its belief (i.e., under a familiar subjective-expected-utility objective — in short SEU). In this mechanism, each type of the seller provides a quantity distorted downward relative to the efficient level (for example, by applying a markup to the marginal cost), with the distortion vanishing for the most efficient type (for which the cost is the lowest). We show that a government seeking robustness should offer the same schedule of quantity choices but putting a floor that guarantees that any type supplies at least as much as the efficient output for the least efficient type (the one whose cost is the highest). The floor is necessary to protect the government against the possibility that Nature selects a high-cost firm with a higher probability than the one perceived by the government. Downward distortions in output under SEU serve to reduce rents for more efficient types. The value of such distortions is significantly reduced when the government is concerned that her belief could be wrong. As a result, robustness calls for an upward revision of the quantity procured from the least efficient types. Hence, contrary to what emerges under the familiar SEU benchmark, the optimal mechanism features *efficiency at both the top and the bottom* of the conjectured cost distribution. Importantly, while this mechanism, by design, is worst-case-optimal, it differs from other worst-case optimal mechanisms that naturally arise when the government only seeks worst-case optimality. For example, a singleton (degenerate) menu specifying a quantity equal to the efficient level for the least efficient type and a transfer equal to the cost for the least efficient type of supplying such a quantity is worst-case optimal but not robustly optimal, according to our criterion. In fact, it is dominated by the Baron-Myerson-with-quantity-floor mechanism: It delivers a lower welfare for all costs, strictly for a subset of the relevant range.

The optimality of the Baron-Myerson-with-quantity-floor mechanism extends to certain environments in which the government faces uncertainty also over consumers' demand. This is the case, for example, when the government's conjecture is that the demand curve is the "smallest" (in a sense made precise below) within the set of demands deemed plausible. More generally, we characterize necessary and sufficient conditions for the Baron-Myerson-with-quantity-floor mechanism to remain robustly optimal under demand uncertainty and then identify qualitative properties of robustly-optimal mechanisms when the Baron-Myerson-with-quantity-floor is not worst-case optimal. When this is the case, we show that any robustly optimal mechanism has a quantity floor that is binding over the same interval of cost levels for which the floor is binding under the Baron-Myerson-with-quantity-floor mechanism. However, robustness calls for a downward adjustment in the quantity procured from firms with an intermediate cost (vis-a-vis the SEU optimum, i.e., the original Baron-Myerson mechanism). In the special case of private procurement (alternatively, of public procurement but where the government maximizes a welfare function assigning zero weight to producer surplus), the robustly optimal mechanism procures the same output as the Baron-Myerson-with-quantity-floor mechanism from low- and high-cost firms and a strictly smaller amount for intermediate costs.

The intuition for these properties is the following. When the firm's cost is the highest, robustness calls for procuring a level of output equal to the efficient level when demand is the lowest and cost is the highest. Monotonicity of the output schedule (which is necessary for incentives) then requires that the government procures the same level of output also from an entire interval of costs around the highest level. For costs for which the floor is not binding, the reason for downward adjustments vis-a-vis the SEU-optimal benchmark is that procuring the SEU-optimal output (as prescribed by the Baron-Myerson-with-quantity-floor mechanism) would lead to a welfare level below the welfare guarantee if demand turns out to be lower than the one conjectured by the government. Without any downward correction, the Baron-Myerson-with-quantity-floor mechanism is thus not worst-case optimal. For very low costs, however, there is no need for any output adjustment. When cost is low, welfare when procuring the SEU-optimal level of output exceeds the guarantee irrespective of the realized demand, making any adjustment unnecessary.

Next, we consider more general forms of cost uncertainty, captured by the set of distributions from which the firm's marginal cost is drawn. It turns out that only the "smallest" distribution in this set (along with the distribution conjectured by the government) matter in the design of the optimal mechanism. In the benchmark case, the smallest distribution is a Dirac measure assigning probability one to the least efficient type (the one with the largest cost). When, instead, the smallest distribution has full support, the unique robustly optimal mechanism is the Baron-Myerson SEU mechanism for the lowest demand and the smallest distribution. In other words, the government disregards its conjecture and offers the same mechanism it would have offered in a world without uncertainty had its conjecture been the most pessimistic one. An immediate implication of this result is that, when uncertainty is only over the demand, the unique robustly optimal mechanism is the same as in Baron and Myerson, with no floor but with the government's conjecture over consumer demand replaced by the lowest demand in the feasible set.

More generally, when the support of the smallest distribution is a strict subset (of positive Lebesgue measure) of the support of the distribution conjectured by the government, possibly with an atom at the top of the distribution (i.e., at the least efficient type), the robustly optimal quantity schedule is a *bridge between two different Baron-Myerson schedules*, the one under the government's conjecture (for low costs) and the one under the lowest distribution (for high costs). In this case, robustness calls for efficiency at the top (the lowest cost) and the middle (intermediate cost).

Equipped with these results, we then investigate how the robustly-optimal mechanism changes when uncertainty changes. We capture these changes by considering alternative sets of demand and cost functions that the government deems possible. We first consider changes in cost uncertainty and then changes in demand uncertainty. In either case, we hold the government's conjecture fixed and investigate how the robustly optimal mechanism changes when the set containing the government's conjecture changes.

Consider first cost uncertainty. We show that the output the government procures need not be monotone in the smallest distribution the government deems possible. We illustrate this possibility by considering a sequence of such distributions converging to the Dirac measure at the least efficient type. Along the sequence, we hold fixed the interval of cost levels that are feasible and characterize the robustly optimal mechanism as a function of the smallest cost distributions. When the support of the smallest distribution is the full range of feasible cost levels and the distribution "shifts to the right" in the hazard-rate order, i.e., assigns more measure to high cost levels, the government procures more output from low-cost firms. This is because such changes make the government less concerned with rent extraction and more concerned with efficiency. However, once the support of the smallest distribution starts shrinking by excluding an interval of low cost levels, the government responds by procuring less output from low-cost firms. This is because, when the support of the smallest distribution does not contain the most efficient types, the government can afford to procure the quantity that is optimal under its conjecture from the most efficient types levering the fact that welfare under such a quantity exceeds the guarantee even if Nature selects these types (by picking a distribution above the smallest). As a result, the government "follows its instincts" and procures the SEU-optimal level of output from low-cost firms.

Next, consider variations in demand uncertainty. We capture these variations by considering changes in the "smallest" demand within the set containing the government's conjecture. Again, the reason for focusing on the lowest demand is that only this demand, together with the government's conjecture, play a role in the design of the robustly optimal mechanism. We show that, as the lowest demand increases, the government procures more output from high-cost firms. This is because the quantity floor, which is the efficient output when cost is the highest and demand is the lowest, increases as the lowest demand increases. Furthermore, the interval of intermediate cost levels for which robustness calls for a downward adjustment vis-a-vis the quantity that is optimal under the government's conjecture (the one specified by the SEU-optimal mechanism) shrinks as the smallest demand increases. We conjecture that, as the distance between the lowest demand and the government's conjecture diminishes, the entire quantity schedule moves upwards (the quantity asked to each type weakly increases, with the inequality strict for some types); however, we did not prove the result.

Finally, we consider a different class of mechanisms which we call price mechanisms. In these mechanisms, the government sets the price for each type and instructs the firm to supply any quantity demanded by consumers at that price. In a price mechanism, the quantity procured is responsive to the changes in demand, and this responsiveness can help in safeguarding the government's welfare in case the conjecture turns out to be wrong. However, these mechanisms have the potential downside of exposing the firm to demand uncertainty. We follow a robust approach whereby the government restricts attention to price mechanisms that are individually rational and incentive compatible no matter the beliefs that the firm may have over the demand and no matter the firm's ambiguity aversion. We do so by considering transfers to the firm that depend on the realized demand and restricting attention to mechanisms that are ex-post individually rational and incentive compatible. Under such mechanisms, the firm maximizes its profits by setting the price asked by the government, no matter its beliefs over the demand and its attitude towards uncertainty.

We then show that robustly-optimal price mechanisms have a simple structure. They consist in fixing a markup for each type that coincides with the SEU-optimal one (under the government's conjecture) but with a cap equal to the largest possible cost (this cap is binding under the SEU-optimal mechanism). The cap is necessary to protect the government against the possibility that Nature selects the highest-cost firm with a high probability.

We then investigate conditions under which price mechanisms do better than their quantity counterparts. Under SEU, price and quantity regulation are equivalent. This is because any incentive-compatible-and-individually-rational quantity schedule can be implemented by offering the firm a menu of prices appropriately designed to induce each type to supply the same output as under quantity regulation. The two classes of mechanisms are also equivalent under worst-case optimality: the largest welfare guarantee is the same for price and quantity mechanisms. However, that the guarantee is the same does not mean that the maximal welfare under the government's conjecture is the same when optimizing over the short list of worst-case-optimal mechanisms. We show that quantity regulation dominates when the Baron-Myerson-with-quantity-floor is robustly optimal (strictly so when the demand the government expects at a price equal to the largest cost exceeds the one under the lowest feasible demand curve). This is because, in this case, the robustly optimal price and quantity mechanisms yield the same output when cost is low but price regulation results in a larger quantity procured from high-cost firms. The government over-procures (relative to the SEU-optimal benchmark) from these firms but more so under price regulation. When, instead, the demand the government expects from setting a price equal to the largest marginal cost level is the same as under the lowest feasible demand, price regulation dominates (strictly so when the Baron-Myerson-with-quantity-floor mechanism is not robustly optimal). This is because, in this case, both price and quantity regulation implement the same quantity floor but quantity regulation requires a downward quantity adjustment (relative to the SEU-optimal benchmark) for intermediate costs that price regulation does not necessitate. Under price regulation, the quantity supplied by intermediate-cost firms is adjusted downwards when Nature selects the lowest possible demand, which protects the government in case its conjecture is wrong. However, if the government's conjecture turns out to be correct, the quantity procured from intermediate types under price regulation is the SEU one, which is optimal under the conjecture. This is not the case under quantity regulation; the quantity procured from intermediate-cost firms is downward adjusted relative to the SEU benchmark to protect the government against the possibility that the true demand is below the conjectured one. As a result, in this case, price regulation dominates.

Organization. The rest of the paper is organized as follows. We wrap up the introduction below with a brief discussion of the most pertinent literature. Section 2 describes the environment and the government's problem for quantity regulation. Section 3 characterizes the short list of worst-case-optimal quantity mechanisms. Section 4 identifies properties of robustly-optimal quantity mechanisms. Section 5 considers the implications of general forms of cost uncertainty. Section 6 studies the implications of changes in cost and demand uncertainty, under quantity regulation. Section 7 studies price mechanisms and identifies conditions under which price regulation dominates (alternatively, is dominated by) quantity regulation. Section 8 concludes. Proofs omitted in the main text are in the Appendix at the end of the document.

Related literature. A vast body of work in information economics investigates properties of optimal mechanisms in settings where agents possess private information about relevant aspects of the environment (preferences, costs, technology, productivity). The closest papers in this literature to ours (given the focus on procurement and regulation) are Baron and Myerson (1982) and Laffont and Tirole (1986). See also Armstrong (1999), Amador and Bagwell (2022), Armstrong and Sappington (2004), Armstrong and Sappington (2006), Biglaiser and Ma (1995), Dana (1993), Lewis and Sappington (1988a), Lewis and Sappington (1988b), and Yang and Zentefis (2023) for various analyses of how optimal mechanisms reflect different knowledge by the designer of the primitives of the environment. These papers look at different combinations of asymmetric information on demand and firms' costs. They assume the designer has a belief over the part of the environment it does not observe and fully trust the belief (what we refer to as the SEU-optimal benchmark).

The last few years, instead, have witnessed interest in relaxing the key assumptions behind SEU analysis, with the intent of providing a more robust approach to contract design (see, for example, Carroll (2017) for an analysis of robustness in the design of screening mechanisms when the designer lacks information about the correlation in the distribution from which consumers' preferences for different goods are drawn; see also Carroll (2019) for an overview of robustness in contract design). The closest papers in this literature to ours are Garrett (2014), Bergemann et al. (2023), and Guo and Shmaya (2024). The first paper identifies properties of optimal menus of contracts when the designer lacks information about the firm's disutility from reducing its production cost in the Laffont and Tirole (1986)'s model. The second paper looks at contract design when the designer maximizes the worst-case competitive ratio (the ratio between the designer's payoff under the selected mechanism in any given environment and the designer's payoff under the optimal mechanism for the realized environment). The third paper looks at min-max regret. Under our approach, instead, the designer maximizes her payoff under a conjecture but over a short list of mechanisms that are worst-case-optimal. In this respect, our approach is identical to the one in Dworczak and Pavan (2022). That paper focuses on information design in a setting without transfers and private information. This paper, instead, focuses on procurement and regulation in a setting where the designer possesses no private information, must screen a privately-informed agent, and uses transfers to incentivize the agent.

The analysis is also related to the work on model-mis-specification and robust control. See Cerreia Vioglio et al. (2022) for a discussion of the underpinnings of this approach and some of its key contributions. What we call a conjecture in our analysis can be interpreted as the designer's model in the language of Cerreia Vioglio et al. (2022). The designer does not trust this model and seeks worst-case optimality against a set of alternative models perceived as plausible. The key difference is that in Cerreia Vioglio et al. (2022) the designer stops after it identifies a worst-case optimal policy whereas under our approach the designer goes back to its original model and optimizes (under this model) over the set of worst-case optimal policies.

Importantly, none of the predictions about the structure of robustly optimal mechanisms identified in the present paper have counterparts in the works mentioned above.

2 Model

2.1 Environment

A (local or federal) government, in the role of a buyer, procures a product, or service, for its citizens. The good is supplied by a monopolistic seller who can provide any quantity $\mathrm{q} \in [0, \bar{\mathrm{q}}]$ of the good, with $\bar{\mathrm{q}} \in \mathbb{R}_{++}$ large but finite.[1] The cost to the seller of supplying any quantity $\mathrm{q} \geq 0$ is $\theta\mathrm{q}$, where θ is the marginal cost.

The government faces uncertainty about the social value of procuring q units of the good

[1] As usual, the assumption that q is bounded is made to validate a certain envelope-theorem representation of the equilibrium payoffs.

for its citizens. The government has a *conjecture* about this value, given by the increasing, strictly concave, and differentiable function $V^\star : \mathbb{R}_+ \to \mathbb{R}_+$, with derivative at any q given by $P^\star(\mathrm{q})$.[2] In other words, the government believes that the gross consumer surplus from procuring q units of the good is equal to

$$V^\star(\mathrm{q}) = \int_0^{\mathrm{q}} P^\star(s)ds.$$

Importantly, the government is not sure about its conjecture and is concerned that the true value may be determined by a different function V satisfying

$$V(\mathrm{q}) = \int_0^{\mathrm{q}} P(s)ds \tag{1}$$

for all $\mathrm{q} \geq 0$. The government believes that the set of possible functions describing gross consumer surplus is $\mathcal{V}$, with each $V \in \mathcal{V}$ strictly increasing, strictly concave, and differentiable, and with $\mathcal{P}$ denoting the set of corresponding inverse demand functions (that is, for each $V \in \mathcal{V}$, $P \in \mathcal{P}$ is the inverse demand function corresponding to V, with the two linked by the relationship in (1)). The sets $\mathcal{P}$ and $\mathcal{V}$ are such that they contain the conjectures: $P^\star \in \mathcal{P}$ and $V^\star \in \mathcal{V}$. We assume that there exists an (inverse) demand function $\underline{P}$ such that, for any $\mathrm{q} \geq 0$ and any $P \in \mathcal{P}$, $P(\mathrm{q}) \geq \underline{P}(\mathrm{q})$. The function $\underline{P}$ is thus the "smallest" inverse demand function that the government considers possible. We then let $\underline{V}$ be the surplus function associated with $\underline{P}$. We assume $\underline{P}$ is strictly decreasing and continuous.

The government is also uncertain about the distribution from which the seller's marginal cost θ is drawn. Its conjecture is that θ is drawn from an absolutely continuous cdf $F^\star$ with density $f^\star$ strictly positive over $\Theta = [\underline{\theta}, \overline{\theta}]$. However, the government is concerned that the true cdf may be different and given by $F \in \mathcal{F}$, where $\mathcal{F}$ is a set of cdfs supported on Θ considered possible. Till Section 4, we assume that $\mathcal{F} = \mathrm{CDF}(\Theta)$, where $\mathrm{CDF}(\Theta)$ is the set of all cdfs with support Θ; that, is, each $F \in \mathrm{CDF}(\Theta)$ is a non-decreasing, right-continuous function $F : \Theta \to [0,1]$ such that $F(\theta) = 0$ for all $\theta < \underline{\theta}$, and $F(\theta) = 1$ for all $\theta \geq \overline{\theta}$.

Remark. As it will become clear in a moment, our analysis admits two equivalent interpretations of the uncertainty faced by the government. In the first interpretation, the

[2]A function V is said to be increasing (alternatively, decreasing) if $V(q) > V(q')$ (alternatively, $V(q) < V(q')$) whenever $q > q'$. In other words, the definition assumes the inequality is strict. We refer to a function V such that $V(q) \geq V(q')$ for all $q > q'$ as non-decreasing. Similarly, V is non-increasing if $V(q) \leq V(q')$ for all $q > q'$.

true state of the world is $(\theta, V) \in \Theta \times \mathcal{V}$. The government has a conjecture $\rho \in \Delta(\Theta \times \mathcal{V})$ with marginal over Θ equal to $F^\star$ and marginal over $\mathcal{V}$ equal to σ. The government believes θ and V (equivalently, θ and P) to be independent. The function $V^\star$ then corresponds to the expected value of procuring output, with the expectation taken over $\mathcal{V}$ under the distribution σ. That is, for any q, $V^\star(q) = \int V(q)\sigma(\mathrm{d}V)$. Under such an interpretation, the pair $(V^\star, F^\star)$, along with the assumption that θ and V are independent, is then a description of the government's beliefs over $\Theta \times \mathcal{V}$.

The second interpretation is that $(V^\star, F^\star)$ is the government's model of the world (in the sense of Cerreia Vioglio et al. (2022)). The government is concerned that its model may be mis-specified and that the true model is some alternative $(V, F) \in \mathcal{V} \times \mathcal{F}$. Under this interpretation, $\mathcal{V} \times \mathcal{F}$ is the set of all models deemed plausible by the government when doing robust decision making, and there is no stochastic structure over $\mathcal{V} \times \mathcal{F}$. Our results are consistent with both interpretations.

2.2 Quantity mechanisms

The seller is perfectly informed about the marginal cost of production θ. To elicit the seller's private information and discipline the supply, the government offers the seller (a firm) a (direct) mechanism $M = (q, t)$. The mechanism consists of a pair of mappings. The first one, $q : \Theta \to \mathbb{R}_+$, specifies the amount of the good procured by the government when the seller reports the marginal cost to be θ. The second mapping, $t : \Theta \to \mathbb{R}$, specifies the total transfer to the seller, for each possible report of the marginal cost.

The mechanism $M = (q, t)$ is *incentive compatible* (IC) if, for all $\theta, \theta' \in \Theta$,

$$u(\theta) := t(\theta) - \theta q(\theta) \geq t(\theta') - \theta q(\theta').$$

It is *individually rational* (IR) if $u(\theta) \geq 0$ for all θ. Because, given the quantity schedule q, there is a bijection between t and u, we will often refer to a mechanism by (q, u) instead of (q, t).

As is well known, $M = (q, u)$ is IC and IR if and only if q is non-increasing and, for all $\theta \in \Theta$, $u(\theta) = u(\overline{\theta}) + \int_\theta^{\overline{\theta}} q(y)dy$, with $u(\overline{\theta}) \geq 0$.

Let $\mathcal{M}$ be the set of all IC and IR mechanisms. If the consumers' gross surplus from consuming q units of the good is given by the function $V \in \mathcal{V}$, with associated inverse demand function P, and the distribution from which the seller's marginal cost is drawn is $F \in \mathcal{F}$, then (ex-ante) welfare under the mechanism $M \in \mathcal{M}$ is given by

$$W(M; V, F) := \int w(\theta, M; V) F(\mathrm{d}\theta),$$

where, for any θ, $w(\theta, M; V) := \mathrm{CS}(\theta, M; V) + \alpha u(\theta)$ is total ex-post welfare at state θ under the mechanism M when the gross consumer surplus function is V, with $\alpha \in [0,1]$ denoting the weight that the government assigns to producer surplus, and with $\mathrm{CS}(\theta, M; V) := V(q(\theta)) - t(\theta)$ denoting net consumer surplus. Note that total surplus at state θ under the mechanism M when the gross consumer surplus function is V is equal to $\mathrm{TS}(\theta, M; V) := V(q(\theta)) - \theta q(\theta)$. Thus, welfare under the mechanism $M = (q, u)$ when the monopolist has type θ and the gross consumer surplus function is V (with associated inverse demand P) is given by

$$\begin{aligned} w(\theta, M; V) &= \mathrm{TS}(\theta, M; V) - (1-\alpha) u(\theta) \\ &= V(q(\theta)) - \theta q(\theta) - (1-\alpha) u(\theta). \end{aligned} \tag{2}$$

For the rest of the paper, we assume that $\lim_{\mathrm{q} \downarrow 0} \underline{P}(\mathrm{q}) > \overline{\theta}$. This assumption, combined with $\underline{P}$ being continuous, ensures that there is a positive total surplus even when the monopolist has the highest marginal cost and the inverse demand is the smallest. Therefore, there are gains from procuring the good even from the least efficient type of the monopolist, no matter what the inverse demand function is.

The reason to focus on quantity mechanisms is twofold. In some problems of interest, the seller may not be able to wait till V is realized to finalize the quantity it supplies to the buyer. In these situations, the choice to run a quantity mechanism is dictated by the constraints on the timing. The second reason why quantity mechanism may be appealing is that they guarantee that the seller has incentives to participate and report truthfully, irrespective of their beliefs over the demand and/or their attitude towards ambiguity when the seller faces uncertainty over $\mathcal{V}$. In this sense, quantity mechanisms are robust.

2.3 Government's problem

The government follows a two-step lexicographic procedure to select the optimal mechanism. In the first step, the government evaluates any IC and IR mechanism by its welfare guarantee, defined as follows:

Definition 1 *Given any mechanism $M \in \mathcal{M}$, the* ***welfare guarantee*** *from M is given by*

$$G(M) := \inf_{V \in \mathcal{V}, F \in \mathcal{F}} W(M; V, F).$$

Definition 2 *The* ***short-list*** *of IC and IR mechanisms for which the welfare guarantee is maximal is given by*

$$\mathcal{M}^{\mathrm{SL}} := \{M \in \mathcal{M} : G(M) \geq G(M') \ \forall \ M' \in \mathcal{M}\}.$$

A mechanism M is ***worst-case optimal*** *if $M \in \mathcal{M}^{\mathrm{SL}}$.*

In the second step, the government chooses a mechanism from the short list $\mathcal{M}^{\mathrm{SL}}$ that maximizes welfare under its conjecture $(V^\star, F^\star)$.

Definition 3 *A mechanism $M \in \mathcal{M}^{\mathrm{SL}}$ is* ***robustly optimal*** *if, for every $M' \in \mathcal{M}^{\mathrm{SL}}$,*

$$W(M; V^\star, F^\star) \geq W(M'; V^\star, F^\star).$$

As anticipated in the introduction, this two-step procedure captures the idea that the government first seeks to protect itself against the possibility that its conjecture is wrong by dismissing all mechanisms that are not worst-case-optimal. When there are multiple mechanisms that are worst-case-optimal (as we show below, this is typically the case), the government then uses its conjecture to select the mechanism for which welfare, under the conjecture, is the highest among all worst-case-optimal mechanisms. A robustly optimal mechanism is one that maximizes ex-ante welfare, under the government's conjecture $(V^\star, F^\star)$, over all mechanisms in the short list $\mathcal{M}^{\mathrm{SL}}$.

3 Short list characterization

In this section, we establish preliminary results characterizing the short-list $\mathcal{M}^{\mathrm{SL}}$ of quantity mechanisms. We start by characterizing the maximal welfare guarantee of an arbitrary IC and IR mechanism, and show that the worst-case welfare for any IC and IR mechanism need not occur under the distribution that puts all the mass at $\overline{\theta}$, but always occurs under the lowest possible inverse demand in $\mathcal{V}$. Let

$$\mathrm{q}_\ell := \arg\max_{\mathrm{q}} \left\{ \underline{V}(\mathrm{q}) - \overline{\theta}\mathrm{q} \right\}$$

be the unique quantity that maximizes total surplus when $V = \underline{V}$ and $\theta = \overline{\theta}$, i.e., q_ℓ is the efficient quantity at the lowest demand and highest type and is equal to $\underline{P}^{-1}(\overline{\theta})$.

Lemma 1 *The welfare guarantee of any mechanism $M = (q, u) \in \mathcal{M}$ is given by*

$$G(M) = \inf_{\theta \in \Theta} w(\theta, M; \underline{V}). \tag{3}$$

Furthermore, for any $M \in \mathcal{M}$,

$$G(M) \leq G^* := \underline{V}(\mathrm{q}_\ell) - \overline{\theta}\mathrm{q}_\ell. \tag{4}$$

The first part of Lemma 1 highlights that, in general, Nature can cause more harm to the government by choosing a cost $\theta < \overline{\theta}$. Intuitively, this is because, incentive compatibility requires q to be non-increasing in θ. By selecting a low θ, along with an inverse demand below $V^\star$, Nature can then inflict more harm to the government than by selecting the largest possible cost $\overline{\theta}$. The second part of the lemma says that the guarantee of any IC and IR mechanism is no greater than the total surplus that the government can obtain by procuring the efficient output q_ℓ when demand is the lowest and the cost is the highest. This follows directly from the fact that Nature can always select $(\underline{V}, \overline{\theta})$, in which case the best the government can do is to purchase the efficient output q_ℓ.

The next Proposition shows that the upper bound on the maximal welfare guarantee is tight and fully characterizes the short-list $\mathcal{M}^{\mathrm{SL}}$.

Proposition 1 (Short-list characterization) *A mechanism $M \equiv (q, u) \in \mathcal{M}^{\mathrm{SL}}$ if and only if (1) q is non-increasing, (2) for all θ, $u(\theta) = \int_\theta^{\overline{\theta}} q(y)dy$, and (3) for all θ,*

$$\underline{V}(q(\theta)) - \theta q(\theta) - (1 - \alpha) \int_\theta^{\overline{\theta}} q(y)dy \geq G^*. \tag{5}$$

Thus, worst-case optimality imposes two further constraints with respect to incentive compatibility and individual rationality. First, the highest cost type, $\overline{\theta}$, must receive zero rent. If this is not the case, Nature can select $\overline{\theta}$ resulting in welfare strictly below $G^{\star}$, no matter what the procured output is. Second, ex-post welfare *at any given type* under the lowest possible inverse demand (and no rent for $\overline{\theta}$) has to be weakly above the maximal guarantee $G^{\star}$. The necessity of the latter constraint follows from Lemma 1. The sufficiency part is established in the Appendix by showing existence of a simple (constant) mechanism for which the maximal welfare guarantee is exactly G^{*}.

4 Robustly optimal mechanisms

In this section, we characterize robustly optimal mechanisms. We use Proposition 1 and the standard representation of total welfare under the conjecture $(V^{\star}, F^{\star})$ as "virtual surplus" as in Baron and Myerson (1982) to cast the government's optimization problem as follows. Let $z^{\star} : \Theta \to \mathbb{R}$ be the function defined as

$$z^{\star}(\theta) := \theta + (1-\alpha)\frac{F^{\star}(\theta)}{f^{\star}(\theta)} \qquad \forall\, \theta \in \Theta.$$

For any θ, $z^{\star}(\theta)$ is type-θ's "virtual cost" under the conjectured distribution $F^{\star}$. The robustly optimal quantity schedule is then given by the solution to the following problem:

$$\max_{q} \int_{\underline{\theta}}^{\overline{\theta}} \Big[V^{\star}(q(\theta)) - z^{\star}(\theta)q(\theta)\Big] F^{\star}(\mathrm{d}\theta) \tag{\textbf{ROPT}}$$

subject to

$$q \quad \text{non-increasing},$$

$$\underline{W}(\theta, q) := \underline{V}(q(\theta)) - \theta q(\theta) - (1-\alpha)\int_{\theta}^{\overline{\theta}} q(y)dy \quad \geq G^{*} \qquad \forall\, \theta \in \Theta. \tag{6}$$

Hence, the robustly optimal quantity schedule solves an optimization problem in which the objective function and the monotonicity constraint are the same as in Baron and Myerson (1982), but where, in addition, zero profits are given to the highest type, and for each θ, total ex-post welfare under the lowest possible demand exceeds the welfare guarantee G^{*}.

Note that the value of $\underline{W}(\theta, q)$ depends on θ, $q(\theta)$, and $q(\theta')$ for all $\theta' \geq \theta$. An implication of this property is that if q and q' coincide for all $\theta \geq \hat{\theta}$, then $\underline{W}(\theta, q) = \underline{W}(\theta, q')$ for all $\theta \geq \hat{\theta}$.

Relaxed problem: When applied to $\theta = \overline{\theta}$, constraint (6) implies that $q(\overline{\theta}) = \mathrm{q}_\ell$. Since q is non-increasing, we conclude that, in any mechanism $M = (q, u) \in \mathcal{M}^{\mathrm{SL}}$, we have $q(\theta) \geq \mathrm{q}_\ell$ for all θ. The following is thus a relaxation of the problem (**ROPT**):

$$\max_q \int_{\underline{\theta}}^{\overline{\theta}} \Big[V^\star(q(\theta)) - z^\star(\theta) q(\theta)\Big] F^\star(\mathrm{d}\theta) \tag{\textbf{RP-1}}$$

subject to

$$q \quad \text{non-increasing}$$

$$q(\overline{\theta}) = \mathrm{q}_\ell.$$

Hereafter, we let q^{BM} be the quantity schedule defined, for all θ, by

$$q^{\mathrm{BM}}(\theta) := \arg\max_{\mathrm{q}} \Big[V^\star(\mathrm{q}) - z^\star(\theta)\mathrm{q}\Big].$$

The quantity $q^{\mathrm{BM}}(\theta)$ thus maximizes the virtual surplus function under the conjecture $(V^\star, F^\star)$ at cost θ. As is well known, the function q^{BM} is the solution to the Baron and Myerson (1982) problem when such a function is non-increasing, which is the case if, and only if, $z^\star$ is non-decreasing. However, to ease the exposition, we will impose a slightly stronger restriction on $z^\star$ by requiring $F^\star$ to be *regular*, in the following sense:

Definition 4 *The cdf F is **regular** if it is absolutely continuous over $\mathbb{R}$ with density $f(\theta) > 0$ for all $\theta \in \Theta$ and with $z(\theta) := \theta + (1-\alpha)F(\theta)/f(\theta)$ continuous and increasing over Θ.*

For example, F is regular when f is log-concave. Next, let $q^\star$ be the quantity schedule defined by

$$q^\star(\theta) := \max\{q^{\mathrm{BM}}(\theta), \mathrm{q}_\ell\} \tag{7}$$

and $\theta^\star$ be the threshold cost defined as follows. If $q^{\mathrm{BM}}(\overline{\theta}) \leq \mathrm{q}_\ell$, by continuity of q^{BM} along with the fact that $q^{\mathrm{BM}}(\underline{\theta}) > \mathrm{q}_\ell$ (assured by the regularity of $F^\star$), let $\theta^\star$ be the unique solution to $q^{\mathrm{BM}}(\theta^\star) = \mathrm{q}_\ell$. If, instead, $q^{\mathrm{BM}}(\overline{\theta}) > \mathrm{q}_\ell$ (i.e., if q^{BM} never crosses q_ℓ), then let $\theta^\star := \overline{\theta}$.

Lemma 2 *Suppose $F^\star$ is regular and $q^{\mathrm{BM}}(\overline{\theta}) \leq \mathrm{q}_\ell$. The quantity schedule $q^\star$ solves the relaxed program (**RP-1**).*

Under the condition in the lemma, $q^\star$ satisfies the constraints of the relaxed problem. The result then follows from the fact that the quantity schedule $q^\star$ maximizes the virtual surplus function $V^\star(\mathrm{q}) - z^\star(\theta)\mathrm{q}$ point-wise over $[\mathrm{q}_\ell, +\infty)$.

Next observe that, because $q^{\mathrm{BM}}(\underline{\theta})$ is defined by the condition $P^\star(\mathrm{q}) = \underline{\theta}$, whereas q_ℓ is defined by the condition $\underline{P}(\mathrm{q}) = \overline{\theta}$, we have that $q^{\mathrm{BM}}(\underline{\theta}) > \mathrm{q}_\ell$. On the other hand, because $q^{\mathrm{BM}}(\overline{\theta})$ is defined by the condition $P^\star(\mathrm{q}) = z^\star(\overline{\theta})$, $q^{\mathrm{BM}}(\overline{\theta})$ can be smaller or greater than q_ℓ. Hence, while it is always the case that $\theta^\star > \underline{\theta}$, whether $\theta^\star < \overline{\theta}$ or $\theta^\star = \overline{\theta}$ depends on the conjecture $(V^\star, F^\star)$ and α. It is easy to see that $\theta^\star < \overline{\theta}$ when $V^\star = \underline{V}$ and $\alpha < 1$. We now define a mechanism associated with the quantity schedule $q^\star$ which will play an important role in many of our results.

Definition 5 *The* ***Baron-Myerson-with-quantity-floor*** *is the mechanism $M^\star \equiv (q^\star, u^\star)$, where $q^\star$ is the quantity schedule in (7) and where $u^\star$ is the function given by $u^\star(\theta) = \int_\theta^{\overline{\theta}} q^\star(y)dy$.*

We then have the following result:

Proposition 2 (Optimality of Baron-Myerson-with-quantity-floor) *Suppose $F^\star$ is regular and $V^\star = \underline{V}$. Then, Baron-Myerson-with-quantity-floor is a robustly optimal mechanism.*

The following is an immediate implication of the last proposition:

Corollary 1 *Suppose there is no demand uncertainty ($\mathcal{V} = \{V^\star\}$) and $F^\star$is regular. Then Baron-Myerson-with-quantity-floor is robustly optimal.*

Hence, when the government's conjecture is that consumer surplus is the lowest possible one (which is the case when the government faces only upward or no uncertainty over V) and that the distribution $F^\star$ is regular, the Baron-Myerson quantity schedule with an output floor at q_ℓ is a robustly optimal mechanism. Figure 1 depicts the output schedule $q^\star$ and highlights an important implication of robustness: *efficiency both at the bottom and at the top of the type distribution.*

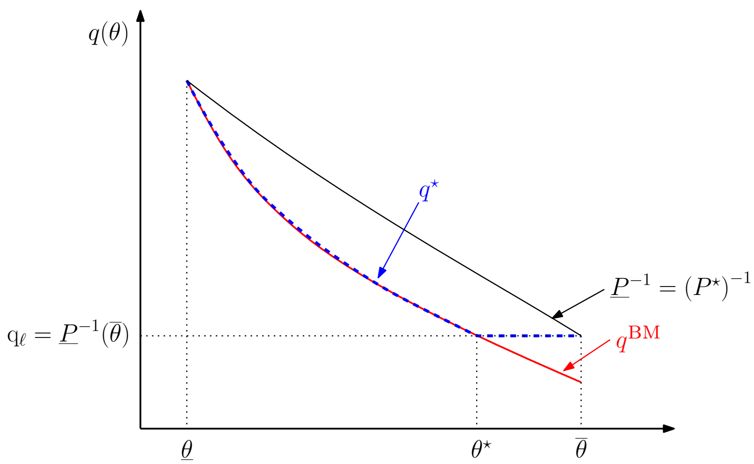

Figure 1: Illustration of Proposition 2.

A key step in the proof of Proposition 2 is to establish that the schedule $q^\star$ satisfies the constraint (6). If this constraint is satisfied at all local minima of $\underline{W}(\theta, q^\star)$, then it is satisfied at all θ. This simple observation allows us to reduce the continuum of constraints to at most countably many inequalities. The next lemma, which is proved in the Appendix, establishes monotonicity properties of $\underline{W}(\theta, q)$ that are central to the analysis of robustly optimal mechanisms. We use this lemma to prove the last proposition and various other results below.

Lemma 3 *Suppose $M \equiv (q, u)$ is an IC mechanism and $I \subseteq \Theta$ is any interval. Then, the following are true.*

A. *Suppose $0 < q(\theta) \leq \underline{P}^{-1}(\theta)$ for all $\theta \in I$. Then $\underline{W}(\theta, q)$ is non-increasing over I (decreasing if $\alpha > 0$, or, when $\alpha = 0$, if q is decreasing with $q(\theta) < \underline{P}^{-1}(\theta)$ for all $\theta \in I$).*

B. *Suppose $q(\theta) > \underline{P}^{-1}(\theta)$ for all $\theta \in I$ and $\alpha = 0$. Then, $\underline{W}(\theta, q)$ is non-decreasing over I. If, in addition, q is decreasing over I, then $\underline{W}(\theta, q)$ is increasing over I.*

The top panel of Figure 2 plots a non-increasing quantity schedule q (dashed line) and $\underline{P}^{-1}$ (solid line). The bottom panel depicts the shape of the associated $\underline{W}(\theta, q)$ function. Part A of Lemma 3 implies that $\underline{W}(\theta, q)$ decreases over $[\theta_1, \theta_2]$ no matter the value of α. This is because, $q(\theta) \leq \underline{P}^{-1}(\theta)$ is a sufficient condition for $\underline{W}(\theta, q)$ to be non-increasing

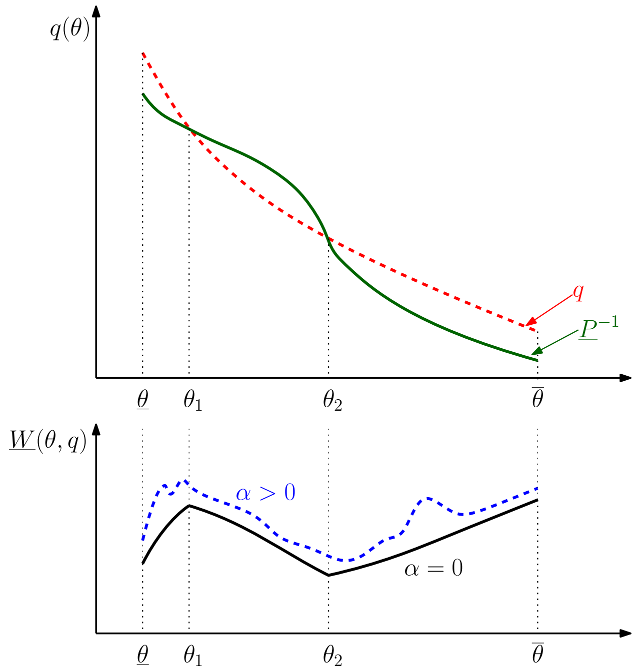

Figure 2: Illustration of Lemma 3.

for all α. On the other hand, Part B shows that this same condition is also necessary for $\underline{W}(\theta, q)$ to be non-increasing when $\alpha = 0$. As the $\underline{W}$ (solid) curve for $\alpha = 0$ in the bottom panel of the figure illustrates, $\underline{W}(\theta, q)$ is increasing on the intervals $[\underline{\theta}, \theta_1]$ and $[\theta_2, \overline{\theta}]$, where $q(\theta) > \underline{P}^{-1}(\theta)$. This property, however, does not necessarily extend to $\alpha > 0$ as illustrated in the bottom panel of Figure 2 (the dashed line depicts $\underline{W}(\theta, q)$ when $\alpha > 0$).

The proof of Proposition 2 in the Appendix uses Lemma 3 to establish that $\underline{W}(\theta, q^\star)$ is non-increasing over Θ, which, together with the fact that $\underline{W}(\overline{\theta}, q^\star) = G^*$ guarantees that the robustness constraint (6) is satisfied.

We now provide an example in which $V^\star \neq \underline{V}$, and for which Baron-Myerson-with-quantity-floor is not robustly-optimal. In this example, $\underline{W}(\theta, q^\star)$ is increasing in θ and its value is below G^* at all $\theta < \overline{\theta}$.

Example 1 *Suppose that* $\alpha = 0$ *and* $\Theta = [4, 5]$. *The government's conjecture about the inverse demand function is that, for any* $q \geq 0$, $P^\star(q) = \max\{10 - q; 0\}$. *As for the government's conjecture* $F^\star$ *about the distribution from which* θ *is drawn, the conjecture is that* $F^\star$ *is the cdf of a uniform distribution over* $[4, 5]$. *Finally, in this example* $\underline{P}(q) = \max\{10 - \frac{5}{4} q; 0\}$.

Note that, under the above conjecture, for any $\theta \in \Theta$, $z^\star(\theta) = 2\theta - 4$, *which is increasing in* θ. *In this case, for any* $\theta \in \Theta$, $q^{\mathrm{BM}}(\theta)$ *is given by the solution to the optimality condition* $P^\star(q^{\mathrm{BM}}(\theta)) = 2\theta - 4$, *from which we obtain that* $q^{\mathrm{BM}}(\theta) = 14 - 2\theta$. *Note that* $q^{\mathrm{BM}}(\theta) \geq q_\ell = 4$ *for all* θ *with equality at* $\overline{\theta} = 5$. *This implies that the quantity floor* q_ℓ *in the relaxed program is not binding and that, for all* θ, $q^\star(\theta) = q^{\mathrm{BM}}(\theta)$.

Note that $\underline{P}^{-1}(\theta) = 8 - \frac{4}{5}\theta$ *for all* $\theta \in \Theta = [4, 5]$. *As a result,* $q^\star(\theta) - \underline{P}^{-1}(\theta) = q^{\mathrm{BM}}(\theta) - \underline{P}^{-1}(\theta) = 6 - \frac{6}{5}\theta \geq 0$ *for all* $\theta \in [4, 5]$ *with strict inequality holding if* $\theta < 5$. *Lemma 3 then implies that* $\underline{W}(\theta, q^\star)$ *is increasing over* $[4, 5)$. *Since* $\underline{W}(5, q^\star) = G^*$, *this implies that* $\underline{W}(\theta, q^\star) < G^*$ *and (6) does not hold for all* $\theta < 5$. ■

Let $M^{\mathrm{OPT}} \equiv (q^{\mathrm{OPT}}, u^{\mathrm{OPT}})$ be a robustly optimal mechanism. Recall that Baron-Myerson-with-quantity-floor is a solution to the relaxed program (**RP-1**).

The following type plays an important role in characterizing robustly optimal mechanisms. Let

$$\theta^m := \max\{\theta : \theta \in \arg\min_{y \in \Theta} \underline{W}(y, q^\star)\}.$$

That is, θ^m is the largest cost at which the function $\underline{W}(\cdot, q^\star)$ attains a minimum. When $F^\star$ is regular, θ^m is well defined. This is because $q^\star$ is continuous over Θ, which implies that $\underline{W}(\theta, q^\star)$ is also continuous on Θ. Because Θ is compact, the set $\{\theta : \underline{W}(\theta, q^\star) \leq \underline{W}(\theta', q^\star)\ \forall\ \theta'\}$ is non-empty and compact. This ensures existence of θ^m. Type θ^m plays an important role in the characterization of robustly optimal mechanisms. If $q^\star$ violates the robustness constraint, it must violate it at θ^m. One can then construct modifications to $q^\star$ using the violation of the robustness constraint at θ^m to characterize robustly optimal mechanisms.

The following proposition (whose proof is in the Appendix) identifies general properties of robustly optimal mechanisms:

Proposition 3 *Suppose* $F^\star$ *is regular. Then, the following are true.*

1. *Baron-Myerson-with-quantity-floor mechanism is robustly optimal if and only if* $\theta^m = \overline{\theta}$ *and* $q^{\mathrm{BM}}(\overline{\theta}) \leq \mathrm{q}_\ell$.

2. *If $\theta^m < \overline{\theta}$ or $\theta^m = \overline{\theta}$ and $q^{\mathrm{BM}}(\overline{\theta}) > \mathrm{q}_\ell$, then $\theta^m \leq \theta^\star$, and every robustly optimal mechanism $M^{\mathrm{OPT}} = (q^{\mathrm{OPT}}, u^{\mathrm{OPT}})$ satisfies the following properties:*

   (a) *$q^{\mathrm{OPT}}(\theta) = \mathrm{q}_\ell$ for all $\theta \in [\theta^\star, \overline{\theta}]$, and*

   (b) *$q^{\mathrm{OPT}}(\theta) \leq q^{\mathrm{BM}}(\theta)$ for almost all $\theta < \theta^\star$, with the inequality strict over a Lebesgue positive measure set of types $I \subseteq [\underline{\theta}, \theta^\star)$.*

Hence, robustness calls for an upward adjustment of the quantity procured from high cost sellers and a downward adjustment of the quantity procured from the intermediate-cost sellers. The upward adjustment (from $q^{\mathrm{BM}}(\theta)$ to q_ℓ) is necessary to avoid the loss in welfare that could originate from Nature selecting a high cost.[3] The downward distortion (over and above the distortion the government would make under SEU to reduce rents) is necessary to guarantee that, if the demand is below the conjectured one, the government does not lose too much by procuring a quantity whose value is below the conjectured one. The benefit of such downward adjustments, however, vanish when θ is very close to $\underline{\theta}$ for, as explained in the Introduction, welfare at such low costs exceeds G^* even when Nature selects the lowest demand $\underline{V}$.

The following proposition (whose proof is also in the Appendix) further tightens the characterization of the optimal quantity mechanism of Proposition 3 for the case $\alpha = 0$.

Proposition 4 *Suppose $F^\star$ is regular and $\alpha = 0$.*

1. *If $\theta^m = \overline{\theta}$, then $q^{\mathrm{BM}}(\overline{\theta}) \leq \mathrm{q}_\ell$.*

2. *The following conditions, when holding jointly, imply that $\theta^m = \overline{\theta}$:*

   (a) *$\underline{W}(\underline{\theta}, q^\star) \geq G^*$;*

   (b) *there exists $\hat{\theta} \in \Theta$ such that $q^\star(\theta) > \underline{P}^{-1}(\theta)$ if $\theta < \hat{\theta}$ and $q^\star(\theta) \leq \underline{P}^{-1}(\theta)$ if $\theta \geq \hat{\theta}$.*

3. *If $\theta^m < \bar{\theta}$, then $q^{\mathrm{OPT}}(\theta) = q^{\mathrm{BM}}(\theta)$ for almost all $\theta \in [\underline{\theta}, \theta^m)$.*

4. *The following conditions, when holding jointly, imply that $\theta^m = \underline{\theta}$:*

[3] The downward distortion in $q^{\mathrm{BM}}(\theta)$ is meant to reduce the rent $u(\theta)$ of low types, but this consideration is not warranted if Nature selects a high cost.

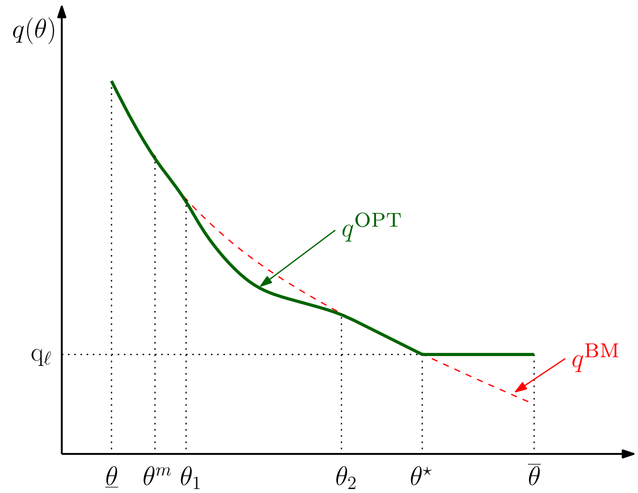

Figure 3: Illustration of Proposition 3

(a) $P^\star(q) - \underline{P}(q) > 1/f^\star(\overline{\theta})$ *for all* q*;*

(b) $F^\star(\theta)/f^\star(\theta)$ *non-decreasing and continuous over* Θ*.*

Figure 3 illustrates the structure of the robustly optimal quantity schedule when it differs from Baron-Myerson-with-quantity-floor, under the assumption that $\alpha = 0$.

Together, Propositions 3 and 4 imply that, when $\alpha = 0$, the Baron-Myerson-with-quantity-floor mechanism is optimal if and only if welfare under the lowest possible demand $\underline{V}$ and the Baron-Myerson-with-quantity-floor output schedule $q^\star$ attains a minimum at the highest possible cost, $\overline{\theta}$. An intuitive sufficient condition for $\theta^m = \overline{\theta}$ is that $\underline{W}(\theta, q^\star)$ is single peaked with $\underline{W}(\underline{\theta}, q^\star) \geq G^*$. When $\alpha = 0$, the monotonicity properties of Lemma 3 imply that $\underline{W}(\theta, q^\star)$ is single peaked when $q^\star$ single crosses $\underline{P}^{-1}$ from above.

Another key strengthening of the result in Proposition 3 is that, when $\theta^m < \bar{\theta}$, the downward adjustments in the quantity schedule required by robustness occur only in the interval $[\theta^m, \theta^\star]$; over the interval $[\underline{\theta}, \theta^m]$, the quantity schedule coincides with the one in Baron-Myerson. Intuitively, ensuring the robustness constraint (6) at θ^m requires reducing the rent $u(\theta^m)$, which in turn implies reducing the rents left to all $\theta \leq \theta^m$. When $\alpha = 0$, this adjustment suffices to guarantee that the robustness constraint is satisfied for all $\theta < \theta^m$. As a result, no quantity adjustment (vis-a-vis the original Baron-Myerson quantity schedule q^{BM}) is necessary for $\theta \leq \theta^m$. However, the region $[\underline{\theta}, \theta^m]$ can be empty. This happens when welfare under the lowest possible demand $\underline{V}$ and the output schedule $q^\star$ attains a minimum at $\underline{\theta}$. In this case, the robustly optimal quantity schedule differs significantly from $q^{\star}$. A sufficient condition for this to happen is that the difference between the conjectured inverse demand function and the lowest possible inverse demand function is sufficiently large at all quantities, along with $F^{\star}$ having a non-decreasing and continuous reverse hazard rate.

5 More general forms of cost uncertainty

We now consider the possibility that the set $\mathcal{F}$ of distributions that the government deems plausible is a subset of the set CDF(Θ) of all cdfs supported on Θ. This possibility is inspired by the second interpretation of the uncertainty faced by the government mentioned in Section 2, along the lines of Cerreia Vioglio et al. (2022). In this interpretation, $(V^{\star}, F^{\star})$ is the government's "model of the world" and $\mathcal{V} \times \mathcal{F}$ is the set of alternative models. Contrary to Cerreia Vioglio et al. (2022), and consistently with what assumed above, the government first seeks to protect itself against the possibility that its model is mis-specified (namely, that the true model is $(V, F) \neq (V^{\star}, F^{\star})$ and then uses its model to select a mechanism that maximizes its payoff (under the model $(V^{\star}, F^{\star})$) over all mechanisms that yield the largest guarantee.

Let $\mathcal{F}$ be the set of cdfs the government believes to be feasible. Assume that $\mathcal{F}$ contains a cdf $\underline{F}$ such that for all $F \in \mathcal{F}$, we have $F(\theta) \geq \underline{F}(\theta)$ for all $\theta \in \Theta$. Note that in the analysis so far, we assumed that $\mathcal{F} = \text{CDF}(\Theta)$ which amounts to $\underline{F}(\theta) = \mathbb{I}(\theta \geq \bar{\theta})$.

Let $\underline{M}^{\text{BM}} := (\underline{q}^{\text{BM}}, \underline{u}^{\text{BM}})$ be the optimal mechanism when θ is drawn from the distribution $\underline{F}$ and the inverse demand function is $\underline{P}$ (equivalently, the value of procuring the good is $\underline{V}$). The following is then true:

Proposition 5 *Suppose $\underline{F}$ is regular and $M^{\text{OPT}} = (q^{\text{OPT}}, u^{\text{OPT}})$ is a robustly optimal mechanism. Then $q^{\text{OPT}}(\theta) = \underline{q}^{BM}(\theta)$ for all $\theta \in (\underline{\theta}, \overline{\theta})$.*

The proof in the Appendix first shows that, when $\underline{F}$ is regular, then any mechanism in the short list is such that $q(\theta) = \underline{q}^{\text{BM}}(\theta)$ for almost all $\theta \in \Theta$. The proposition then follows from this property together with the monotonicity of q and the continuity of $\underline{q}^{\text{BM}}$.

A key implication of the above result — namely, that, when $\underline{F}$ is regular, the shortlist contains essentially a unique mechanism — is that the conjecture $(V^{\star}, F^{\star})$ plays no role in determining the robustly optimal mechanism. The following result is then an immediate implication of the previous proposition.

Corollary 2 *Suppose $F^\star$ is regular and the government faces no uncertainty over the cost (that is, $\mathcal{F} = \{F^\star\}$). The unique robustly optimal mechanism is the Baron-Myerson one for the lowest possible demand $\underline{V}$.*

In contrast, Propositions 2 and 3 establish that, when the government faces uncertainty over the cost and the lowest distribution $\underline{F}$ is the cdf of a Dirac measure assigning probability one to $\theta = \overline{\theta}$, the conjecture $(V^\star, F^\star)$ plays an important role in the determination of the robustly optimal mechanism. To illustrate how the two cases are related, it is useful to consider more general distributions $\underline{F}$ satisfying the following properties. Throughout this section, we fix two parameters of $\underline{F}$ and define a key property of $\underline{F}$ with respect to these parameters: (i) θ_s, the lowest type in the support of $\underline{F}$; and (ii) δ_s, the probability mass point at $\overline{\theta}$ in $\underline{F}$.

Definition 6 *The cdf $\underline{F}$ is **partially regular** with respect to (θ_s, δ_s) if the following properties are true:*

(a) $\underline{F}$ is absolutely continuous over $(-\infty, \overline{\theta})$, with density $\underline{f}(\theta) > 0$ for all $\theta \in [\theta_s, \overline{\theta}]$,

(b) $\underline{F}(\theta) = 0$ for all $\theta \leq \underline{\theta}$, $\underline{F}(\theta) = 1$ for all $\theta \geq \overline{\theta}$,

(c) $\lim_{\theta \uparrow \overline{\theta}} \underline{F}(\theta) = 1 - \delta_s$,

(d) the function $\underline{z} : [\theta_s, \overline{\theta}] \to \mathbb{R}$ defined by

$$\underline{z}(\theta) := \begin{cases} \theta + (1-\alpha)\frac{\underline{F}(\theta)}{\underline{f}(\theta)} & \forall\ \theta \in [\theta_s, \overline{\theta}) \\ \overline{\theta} + (1-\alpha)\frac{1}{\underline{f}(\overline{\theta})} & \text{if } \theta = \overline{\theta} \text{ and } \delta_s = 0 \\ \overline{\theta} + (1-\alpha)\frac{1-\delta_s}{\delta_s} & \text{if } \theta = \overline{\theta} \text{ and } \delta_s > 0 \end{cases}$$

is increasing over $[\theta_s, \overline{\theta}]$ and continuous over $[\theta_s, \overline{\theta})$.

We then generalize the definitions of q_ℓ and $q^\star$ as follows. Let $\mathrm{q}_\ell^s := \underline{P}^{-1}(\theta_s)$ denote the efficient output when the inverse demand is $\underline{P}$ and the cost is θ_s. Then let $q_s^\star$ be the quantity schedule defined by

$$q_s^\star(\theta) := \begin{cases} \max\{q^{\text{BM}}(\theta), \text{q}_\ell^s\} & \theta < \theta_s \\ \underline{q}^{\text{BM}}(\theta) & \theta \geq \theta_s, \end{cases} \tag{8}$$

where q^{BM} continues to denote the optimal quantity schedule of Baron and Myerson (1982) when the conjecture is $(V^\star, F^\star)$, with $F^\star$ regular.

Proposition 6 *Suppose $F^\star$ is regular, $\underline{F}$ is partially regular with respect to (θ_s, δ_s), and $V^\star = \underline{V}$. The mechanism $M_s^\star = (q_s^\star, u_s^\star)$ where $q_s^\star$ is the quantity schedule in (8) and $u_s^\star$ is the function given by $u_s^\star(\theta) = \int_\theta^{\bar{\theta}} q_s^\star(y)dy$ for all θ is robustly optimal.*

The proof in the Appendix first shows that the mechanism in the proposition belongs to the short list. It then shows that any other mechanism in the short list yields a lower welfare under the conjecture $(V^\star, F^\star)$.

Figure 4 illustrates the structure of the quantity schedule $q^{\text{OPT}} = q_s^\star$ identified in Proposition 6. It highlights that the conjecture $(V^\star, F^\star)$ shapes the quantity procured under a robustly optimal mechanism but only outside the support of $\underline{F}$.

6 Changes in uncertainty

Equipped with the results in the previous sections, we now investigate how robustly optimal mechanisms change when the government's uncertainty over the seller's cost and consumers' demand change. In either case, we hold the government's conjecture fixed at $(V^\star, F^\star)$.

6.1 Variations in cost uncertainty

As the results above indicate the optimal mechanism depends on $\mathcal{F}$ only through $F^\star$ and $\underline{F}$. To understand how changes in cost uncertainty affect the properties of robustly optimal mechanisms, it is thus instructive to consider a sequence $(\underline{F}_n)$ of cdfs defining the lowest elements of the set $\mathcal{F}$ of cdfs considered plausible by the government.

Let $(\underline{F}_n)$ be any sequence satisfying the following properties:

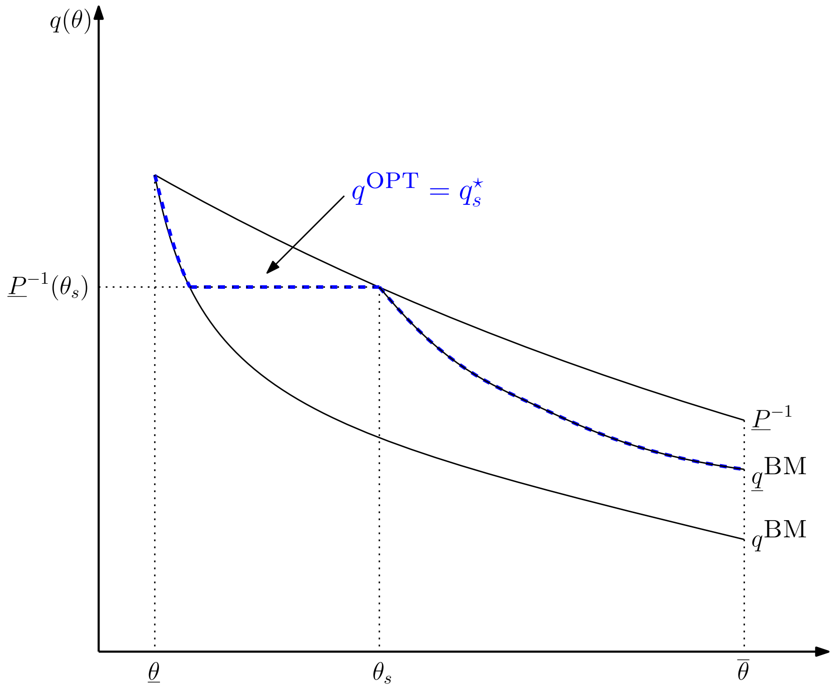

Figure 4: q^{OPT} identified in Proposition 6

(a) for any n there exists $\theta_n \in \Theta$ such that $\underline{F}_n$ is partially regular with respect to $\theta_s = \theta_n$ and $\delta_s = \delta_n$,

(b) $\underline{\theta}_{n+1} \geq \underline{\theta}_n$, with $\lim_{n\to\infty} \underline{\theta}_n = \overline{\theta}$,

(c) for all $\theta \in [\underline{\theta}_{n+1}, \overline{\theta}]$,

$$\frac{\underline{F}_{n+1}(\theta)}{\underline{f}_{n+1}(\theta)} \leq \frac{\underline{F}_n(\theta)}{\underline{f}_n(\theta)}, \tag{9}$$

and with $\delta_{n+1} \geq \delta_n$,

(d) $\underline{\theta}_n = \underline{\theta}$ if, and only if, $n \leq \overline{n}$, and $\delta_n > 0$ if, and only if, $n \geq \overline{\overline{n}}$, for some $\overline{n}, \overline{\overline{n}} \in \mathbb{N} \cup \{+\infty\}$, with $\overline{\overline{n}} \geq \overline{n}$,

(e) for any n and any $\theta \in [\underline{\theta}_n, \overline{\theta}]$,

$$\frac{\underline{F}_n(\theta)}{\underline{f}_n(\theta)} \leq \frac{F^\star(\theta)}{f^\star(\theta)}. \tag{10}$$

Figure 5 provides an illustration of the sequence $(\underline{F}_n)$. Note that property (c) above means that the distributions are ranked in the reverse-hazard-rate order. The sequence can thus

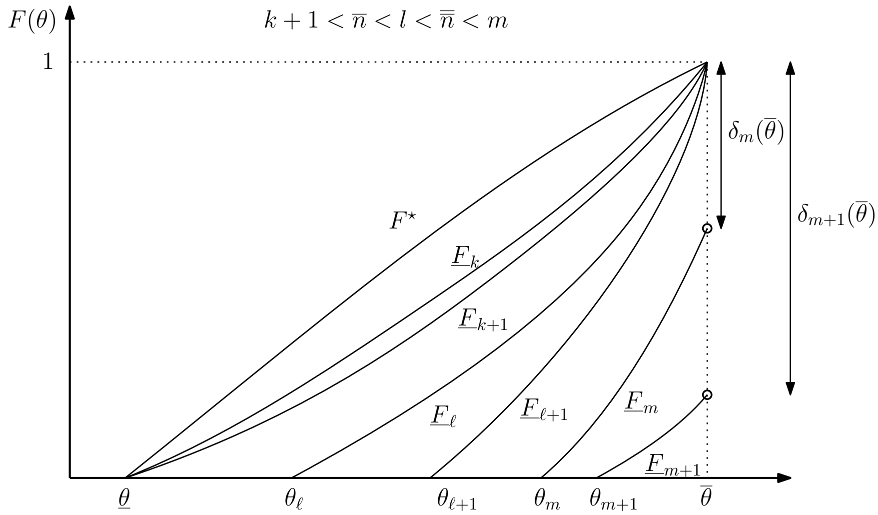

Figure 5: Pictorial depiction of the sequence (F_n)

be interpreting as capturing an increase in the severity of the government's uncertainty over the seller's cost.

Let q_n^{OPT} be a robustly optimal quantity schedule when the lowest distribution in $\mathcal{F}$ is $\underline{F}_n$. The following proposition establishes that the quantity procured under a robustly optimal mechanism is not monotone in the government's pessimism, that is, in $\underline{F}_n$. This property holds despite the fact that, as is well known, the Baron-Myerson quantity schedule $\underline{q}_n^{\text{BM}}$ defined, for all $\theta \in [\underline{\theta}_n, \overline{\theta})$, by

$$\underline{q}_n^{\text{BM}}(\theta) := \arg\max_{\text{q}} \{V^\star(\text{q}) - \underline{z}_n(\theta)\text{q}\}$$

is increasing in the inverse-hazard rare order: for any $n, n' \in \mathbb{N}$, with $n' > n$ and any $\theta \geq \underline{\theta}_{n'}$, $\underline{q}_{n'}^{\text{BM}}(\theta) \geq \underline{q}_n^{\text{BM}}(\theta)$. That is, when the government's conjecture over the seller's cost coincides with the distribution $\underline{F}_n$, an increase in the distribution (in the inverse-hazard-rate order) leads to an increase in the output procured.

Proposition 7 (Non-monotonicity of output in severity of cost uncertainty) *Suppose $V^\star = \underline{V}$ and $F^\star$ is regular. Let $(\underline{F}_n)$ be any sequence of cdfs satisfying properties (a)-(e) above and let (M_n^{OPT}) be any sequence of mechanisms such that, for each n, $M_n^{\text{OPT}} := (q_n^{\text{OPT}}, u_n^{\text{OPT}})$ is a robustly optimal mechanism when the lowest distribution in $\mathcal{F}$ is $\underline{F}_n$. Then,*

1. *For every $\theta \in (\underline{\theta}, \overline{\theta})$, there exists $n(\theta) \in \mathbb{N}$ such that $q_n^{\text{OPT}}(\theta)$ is non-decreasing (alternatively, non-increasing) on $n \leq n(\theta) - 1$ (alternatively, $n > n(\theta)$).*

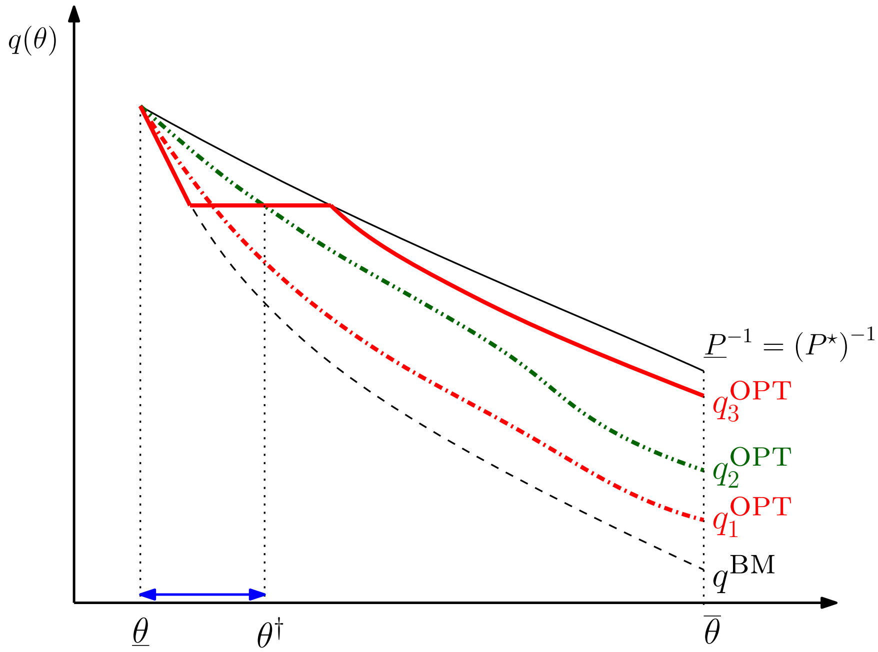

Figure 6: Illustration of Proposition 7

2. Moreover, for every θ, there exists $j, k \in \mathbb{N}$ with $j < k$ such that $q_j^{\mathrm{OPT}}(\theta) > q_k^{\mathrm{OPT}}(\theta)$.

Figure 6 illustrates the result in Proposition 7. For any $\theta \in [\underline{\theta}, \theta^{\dagger}]$, as the lowest distribution changes from $\underline{F}_1$ to $\underline{F}_2$, the quantity procured increases. In fact, the robustly optimal quantity schedule changes from the dash-dotted line to the dash-double-dotted line. Note that both $\underline{F}_1$ to $\underline{F}_2$ have support Θ; a reduction in the inverse hazard rate then implies a reduction in the value of reducing the rents paid to the most efficient types and hence an increase in the output procured under the optimal mechanism. When the lowest distribution changes from $\underline{F}_2$ to $\underline{F}_3$, the robustly optimal quantity schedule changes from the dash-double-dotted line to the solid line and the quantity procured from types in the range $[\underline{\theta}, \theta^{\dagger}]$ goes down. This is because the support of $\underline{F}_3$ no longer contains low-cost types. The government can then afford to procure a positive output from these types without jeopardizing robustness. Thus, the quantity procured from types in the range $[\underline{\theta}, \theta^{\dagger}]$ is not monotone in n, equivalently, in the lowest possible cost distribution. The formal proof is in the Appendix.

6.2 Variations in demand uncertainty

Recall that, fixing $P^\star$, the optimal mechanism depends on the set of inverse demand functions $\mathcal{P}$ defining the government's demand uncertainty only through the smallest demand $\underline{P}$ of the set $\mathcal{P}$. Suppose the lowest demand increases from $\underline{P}$ to $\underline{P}_N$, with $\underline{P}_N(\mathrm{q}) \geq \underline{P}(\mathrm{q})$ for all q and with $P^\star \in \mathcal{P}_N \cap \mathcal{P}$ — the subscript "N" is meant to be mnemonic for "new". As before, assume $\underline{P}_N$ is decreasing and continuous. The robustness constraint (6) then becomes $\underline{W}_N(\theta, q) \geq G_N^*$, with $\underline{W}_N$ defined, for all (θ, q), by

$$\underline{W}_N(\theta, q) := \underline{V}_N(q(\theta)) - \theta q(\theta) - (1-\alpha) \int_{\theta}^{\overline{\theta}} q(y)dy.$$

Let $\mathrm{q}_\ell^N := \underline{P}_N{}^{-1}(\overline{\theta})$, and $q_N^\star(\theta) := \max\{q^{\mathrm{BM}}(\theta), \mathrm{q}_\ell^N\}$. Let $\theta_N^\star$ be the threshold defined analogously with $\theta^\star$ but for the inverse demand $\underline{P}_N$. That is, if $q^{\mathrm{BM}}(\overline{\theta}) \leq \mathrm{q}_\ell^N$, let $\theta_N^\star$ be the unique solution to $q^{\mathrm{BM}}(\theta_N^\star) = \mathrm{q}_\ell^N$. If, instead, $q^{\mathrm{BM}}(\overline{\theta}) > \mathrm{q}_\ell^N$, let $\theta_N^\star := \overline{\theta}$. Finally, let

$$\theta_N^m := \max\{\theta : \theta \in \arg\min_{y \in \Theta} \underline{W}_N(y, q_N^\star)\}.$$

We then have the following result:

Proposition 8 *Assume $F^\star$ is regular. Suppose that $\alpha = 0$ and the government's demand uncertainty changes from $\mathcal{P}$ (with lowest element $\underline{P}$) to $\mathcal{P}_N$ (with lowest element $\underline{P}_N$), with $\underline{P}_N(\mathrm{q}) \geq \underline{P}(\mathrm{q})$ for all q, and with $P^\star \in \mathcal{P}_N \cap \mathcal{P}$. The following are true:*

1. *$\mathrm{q}_\ell^N \geq \mathrm{q}_\ell$, and $\theta_N^\star \leq \theta^\star$, with the first inequality strict if, and only if, $\underline{P}_N(\mathrm{q}_\ell) > \underline{P}(\mathrm{q}_\ell)$, and the second inequality strict if $\mathrm{q}_\ell^N > \mathrm{q}_\ell$, $\theta^\star < \overline{\theta}$.*

2. *Furthermore, if $\theta_N^\star \geq \theta^m$, then $\theta_N^m \geq \theta^m$.*

The result in the proposition says that, as the downside uncertainty shrinks, the government responds by procuring more output when the monopolist's cost is high. Furthermore, if the optimal mechanism before the reduction in uncertainty is different from a Baron-Myerson mechanism with a floor (that is, if $\theta^m < \theta^\star$), and the reduction in uncertainty is small (in which case $|\mathrm{q}_\ell^N - \mathrm{q}_\ell|$ and $|\theta_N^\star - \theta^\star|$ are small), the cost region over which the government procures less output relative to the SEU optimum shrinks as uncertainty is reduced.

7 Price vs quantity regulation

7.1 Price mechanisms

In this section, we consider an alternative class of procurement mechanisms in which the government responds to the seller's cost report by fixing the price for the seller's output instead of committing to procuring a given quantity. The seller is then asked to supply any output demanded by the consumers at the specified price. The final transfer to the seller is determined ex-post, once the realized demand curve becomes common knowledge between the seller and the government. By conditioning the transfer on the realized demand, the government guarantees that the seller has incentives to participate and report truthfully, irrespective of the sellers' beliefs over the realized demand and its attitude towards uncertainty and ambiguity, as we show below.

Such price mechanisms have a potential advantage over the quantity mechanisms considered in the previous sections; the quantity procured under a price mechanism is responsive to the realized demand, in case the conjecture turns out to be wrong. The feasibility of these mechanisms also hinges on the government being able to ask the monopolist to wait till the uncertainty over the demand is resolved before the final transfer is determined. This possibility may be appropriate in certain environment in which the role of the government is to regulate the interaction between the monopolist and the consumers. It may not be appropriate in settings in which the uncertainty over V reflects the government's inability to determine the social value of procuring the output, with this value possibly unverifiable to third parties. As we show in a moment, price mechanisms, even when feasible, also come with certain disadvantages that make their attractiveness vis-a-vis to quantity mechanisms unclear (we provide conditions for each class to dominate below).

Let $\mathcal{D}$ be the set of possible demand functions that the government considers possible. We assume that there exists a lowest demand function $\underline{D} \in \mathcal{D}$ such that $D(\mathrm{p}) \geq \underline{D}(\mathrm{p})$ for all p. To each $D \in \mathcal{D}$ corresponds a unique inverse demand function $P \in \mathcal{P}$, and hence a unique value function $V \in \mathcal{V}$. Let $D^\star$ denote the demand function corresponding to the government's conjecture $V^\star$.

A price mechanism consists of a price function p along with a transfer function t. The price function $p : \Theta \to \mathbb{R}$ specifies a price for each possible cost report. The seller is required to supply any quantity demanded by consumers at that price. Because there is uncertainty over the demand, when the government sets the price instead of the quantity it exposes the seller to uncertainty over its profits. To guarantee that the seller has incentives to participate and report truthfully no matter its beliefs and attitude towards uncertainty, the government must condition the transfer to the seller on the demand $D \in \mathcal{D}$, which is learned ex-post.

This ex-post approach seems the closest to the objective of guaranteeing robustness of the selected mechanism.

Definition 7 *A **price mechanism** $\widetilde{M} = (p, t)$ is a pair of mapping*

$$p : \Theta \to \mathbb{R}_+$$
$$t : \Theta \times \mathcal{D} \to \mathbb{R}_+$$

where $p(\theta)$ is the price charged to the consumers and $t(\theta, D)$ is the transfer to the seller when the cost report is θ and the realized demand is D.

The timing of events is the following:

- The seller learns its type θ;
- The government commits to a price mechanism $\widetilde{M} = (p, t)$;
- After observing $\widetilde{M}$, the seller reports $\theta' \in \Theta$ to the government and is required to supply any output demanded at price $p(\theta')$;[4]
- After learning the entire demand curve D, the government transfers $t(\theta', D)$ to the seller.[5]

Consistent with what was assumed in the previous section, we maintain that, at the time the government and the seller learn the quantity $D(p(\theta'))$ (which can occur concurrently or before they learn the entire demand curve D), it is not worth adjusting the quantity supplied.

[4]This interpretation is close to regulation where the government asks the seller to sell the good directly but regulates the price.

[5]For simplicity, we assume that the seller transfers the revenues $p(\theta')D(p(\theta'))$ to the government. Alternatively, one can assume the seller keeps the revenues for itself and the government transfers $\tilde{t}(\theta', D) := t(\theta', D) - p(\theta')D(p(\theta'))$ to the seller. In this case a price mechanism is given by the pair $(p, \tilde{t})$.

For example, the cost of any ex-post adjustment could be prohibitively high or the value consumers assign to extra output at the stage at which the demand becomes known to the government and the seller could be too low to justify the production cost. In the absence of these frictions, uncertainty over the demand is inconsequential.

The price mechanism $\widetilde{M} = (p, t)$ is ex-post *incentive compatible (EPIC)* if, for all $\theta, \theta' \in \Theta$ and $D \in \mathcal{D}$,

$$t(\theta, D) - \theta D(p(\theta)) \geq t(\theta', D) - \theta D(p(\theta')).$$

It is *ex-post individually rational (EPIR)* if, for all $\theta \in \Theta$ and $D \in \mathcal{D}$,

$$\tilde{u}(\theta, D) := t(\theta, D) - \theta D(p(\theta)) \geq 0.$$

The following lemma characterizes EPIC and EPIR price mechanisms. We omit the proof since it follows from standard arguments.

Lemma 4 *$\widetilde{M} = (p, t)$ is EPIC and EPIR if and only if p is non-decreasing, and for every $\theta \in \Theta$ and $D \in \mathcal{D}$,*

$$\tilde{u}(\theta, D) = \tilde{u}(\overline{\theta}, D) + \int_{\theta}^{\overline{\theta}} D(p(y))dy, \tag{11}$$

with $\tilde{u}(\overline{\theta}, D) \geq 0$.

Let $\widetilde{\mathcal{M}}$ be the set of all EPIC and EPIR price mechanisms. For any $\widetilde{M} \in \widetilde{\mathcal{M}}$, any $F \in \mathcal{F}$, and any demand $D \in \mathcal{D}$, welfare is given by

$$\widetilde{W}(\widetilde{M}; D, F) := \int \left[\widetilde{V}(p(\theta); D) - \theta D(p(\theta)) - (1 - \alpha)\tilde{u}(\theta, D)\right] F(\mathrm{d}\theta),$$

where $\tilde{u}(\theta, D)$ is as in (11) and where, for any p, and any D,

$$\widetilde{V}(\mathrm{p}; D) := \int_{0}^{D(\mathrm{p})} D^{-1}(y)dy. \tag{12}$$

The welfare guarantee of any price mechanism $\widetilde{M} \in \widetilde{\mathcal{M}}$ is given by

$$G(\widetilde{M}) := \inf_{D \in \mathcal{D}, F \in \mathcal{F}} \widetilde{W}(\widetilde{M}; D, F).$$

The shortlist of price mechanisms is given by

$$\widetilde{\mathcal{M}}^{\text{SL}} := \{\widetilde{M} \in \widetilde{\mathcal{M}} : G(\widetilde{M}) \geq G(\widetilde{M}') \; \forall \; \widetilde{M}' \in \widetilde{\mathcal{M}}\}.$$

Recall that the maximal welfare guarantee for quantity mechanisms is $G^* := \underline{V}(\mathrm{q}_\ell) - \overline{\theta}\mathrm{q}_\ell$, as shown in Lemma 1, where $\mathrm{q}_\ell := \underline{P}^{-1}(\overline{\theta}) = \underline{D}(\overline{\theta})$ is the efficient quantity when $\theta = \overline{\theta}$ and $P = \underline{P}$ (equivalently, when $D = \underline{D}$ and $V = \underline{V}$). This guarantee also applies to price mechanisms.

Lemma 5 (Welfare Guarantee) *For any price mechanism $\widetilde{M} \in \widetilde{\mathcal{M}}$, $G(\widetilde{M}) \leq G^*$. There exists a price mechanism $\underline{\widetilde{M}} \in \widetilde{\mathcal{M}}$ such that $G(\underline{\widetilde{M}}) = G^*$. Any $\widetilde{M} \in \widetilde{\mathcal{M}}$ for which $G(\widetilde{M}) = G^*$ is such that $p(\overline{\theta}) = \overline{\theta}$ and $\tilde{u}(\overline{\theta}, \underline{D}) = 0$.*

Lemma 5 establishes that the maximal guarantee over all price and quantity mechanisms is the same. This follows from the fact that, no matter whether the government offers a price or quantity mechanism, Nature can always select the lowest possible demand $\underline{D}$ and the highest possible cost $\overline{\theta}$. The maximal welfare when $\theta = \overline{\theta}$ and $D = \underline{D}$ is attained by procuring the efficient output q_ℓ. Whether the government induces this outcome by fixing the price or by dictating that the seller produces q_ℓ is inconsequential. The lemma also shows that the government can guarantee G^* through a price mechanism that fixes the price at $\overline{\theta}$ for all cost reports.

The following is an immediate implication of the above observations:

Remark 1 *Price and quantity regulation are equivalent under worst-case optimality: The maximal welfare guarantee over all price mechanisms is the same as over all quantity mechanisms.*

As we show in Subsection 7.2, things are different when, instead, the government uses its conjecture to select a mechanism among those for which the guarantee is the highest.

Lemma 5 also establishes that any price mechanism yielding the maximal guarantee G^* must require that the seller sets a price equal to $\overline{\theta}$ when $\theta = \overline{\theta}$ and must provide no rent to the seller when $\theta = \overline{\theta}$ and $D = \underline{D}$. This is because only such choices yield G^* when the cost is the highest and demand is the lowest.

An immediate implication of the last lemma is that any price mechanism $\widetilde{M} = (p, t) \in \widetilde{\mathcal{M}}^{\text{SL}}$ in the short list is such that (a) p is non-decreasing, (b) $p(\overline{\theta}) = \overline{\theta}$, (c) $\tilde{u}(\overline{\theta}, \underline{D}) = 0$, and (d)

$$\widetilde{V}(p(\theta); D) - \theta D(p(\theta)) - (1-\alpha)\tilde{u}(\theta, D) \geq G^*, \qquad \forall\ \theta \in \Theta \text{ and } D \in \mathcal{D}. \tag{13}$$

Furthermore, any mechanism $\widetilde{M} = (p, t)$ satisfying properties (a)-(d) is in the short list.

Now recall from Lemma 3 that, when given D, the quantity procured at each θ is below the efficient level $D(\theta) = P^{-1}(\theta)$, then welfare is decreasing in θ. Hence, given any price mechanism $\widetilde{M} \in \widetilde{\mathcal{M}}$ such that $p(\theta) \geq \theta$ for all θ, the robustness constraint in (13) is satisfied if, and only if, for any $D \in \mathcal{D}$,

$$\widetilde{V}(\overline{\theta}; D) - \overline{\theta} D(\overline{\theta}) - (1-\alpha)\tilde{u}(\overline{\theta}, D) \geq G^*, \tag{14}$$

which is the case if and only if $\tilde{u}(\overline{\theta}, D) = 0$.

Finally, observe that, given any $\widetilde{M} \in \widetilde{\mathcal{M}}$, the government's payoff under the conjecture $(V^\star, F^\star)$ (equivalently, $(D^\star, F^\star)$) is equal to

$$\int_{\underline{\theta}}^{\overline{\theta}} \Big[V^\star(D^\star(p(\theta))) - z^\star(\theta) D^\star(p(\theta)) \Big] F^\star(\mathrm{d}\theta) - (1-\alpha)\tilde{u}(\overline{\theta}, D^\star). \tag{15}$$

Using the above results, we can then provide a simple program yielding all robustly optimal price mechanisms.

Lemma 6 *Suppose $F^\star$ is regular. The price mechanism $\widetilde{M}^{\text{OPT}} = (p^{\text{OPT}}, t^{\text{OPT}})$ is robustly optimal if and only if the price schedule p^{OPT} solves the following program*

$$\max_p \int_{\underline{\theta}}^{\overline{\theta}} \Big[V^\star(D^\star(p(\theta))) - z^\star(\theta) D^\star(p(\theta)) \Big] F^\star(d\theta) \tag{\textbf{ROPT-P}}$$

subject to

$$p \ \ \text{non} - \text{decreasing}$$

$$p(\overline{\theta}) = \overline{\theta} \tag{16}$$

$$p(\theta) \geq \theta \qquad \forall\ \theta \in \Theta, \tag{17}$$

and t^{OPT} *is such that the function* $\tilde{u}^{\mathrm{OPT}}$ *defined, for all* $\theta \in \Theta$ *and* $D \in \mathcal{D}$, *by* $\tilde{u}^{\mathrm{OPT}}(\theta, D) := t^{\mathrm{OPT}}(\theta, D) - \theta D(p^{OPT}(\theta))$ *satisfies the following properties:*

$$\begin{aligned} \tilde{u}^{\mathrm{OPT}}(\theta, D) &= \tilde{u}^{\mathrm{OPT}}(\overline{\theta}, D) + \int_{\theta}^{\overline{\theta}} D(p^{\mathrm{OPT}}(y))dy \qquad \forall\ \theta \in \Theta,\ \forall\ D \in \mathcal{D} \\ 0 \leq (1-\alpha)\tilde{u}^{\mathrm{OPT}}(\overline{\theta}, D) &\leq \widetilde{V}(\overline{\theta}; D) - \overline{\theta} D(\overline{\theta}) - G^* \qquad \forall\ D \in \mathcal{D} \\ \tilde{u}^{\mathrm{OPT}}(\overline{\theta}, \underline{D}) &= \tilde{u}^{\mathrm{OPT}}(\overline{\theta}, D^\star) = 0. \end{aligned} \tag{18}$$

It is easy to see that, under SEU, price and quantity regulation are equivalent. Under the conjecture $(V^\star, F^\star)$, the following mechanism implements the same quantity schedule q^{BM} as the Baron-Myerson original mechanism. The same is true when the original mechanism is amended to incorporate a quantity floor. Under the conjecture $(V^\star, F^\star)$, the following mechanism implements the same quantity schedule $q^\star$ as the Baron-Myerson-quantity-floor mechanism defined in the previous section.

Definition 8 *The price mechanism* (p, t) *is a* ***Baron-Myerson-with-price-cap*** *if and only if the price function is given by*

$$p(\theta) = \min(z^\star(\theta), \overline{\theta}) \tag{19}$$

for all θ, *and, for all* (θ, D), *the transfer function* t *is defined as in Lemma 6.*

Note that whereas Baron-Myerson-with-quantity-floor is a unique mechanism, there are many price mechanisms that qualify as Baron-Myerson-with-price-cap. All these mechanisms have the same price schedule, with the latter given by (19). They differ in the transfer schedule t. In fact there are infinitely many transfer schedules satisfying the constraints of Lemma 6. In particular, these mechanisms differ in the rents $\tilde{u}(\overline{\theta}, D)$ given to the highest type $\overline{\theta}$ for demands $D \notin \{\underline{D}, D^\star\}$.

We then have the following result.

Proposition 9 *Suppose* $F^\star$ *is regular.*[6] *The Baron-Myerson-with-price-cap mechanism is robustly optimal. Moreover, every robustly optimal price mechanism has the same price schedule as Baron-Myerson-with-price-cap.*

[6] The result also holds under the weaker requirement that $z^\star$ is non-decreasing.

The result follows from the fact that the virtual surplus function $V^\star(D^\star(\mathrm{p})) - z^\star(\theta)D^\star(\mathrm{p})$ is quasi-concave in p, and attains a maximum at $z^\star(\theta)$. The price function in the proposition is thus the unique one that solves program (**ROPT-P**).

Proposition 9 thus establishes that there exists a unique robustly optimal price schedule. It consists in setting a price for each θ equal to the minimum between the virtual cost $z^\star(\theta)$ and $\overline{\theta}$. We then have the following result:

Corollary 3 *Suppose $F^\star$ is regular.*[7] *The unique robustly optimal price schedule is invariant in both the government's conjectured demand $D^\star$ and the set of demand functions $\mathcal{D}$ the government considers feasible. It consists in setting a markup equal to $(1-\alpha)F^\star(\theta)/f^\star(\theta)$ at each cost θ and then capping the price at $\overline{\theta}$.*

The result follows directly from Proposition 9 and the fact that the price function in (19) is independent of $D^\star$ and $\mathcal{D}$.

Figure 7 illustrates Proposition 9 for $\alpha = 0$. By committing to pay rents contingent on the realized demand D and setting a markup that only depends on the conjecture $F^\star$ over the cost, with a price cap at $\theta = \overline{\theta}$, the regulator maximizes welfare no matter the uncertainty it faces over the demand and its conjecture $F^\star$.

7.2 Price vs quantity mechanisms

As shown above, any robustly optimal price mechanism has the same price schedule as Baron-Myerson-with-price-cap. The question of interest, though, is whether such mechanisms do better than their quantity counterparts, which is what we address in this section.

Lemma 5 clarifies that the maximal guarantee (i.e., the maximal welfare under the worst case scenario) is the same for either type of mechanism. However, as we show next, the maximal welfare attainable under the government's conjecture over the short list of worst-case-optimal mechanisms need not coincide over the two classes of robustly optimal mechanisms.

[7] Again, the result in the corollary continues to hold if one replaces the regularity assumption with the weaker requirement that $z^\star$is non-decreasing.

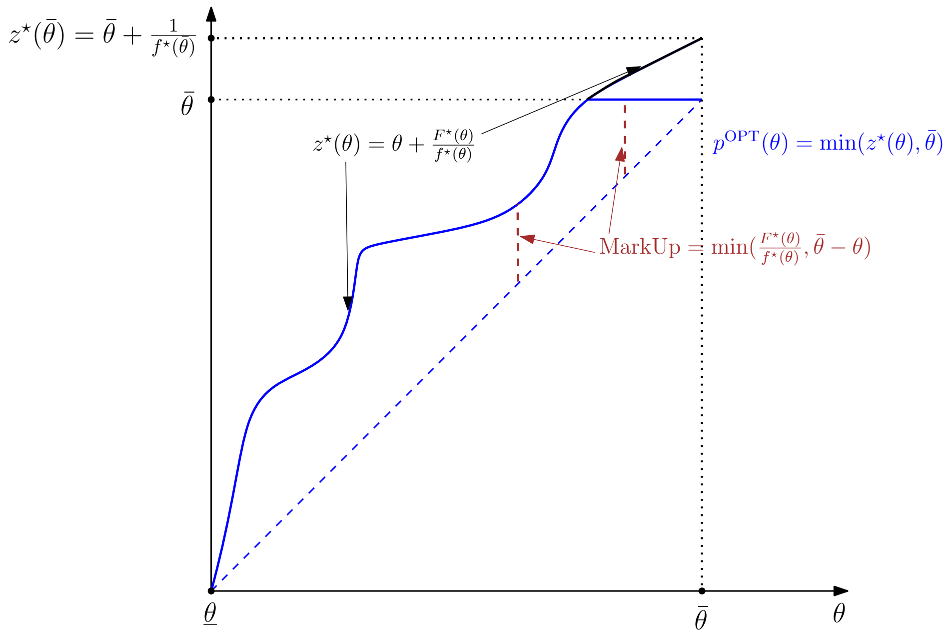

Figure 7: Robustly optimal price schedule for $\alpha = 0$

Definition 9 *Price regulation dominates quantity regulation if*

$$\widetilde{W}(\widetilde{M}^{\mathrm{OPT}}; D^\star, F^\star) \geq W(M^{\mathrm{OPT}}; V^\star, F^\star). \tag{20}$$

Quantity regulation strictly dominates price regulation if the above inequality does not hold. Price regulation strictly dominates quantity regulation if the above inequality is strict. Quantity regulation dominates price regulation if

$$\widetilde{W}(\widetilde{M}^{\mathrm{OPT}}; D^\star, F^\star) \leq W(M^{\mathrm{OPT}}; V^\star, F^\star).$$

Price and quantity regulation are equivalent if the inequality in (20) is an equality.

We then have the following result:

Proposition 10 *Assume $F^\star$ is regular.*

1. *If Baron-Myerson-with-quantity-floor is robustly optimal (i.e., if $M^{\mathrm{OPT}} = M^\star$), then quantity regulation dominates price regulation. Further, if $D^\star(\overline{\theta}) > \underline{D}(\overline{\theta})$, then quantity regulation strictly dominates price regulation.*

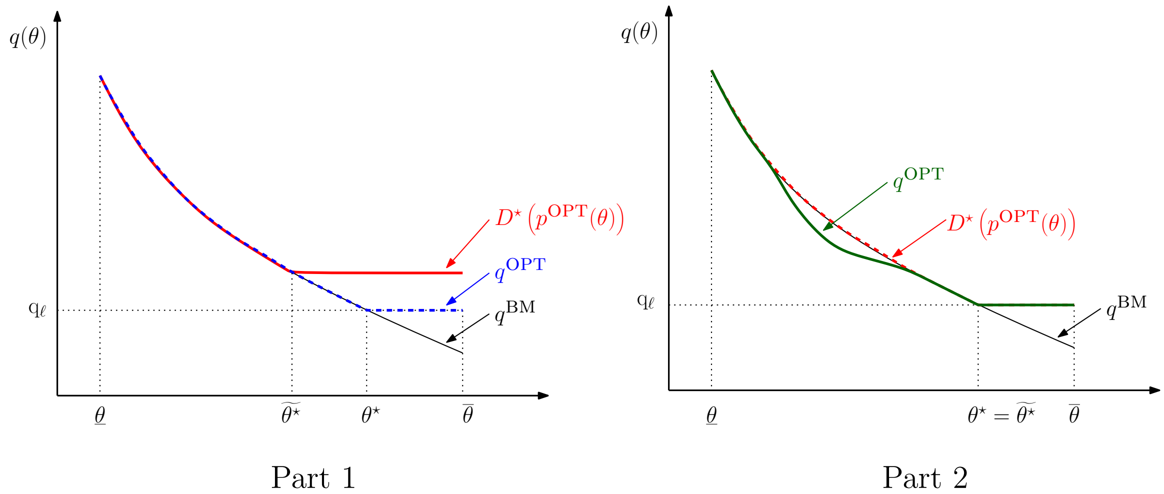

Figure 8: Graphical illustration of Proposition 10.

2. *If Baron-Myerson-with-quantity-floor is not robustly optimal (i.e., if* $M^{\text{OPT}} \neq M^\star$*) and* $D^\star(\overline{\theta}) = \underline{D}(\overline{\theta})$*, price regulation strictly dominates quantity regulation.*

Figure 8 illustrates quantity schedules q^{OPT} and demands $D^\star(p^{\text{OPT}})$ under the conjecture $D^\star$ for each of the two parts of Proposition 10.

The intuition for why quantity mechanisms dominate when the Baron-Myerson-with-quantity-floor mechanism is robustly optimal is the following. Controlling quantity or price leads to the same procurement of output for low costs. For high costs, however, the output procured under the optimal price mechanism can be higher than the one procured under the optimal quantity mechanism. This is because, under a price mechanism, ensuring guarantee G^* calls for capping the price at $\overline{\theta}$. When the government expects the demand at $\text{p} = \overline{\theta}$ to be above the minimal feasible level $\underline{D}(\overline{\theta})$, by capping the price at $\overline{\theta}$ it then ends up procuring too much output when cost is high, compared to what it procures when controlling quantity. In fact, when the Baron-Myerson-with-quantity-floor mechanism is robustly optimal, under quantity control, the government only needs to procure no less than $\text{q}_\ell = \underline{D}(\overline{\theta}) \leq D^\star(\overline{\theta})$ to protect itself under the worst case scenario. In this case, quantity mechanisms thus dominate.

When, instead, the Baron-Myerson-with-quantity-floor mechanism is not robustly optimal, under quantity control, the government needs to reduce the quantity it procures for intermediate costs below the level $q^{\text{BM}}(\theta)$ it would optimally procure in the absence of robustness considerations. This downward distortion (over and above the distortion the government would make under usual SEU analysis to reduce rents) is costly, and is necessary to guarantee that, if the demand ends up to be below the conjectured one, the government does not lose too much by procuring a quantity whose value is below the conjectured one. However, this downward adjustment, can be avoided by fixing the price. This is because, if demand turns out to be below the conjectured level, the quantity procured under a price mechanism is reduced, thus sparing the government from the risk of over-procurement. Provided that the price cap $\mathrm{p} = \overline{\theta}$ does not result in over-procurement for large costs (which is never the case when $\underline{D}(\overline{\theta}) = D^{\star}(\overline{\theta})$), price mechanisms thus strictly dominate when Baron-Myerson-with-quantity-floor is not robustly optimal.

The following is then a direct implication of the previous proposition and can be seen by combining Propositions 2 and 9.

Corollary 4 *Suppose $F^{\star}$ is regular and the government faces no uncertainty over the demand. Then price and quantity mechanisms are equivalent.*

The result should be expected: without demand uncertainty, it is inconsequential whether the government induces the seller to supply the desired quantity by fixing the price or by specifying the output that must be supplied.

8 Conclusions

We consider the procurement problem of a buyer (e.g., a government) concerned with the possibility that its conjecture over the value and cost of procuring a product or service may be wrong. We postulate that the buyer first protects itself by identifying all mechanisms that are worst-case optimal, i.e., that deliver the largest welfare guarantee. This set typically contains multiple mechanisms. The government then selects the mechanism from this set that maximizes its expected payoff under its conjecture. The approach seems quite compelling in many situations of interest and yields novel predictions for the structure of optimal mechanisms.

We show that, when the only uncertainty is over the cost of supplying the good, the optimal mechanism procures the same output as the Baron-Myerson mechanism, but with a floor that protects the buyer in case the cost is higher than conjectured (in which case there is less value in distorting output to reduce rents). When the buyer faces uncertainty also over the value of the good, robustness calls for upward adjustments (relative to the case without uncertainty) in the quantity procured from high-cost sellers and downward adjustments in the quantity procured from intermediate-cost sellers. We also investigate the implications of changes in uncertainty (over cost and demand) and the merits and limitations of regulating prices instead of quantity.

In future work, it would be interesting to study how the buyer ought to respond to the seller's own uncertainty, especially when the latter is the seller's private information. It would also be interesting to investigate when it may be optimal to regulate quantity for a set of possible seller's types and price over the complement set. We also expect the current results, along with those that these enrichments will deliver, to provide valuable insights over the structure of optimal policy interventions in other markets in which uncertainty plays a major role.

9 Appendix: Omitted Proofs

Proof of Lemma 1. Fix any $V \in \mathcal{V}$ and any $F \in \mathcal{F}$, and observe that

$$\begin{aligned} W(M; V, F) &= \int w(\theta, M; V) F(\mathrm{d}\theta) \\ &\geq \int w(\theta, M; \underline{V}) F(\mathrm{d}\theta) \qquad \text{(by definition of } \underline{V}\text{)} \\ &\geq \int \left[\inf_{\theta} w(\theta, M; \underline{V})\right] F(\mathrm{d}\theta) \\ &= \inf_{\theta} w(\theta, M; \underline{V}) \\ &= \inf_{\theta \in \Theta} \left[\underline{V}(q(\theta)) - \theta q(\theta) - (1-\alpha) u(\theta)\right]. \end{aligned}$$

Hence,

$$G(M) \geq \inf_{\theta \in \Theta} \left[\underline{V}(q(\theta)) - \theta q(\theta) - (1-\alpha) u(\theta)\right]. \tag{21}$$

Because $\underline{V} \in \mathcal{V}$ and, for each θ, the Dirac distribution that puts probability mass one at θ is in the set $\mathcal{F}$ of feasible distributions, we have that, for all θ,

$$G(M) \leq \underline{V}(q(\theta)) - \theta q(\theta) - (1-\alpha) u(\theta). \tag{22}$$

Combining the inequality in (22) with the inequality in (21), we obtain condition (3).

Finally, using (3), we obtain that

$$G(M) \leq \underline{V}(q(\overline{\theta})) - \overline{\theta}q(\overline{\theta}) - (1-\alpha)u(\overline{\theta}) \leq \underline{V}(q(\overline{\theta})) - \overline{\theta}q(\overline{\theta}) \leq \underline{V}(\mathrm{q}_\ell) - \overline{\theta}\mathrm{q}_\ell = G^*,$$

where the second inequality follows from IR and the third inequality follows from the definition of q_ℓ. This establishes (4). ■

Proof of Proposition 1. First, we show that there exists an IC and IR mechanism that delivers the welfare guarantee upper bound in (4). Consider the constant mechanism $M_L = (q_L, u_L)$ that asks each type θ to produce q_ℓ and pays $\overline{\theta}\mathrm{q}_\ell$; that is, $q_L(\theta) = \mathrm{q}_\ell$ and $t_L(\theta) = \overline{\theta}\mathrm{q}_\ell$, all θ (yielding a profit $u_L(\theta) = (\overline{\theta} - \theta)\mathrm{q}_\ell$ to each θ). The mechanism M_L is clearly IC and IR. Under the mechanism M_L, welfare when the marginal cost is θ and the gross surplus is $\underline{V}$ is equal to

$$\begin{aligned} w(\theta, M_L; \underline{V}) &= \underline{V}(\mathrm{q}_\ell) - \theta\mathrm{q}_\ell - (1-\alpha)u_L(\theta) \\ &= \underline{V}(\mathrm{q}_\ell) - \overline{\theta}\mathrm{q}_\ell + \alpha(\overline{\theta} - \theta)\mathrm{q}_\ell \\ &= G^* + \alpha(\overline{\theta} - \theta)\mathrm{q}_\ell. \end{aligned}$$

Hence, $\inf_\theta w(\theta, M_L; \underline{V}) = G^*$. Condition (3) in Lemma 1 then implies $G(M_L) = G^*$. By Lemma 1, we get $M_L \in \mathcal{M}^{\mathrm{SL}}$. Condition (4) in Lemma 1 in turn implies that, for any $M \in \mathcal{M}^{\mathrm{SL}}$, $G(M) = G^*$. For a mechanism $M = (q, u)$ to be IC and IR, it must be that q is non-increasing and, for all θ,

$$u(\theta) = u(\overline{\theta}) + \int_\theta^{\overline{\theta}} q(y)dy \tag{23}$$

with $u(\overline{\theta}) \geq 0$. Condition (3) in Lemma 1 in turn implies that, if $M \in \mathcal{M}^{\mathrm{SL}}$, then, for all θ

$$\underline{V}(q(\theta)) - \theta q(\theta) - (1-\alpha)u(\theta) \geq G^*.$$

This is possible only if $u(\overline{\theta}) = 0$ (else, the constraint is violated at $\overline{\theta}$) and, for any θ, constraint (5) holds.

Conversely, suppose $M = (q, u)$ satisfies properties (a)-(c) in the lemma. Then M is IC and IR. Furthermore, by Condition (3) in Lemma 1, $G(M) \geq G^*$. Because every mechanism in $\mathcal{M}^{\mathrm{SL}}$ has a welfare guarantee of G^*, we thus have that $M \in \mathcal{M}^{\mathrm{SL}}$. Q.E.D.

Proof of Lemma 2. If $F^\star$ is regular, then q^{BM} is decreasing. Moreover, $q^{\text{BM}}(\overline{\theta}) \le \text{q}_\ell$ implies that $q^\star$ is non-increasing with $q^\star(\overline{\theta}) = \text{q}_\ell$. Thus, the quantity schedule $q^\star$ satisfies the constraints of the relaxed problem. To complete the proof, observe that for any θ, the function $V^\star(\text{q}) - z^\star(\theta)\text{q}$ is strictly concave in q and attains a maximum at $q^{\text{BM}}(\theta)$, and therefore, the quantity schedule $q^\star$ maximizes the objective function in the relaxed program over all non-increasing functions q satisfying $q(\overline{\theta}) = \text{q}_\ell$. ■

Proof of Lemma 3. Fix $M = (q, u)$ and pick any $\theta, \theta' \in I$, with $\theta' < \theta$. Note that

$$\underline{W}(\theta', q) - \underline{W}(\theta, q) = \int_{q(\theta)}^{q(\theta')} \underline{P}(y)dy - \theta' q(\theta') + \theta q(\theta) - \int_{\theta'}^{\theta} q(y)dy + \alpha \int_{\theta'}^{\theta} q(y)dy. \tag{24}$$

Proof of Part (A). We consider two cases.

Case 1: $\underline{P}(q(\theta')) \ge \theta > \theta'$. Note that the right-hand-side of (24) equals to

$$\int_{q(\theta)}^{q(\theta')} (\underline{P}(y) - \theta)dy + \left\{ (\theta - \theta')q(\theta') - \int_{\theta'}^{\theta} q(y)dy \right\} + \alpha \int_{\theta'}^{\theta} q(y)dy. \tag{25}$$

The first term in (25) is non-negative because, for all $y \in (q(\theta), q(\theta'))$, $\underline{P}(z) > \underline{P}(q(\theta')) \ge \theta$, which follows from $\underline{P}$ being decreasing. Furthermore, if $q(\theta') > q(\theta)$, then this first term in (25) is positive. Next, observe that, because q is non-increasing, the expression in curly brackets in (25) is non-negative. Finally observe that the last term in (25) is non-negative because q is positive over I. We conclude that $\underline{W}(\theta', q) \ge \underline{W}(\theta, q)$, i.e., $\underline{W}(\cdot, q)$ is non-increasing over I (decreasing when $\alpha > 0$, or when q is decreasing over I).

Case 2: $\theta > \underline{P}(q(\theta')) \ge \theta'$. Use Figure 9 to observe that the sum of the first four terms in (24) is equal to

$$\int_{\theta'}^{\underline{P}(q(\theta'))} \Big(q(\theta') - q(y)\Big)dy + \int_{\underline{P}(q(\theta'))}^{\theta} \Big(\underline{P}^{-1}(y) - q(y)\Big)dy + \int_{\theta}^{\underline{P}(q(\theta))} \Big(\underline{P}^{-1}(y) - q(\theta)\Big)dy. \tag{26}$$

Now we argue that each of these three terms in expression (26) is non-negative. The first term is non-negative because q is non-increasing. Next observe that, for all $y \in (\underline{P}(q(\theta')), \theta)$, $\underline{P}^{-1}(y) \ge q(y)$. Hence, the second term in (26) is also non-negative. Finally, the last term in (26) is also non-negative because, for any $y \in (\theta, \underline{P}(q(\theta)))$, $\underline{P}^{-1}(y) \ge q(\theta)$, which follows

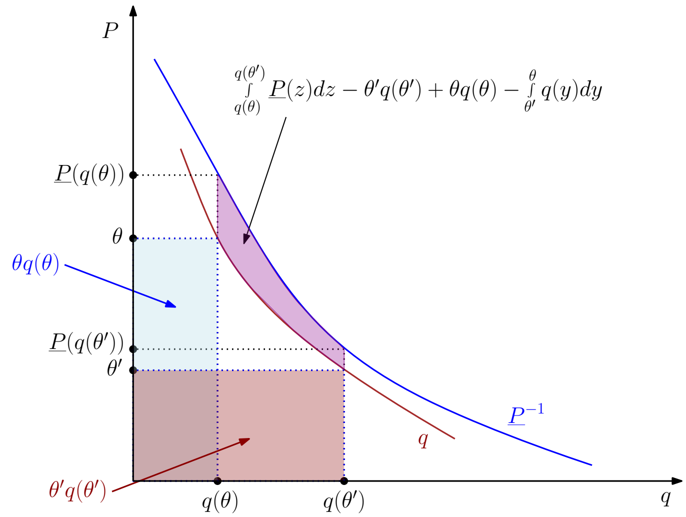

Figure 9: Illustration of Case 2 in Part A.

from $\underline{P}$ being decreasing. We conclude that $\underline{W}(\cdot, q)$ is non-increasing over I (decreasing when $\alpha > 0$, or when q is decreasing and such that $q(y) < \underline{P}^{-1}(y)$ for all $y \in I$).

Proof of Part (B): Because $\alpha = 0$, the difference in welfare $\underline{W}(\theta, q) - \underline{W}(\theta', q)$ across the two states is given by the (negative of the) expression in (24), which can be rewritten as

$$\underline{W}(\theta, q) - \underline{W}(\theta', q) = \int_{\theta'}^{\theta} \Big(q(y) - q(\theta)\Big) dy + \int_{q(\theta)}^{q(\theta')} \Big(\theta' - \underline{P}(z)\Big) dz. \tag{27}$$

We consider two cases.

Case 1: $\underline{P}(q(\theta)) \leq \theta'$. In this case, $\underline{P}(z) < \theta'$ for all $z > q(\theta)$. This implies that the second integral in (27) is non-negative and the first integral is non-negative because q is non-increasing. If q is decreasing, both integrals are positive.

Case 2: $\theta' < \underline{P}(q(\theta)) < \theta$. We then have that $q(\theta') > \underline{P}^{-1}(\theta') > q(\theta)$. Hence, using (27), we have that

$$\underline{W}(\theta, q) - \underline{W}(\theta', q) \geq \int_{\theta'}^{\theta} \Big(q(y) - q(\theta)\Big) dy - \int_{q(\theta)}^{\underline{P}^{-1}(\theta')} \Big(\underline{P}(z) - \theta'\Big) dz \tag{28}$$

See Figure 10 for an illustration of the right-hand-side of the inequality in (28). Changing

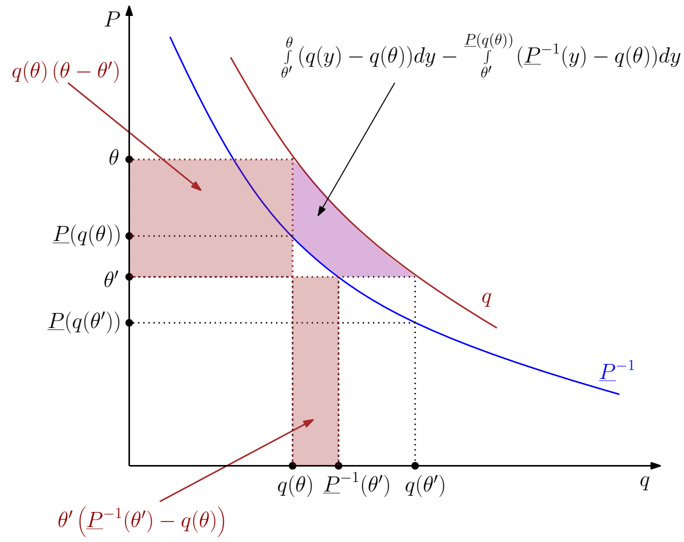

Figure 10: Illustration of Case 2 in Part B.

the variable of integration, the second integral can be written as

$$\int_{q(\theta)}^{\underline{P}^{-1}(\theta')}\Big(\underline{P}(z)-\theta'\Big)dz=\int_{\theta'}^{\underline{P}(q(\theta))}\Big(\underline{P}^{-1}(y)-q(\theta)\Big)dy.$$

Thus, the right-hand-side of (28) reduces to (see Figure 10 for an illustration)

$$\int_{\theta'}^{\theta}\Big(q(y)-q(\theta)\Big)dy-\int_{\theta'}^{\underline{P}(q(\theta))}\Big(\underline{P}^{-1}(y)-q(\theta)\Big)dy, \tag{29}$$

which is non-negative because $\underline{P}(q(\theta))<\theta$ and $\underline{P}^{-1}(y)<q(y)$ for all $y\in I$. Thus, $\underline{W}(\theta,q)\geq\underline{W}(\theta',q)$, i.e., $\underline{W}(\cdot,q)$ is non-decreasing over I. The above inequality also reveals that, when q is decreasing, the expression in (29) is positive, implying that $\underline{W}(\cdot,q)$ is increasing over I. ■

Proof of Proposition 2. If $F^\star$ is regular and $V^\star=\underline{V}$, then $q^{\mathrm{BM}}(\overline{\theta})<(P^\star)^{-1}(\overline{\theta})=\underline{P}^{-1}(\overline{\theta})=\mathrm{q}_\ell$, implying that $q^\star(\overline{\theta})=\mathrm{q}_\ell$. From Lemma 2, we know that the quantity schedule $q^\star$ solves the relaxed program (**RP-1**). To establish the proposition, it thus suffices to show that the quantity schedule $q^\star$ satisfies the robustness constraint (6).

Because $q^{\mathrm{BM}}(\overline{\theta})<\mathrm{q}_\ell$, we have $\theta^\star<\overline{\theta}$. Note that for all $\theta\in[\underline{\theta},\theta^\star]$, $q^\star(\theta)=q^{\mathrm{BM}}(\theta)<(P^\star)^{-1}(\theta)=\underline{P}^{-1}(\theta)$. By Lemma 3, $\underline{W}(\theta,q^\star)$ is non-increasing in θ in the interval $[\underline{\theta},\theta^\star]$.

That $q^\star(\theta) = \mathrm{q}_\ell$ for all $\theta \geq \theta^\star$ implies that $\underline{W}(\theta, q^\star)$ is also non-increasing over $[\theta^\star, \overline{\theta}]$. Hence, we conclude that $\underline{W}(\theta, q^\star) \geq \underline{W}(\overline{\theta}, q^\star) = \underline{V}(\mathrm{q}_\ell) - \overline{\theta}\mathrm{q}_\ell = G^*$ for all θ (with equality at $\theta = \overline{\theta}$). This means that constraint (6) is satisfied. Q.E.D.

Proof of Proposition 3. The proof is divided into different lemmas. The following lemma establishes Part 1 in Proposition 3.

Lemma 7 *Suppose $F^\star$ is regular. Then, Baron-Myerson-with-quantity-floor is robustly optimal if and only if $\theta^m = \overline{\theta}$ and $q^{\mathrm{BM}}(\overline{\theta}) \leq \mathrm{q}_\ell$.*

Proof: If $q^{\mathrm{BM}}(\overline{\theta}) \leq \mathrm{q}_\ell$, then, by Lemma 2, $q^\star$ solves the relaxed problem. If $\theta^m = \overline{\theta}$, then $\underline{W}(\theta, q^\star) \geq \underline{W}(\overline{\theta}, q^\star)$ for all $\theta \in \Theta$. But $u^\star(\overline{\theta}) = 0$ and $q^\star(\overline{\theta}) = \mathrm{q}_\ell$ imply that $\underline{W}(\overline{\theta}, q^\star) = G^*$. Hence, $\underline{W}(\theta, q^\star) \geq G^*$ for all $\theta \in \Theta$, implying that the robustness constraint (6) is satisfied and Baron-Myerson-with-quantity-floor is robustly optimal.

For the converse, suppose Baron-Myerson-with-quantity-floor is robustly optimal. Then, the robustness constraint (6) holds at $\overline{\theta}$, that is, $\underline{W}(\overline{\theta}, q^\star) = G^*$. Because $u^\star(\overline{\theta}) = 0$, we must have that $q^\star(\overline{\theta}) = \mathrm{q}_\ell$ which implies that $q^{\mathrm{BM}}(\overline{\theta}) \leq \mathrm{q}_\ell$. Now, because the robustness constraints hold for all θ, it must be that $\underline{W}(\theta, q^\star) \geq G^* = \underline{W}(\overline{\theta}, q^\star)$. As a result, $\theta^m = \overline{\theta}$. ■

The next three lemmas establish Part 2.

Lemma 8 *If $\theta^m = \overline{\theta}$ and $q^{\mathrm{BM}}(\overline{\theta}) > \mathrm{q}_\ell$, then $\theta^\star = \theta^m$. If, instead, $\theta^m < \overline{\theta}$, then $\theta^m < \theta^\star$ and $q^\star(\theta^m) \geq \underline{P}^{-1}(\theta^m)$. Moreover, there exists $\delta > 0$ such that $q^\star(\theta) > \underline{P}^{-1}(\theta)$ for all $\theta \in (\theta^m, \theta^m + \delta]$.*

Proof: That $\theta^\star = \theta^m$ when $\theta^m = \overline{\theta}$ and $q^{\mathrm{BM}}(\overline{\theta}) > \mathrm{q}_\ell$ follows directly from the definition of $\theta^\star$. Thus suppose that $\theta^m < \overline{\theta}$. Then, by Lemma 7, $M^\star \equiv (q^\star, u^\star)$ is not robustly optimal, meaning that, for some θ, $\underline{W}(\theta, q^\star) < G^*$ and hence $\underline{W}(\theta^m, q^\star) < G^*$. If $\theta^\star < \overline{\theta}$, then for every $\theta \geq \theta^\star$, $q^\star(\theta) = \mathrm{q}_\ell$ and $\underline{W}(\theta, q^\star) = G^\star + \alpha(\overline{\theta} - \theta) \geq G^*$. Thus, $\theta^m < \theta^\star$. Now suppose $\theta^\star = \overline{\theta}$, then $\theta^m < \overline{\theta}$ implies the desired inequality $\theta^m < \theta^\star$. Hence, $\theta^m < \theta^\star$, implying that $q^\star(\theta^m) = q^{\mathrm{BM}}(\theta^m)$.

Now suppose that $q^\star(\theta^m) < \underline{P}^{-1}(\theta^m)$. Because $\underline{P}^{-1}$ is continuous and $q^\star$ is non-increasing and continuous, and $\theta^m < \theta^\star$, there exists $\delta > 0$ such that, for all $\theta \in [\theta^m, \theta^m + \delta]$, $0 < \mathrm{q}_\ell <$

$q^\star(\theta) < \underline{P}^{-1}(\theta)$. Part A of Lemma 3 then implies that the function $\underline{W}(\cdot, q^\star)$ is non-increasing over $[\theta^m, \theta^m + \delta]$, contradicting the definition of θ^m. Hence, $q^\star(\theta^m) \geq \underline{P}^{-1}(\theta^m)$. Because $q^\star$ is continuous at θ^m, the arguments above then imply that there exists $\delta > 0$ such that $q^\star(\theta) > \underline{P}^{-1}(\theta)$ for all $\theta \in (\theta^m, \theta^m + \delta]$. ■

Lemma 9 *Suppose $\theta^m < \overline{\theta}$ or $\theta^m = \overline{\theta}$ and $q^{\text{BM}}(\overline{\theta}) > \mathrm{q}_\ell$. Then every robustly optimal mechanism $M^{\text{OPT}} = (q^{\text{OPT}}, u^{\text{OPT}})$ is such that $q^{\text{OPT}}(\theta) = \mathrm{q}_\ell$ for all $\theta \in [\theta^\star, \overline{\theta}]$.*

Proof: Clearly, because q^{OPT} must satisfy the robustness constraints (6) it must be that $q^{\text{OPT}}(\overline{\theta}) = \mathrm{q}_\ell$. The result thus holds when $\theta^\star = \overline{\theta}$, which is the case when $\theta^m = \overline{\theta}$ and $q^{\text{BM}}(\overline{\theta}) > \mathrm{q}_\ell$.

Now suppose that $\theta^\star < \overline{\theta}$ and there exists $\theta' \in (\theta^\star, \overline{\theta})$ such that $q^{\text{OPT}}(\theta') > \mathrm{q}_\ell$. The monotonicity of q^{OPT} then implies that $q^{\text{OPT}}(\theta) > \mathrm{q}_\ell$ for all $\theta \in [\theta^\star, \theta']$. This means that there exists a non-zero Lebesgue measure of types such that $q^{\text{OPT}}(\theta) > \mathrm{q}_\ell$. Then consider the mechanism $\widetilde{M} = (\tilde{q}, \tilde{u})$ where the quantity schedule is given by

$$\tilde{q}(\theta) = \begin{cases} q^{\text{OPT}}(\theta) & \text{if } \ \theta < \theta^\star \\ \mathrm{q}_\ell & \text{if } \ \theta \geq \theta^\star, \end{cases}$$

and where the rents $\tilde{u}$ are given by the envelope formula together with $\tilde{u}(\overline{\theta}) = 0$. Because $\tilde{q}$ is non-increasing, this ensures that $\widetilde{M}$ is IC and IR. By definition, $\tilde{q}(\theta) \leq q^{\text{OPT}}(\theta)$ for all θ. Clearly, for all $\theta < \theta^\star$,

$$\underline{W}(\theta, \tilde{q}) := \underline{V}(\tilde{q}(\theta)) - \theta\tilde{q}(\theta) - (1-\alpha)\int_\theta^{\bar{\theta}} \tilde{q}(y)dy > \underline{V}(q^{\text{OPT}}(\theta)) - \theta q^{\text{OPT}}(\theta) - (1-\alpha)\int_\theta^{\bar{\theta}} q^{\text{OPT}}(y)dy \geq G^*.$$

The first inequality follows from the fact that (a) for any such θ, $\tilde{q}(\theta) = q^{\text{OPT}}(\theta)$, along with the fact that (b) for any $y > \theta$, $\tilde{q}(y) \leq q^{\text{OPT}}(y)$, with the inequality strict over a Lebesgue positive measure set of types. The second inequality follows from the fact that $M^{\text{OPT}} = (q^{\text{OPT}}, u^{\text{OPT}}) \in \mathcal{M}^{\text{SL}}$ which means that

$$\underline{V}(q^{\text{OPT}}(\theta)) - \theta q^{\text{OPT}}(\theta) - (1-\alpha)\int_\theta^{\bar{\theta}} q^{\text{OPT}}(y)dy \geq G^*.$$

Next, observe that, for any $\theta \geq \theta^\star$,

$$\underline{W}(\theta, \tilde{q}) := \underline{V}(\mathrm{q}_\ell) - \overline{\theta}\mathrm{q}_\ell + \alpha \mathrm{q}_\ell(\overline{\theta} - \theta) = G^* + \alpha \mathrm{q}_\ell(\overline{\theta} - \theta) \geq G^*.$$

So the schedule $\tilde{q}$ satisfies the robustness constraints (6). Hence, the mechanism $\widetilde{M} = (\tilde{q}, \tilde{u}) \in \mathcal{M}^{\mathrm{SL}}$. The government's payoff

$$\int_{\underline{\theta}}^{\overline{\theta}} \Big[V^\star(q(\theta)) - z^\star(\theta) q(\theta) \Big] F^\star(d\theta)$$

under $\widetilde{M}$ is strictly higher than under M^{OPT}. This follows from the fact that, for any $\theta \geq \theta^\star$, q_ℓ maximizes $V^\star(\mathrm{q}) - z^\star(\theta)\mathrm{q}$ over $\mathrm{q} \geq \mathrm{q}_\ell$, along with the fact that $F^\star$ is absolutely continuous. This contradicts the optimality of M^{OPT}. ■

Lemma 10 *Suppose $\theta^m < \overline{\theta}$ or $\theta^m = \overline{\theta}$ and $q^{\mathrm{BM}}(\overline{\theta}) > \mathrm{q}_\ell$. Then every robustly optimal mechanism $M^{\mathrm{OPT}} = (q^{\mathrm{OPT}}, u^{\mathrm{OPT}})$ is such that $q^{\mathrm{OPT}}(\theta) \leq q^{\mathrm{BM}}(\theta)$ for almost all $\theta \in [\underline{\theta}, \theta^\star)$, with the inequality strict over a Lebesgue positive-measure set of types $I \subseteq [\underline{\theta}, \theta^\star)$.*

Proof: From Lemma 9, $q^{\mathrm{OPT}}(\theta) = \mathrm{q}_\ell$ for all $\theta \in [\theta^\star, \overline{\theta}]$. Now suppose there is a positive-Lebesgue-measure set of types $I \subseteq [\underline{\theta}, \theta^\star)$ such that $q^{\mathrm{OPT}}(\theta) > q^\star(\theta) = q^{\mathrm{BM}}(\theta)$. Consider the mechanism $\widetilde{M} = (\tilde{q}, \tilde{u})$ where the quantity schedule is given by

$$\tilde{q}(\theta) = \min\{q^\star(\theta), q^{\mathrm{OPT}}(\theta)\}$$

and where the rents $\tilde{u}$ are given by the envelope formula together with $\tilde{u}(\overline{\theta}) = 0$. Clearly, because $\tilde{q}$ is non-increasing and $\tilde{u}$ satisfies the above properties, the mechanism $\widetilde{M}$ is IC and IR. The next two claims establish that $\widetilde{M} = (\tilde{q}, \tilde{u})$ satisfies the robustness constraints (6).

Claim 1 *Suppose θ is such that either $\tilde{q}(\theta) = q^{\mathrm{OPT}}(\theta)$ or $\underline{P}^{-1}(\theta) \leq \tilde{q}(\theta) = q^\star(\theta) < q^{\mathrm{OPT}}(\theta)$. Then $\underline{W}(\theta, \tilde{q}) \geq G^*$.*

Proof: That $\underline{W}(\theta, \tilde{q}) \geq G^*$ for any θ such that $\tilde{q}(\theta) = q^{\mathrm{OPT}}(\theta)$ follows from the fact that $\tilde{q}(y) \leq q^{\mathrm{OPT}}(y)$ for all $y \geq \theta$, and hence, $\underline{W}(\theta, \tilde{q}) \geq \underline{W}(\theta, q^{\mathrm{OPT}}) \geq G^*$. Thus, consider a θ for which $\underline{P}^{-1}(\theta) \le \tilde{q}(\theta) = q^\star(\theta) < q^{\text{OPT}}(\theta)$. The quasi-concavity of the function $\underline{V}(\text{q}) - \theta\text{q}$ in q implies that

$$\underline{V}(q^\star(\theta)) - \theta q^\star(\theta) > \underline{V}(q^{\text{OPT}}(\theta)) - \theta q^{\text{OPT}}(\theta).$$

Together with the fact that $\tilde{q}(y) \le q^{\text{OPT}}(y)$ for all $y \ge \theta$, this means that $\underline{W}(\theta, \tilde{q}) \ge \underline{W}(\theta, q^{\text{OPT}}) \ge G^*$. ■

Claim 2 *Suppose θ is such that $q^\star(\theta) < \min\{\underline{P}^{-1}(\theta), q^{\text{OPT}}(\theta)\}$. Then, $\underline{W}(\theta, \tilde{q}) \ge G^*$.*

Proof: The proof considers two cases to establish the existence of $\theta' < \theta$ such that $W(\cdot, \tilde{q})$ is non-increasing on $[\theta, \theta']$ with $W(\theta', \tilde{q}) \ge G^*$.

Case 1. Suppose $q^\star(\overline{\theta}) = \text{q}_\ell = \underline{P}^{-1}(\overline{\theta})$. Because $q^\star$ and $\underline{P}^{-1}$ are both continuous, there exists $\theta < \theta' \le \overline{\theta}$ such that $q^\star(y) \le \underline{P}^{-1}(y)$ for all $y \in [\theta, \theta']$, with $q^\star(\theta') = \underline{P}^{-1}(\theta')$. Thus,

$$\tilde{q}(\theta') = \min\{\underline{P}^{-1}(\theta'), q^{\text{OPT}}(\theta')\}.$$

Further, for all $y \in [\theta, \theta']$,

$$\tilde{q}(y) = \min\{q^{\text{OPT}}(y), q^\star(y)\} \le \underline{P}^{-1}(y).$$

Part A of Lemma 3 implies that $\underline{W}(\cdot, \tilde{q})$ is non-increasing over $[\theta, \theta']$ whereas Claim 1 implies that $\underline{W}(\theta', \tilde{q}) \ge G^*$. Hence $\underline{W}(\theta, \tilde{q}) \ge G^*$.

Case 2. Now suppose $q^\star(\overline{\theta}) = q^{\text{BM}}(\overline{\theta}) > \text{q}_\ell = \underline{P}^{-1}(\overline{\theta})$. Then, because $q^\star(\theta) < \underline{P}^{-1}(\theta)$, and $\underline{P}^{-1}$ and $q^\star$ are continuous (latter due to regularity of $F^\star$), there exists $\theta < \hat{\theta} < \overline{\theta}$ such that $q^\star(\hat{\theta}) = \underline{P}^{-1}(\hat{\theta})$ and $q^\star(y) > \underline{P}^{-1}(y)$ for all $y > \hat{\theta}$. Again, just like we argued in Case 1, there exists $\theta < \theta' \le \hat{\theta}$ such that $q^\star(y) \le \underline{P}^{-1}(y)$ for all $y \in [\theta, \theta']$ with $q^\star(\theta') = \underline{P}^{-1}(\theta')$. Repeating the remaining arguments in Case 1 completes the proof. ■

The above two claims establish that $\tilde{q}$ satisfies the robustness constraints (6). Hence $\widetilde{M} = (\tilde{q}, \tilde{u}) \in \mathcal{M}^{\text{SL}}$. That the government's payoff under $\widetilde{M}$ is strictly higher than under M^{OPT} follows from the fact that, for all θ such that $\tilde{q}(\theta) = q^\star(\theta) < q^{\text{OPT}}(\theta)$,

$$V^\star(q^\star(\theta)) - z^\star(\theta) q^\star(\theta) > V^\star(q^{\text{OPT}}(\theta)) - z^\star(\theta) q^{\text{OPT}}(\theta),$$

where the inequality follows from the fact that $q^\star(\theta)$ maximizes $V^\star(\mathrm{q}) - z^\star(\theta)\mathrm{q}$ over $[\mathrm{q}_\ell, +\infty)$. Because $F^\star$ is absolutely continuous, the set $I \subseteq [\underline{\theta}, \theta^\star)$ over which $q^{\mathrm{OPT}}(\theta) > q^{\mathrm{BM}}(\theta)$ has positive $F^\star$-measure, contradicting the optimality of M^{OPT}. Hence, it must be that $q^{\mathrm{OPT}}(\theta) \leq q^{\mathrm{BM}}(\theta)$ for almost all $\theta \in [\underline{\theta}, \theta^\star)$.

We complete the proof by showing that there must exist a set of types $I \subseteq [\underline{\theta}, \theta^\star)$ of positive Lebesgue measure such that $q^{\mathrm{OPT}}(\theta) < q^{\mathrm{BM}}(\theta)$ for all $\theta \in I$. To do that assume for contradiction $q^{\mathrm{OPT}}(\theta) = q^{\mathrm{BM}}(\theta)$ almost everywhere on $[\underline{\theta}, \theta^\star)$. Moreover, because $q^\star$ is continuous and $q^{\mathrm{BM}}(\theta)$ is the unique maximizer of $V^\star(\mathrm{q}) - z^\star(\theta)\mathrm{q}$, it is without loss to assume that $q^{\mathrm{OPT}}(\theta) = q^{\mathrm{BM}}(\theta)$ for all $\theta < \theta^\star$. Consider the following two cases:

Case 1. Suppose $q^{\mathrm{BM}}(\theta^\star) = \mathrm{q}_\ell$. Then $q^{\mathrm{OPT}}(\theta) = q^\star(\theta)$ for all θ, which contradicts Part 1 of Proposition 3.

Case 2. Suppose $q^{\mathrm{BM}}(\theta^\star) > \mathrm{q}_\ell$. Then, by definition of $\theta^\star$, it must be that $\theta^\star = \overline{\theta}$. Hence, $q^{\mathrm{OPT}}(\overline{\theta}) = \mathrm{q}_\ell < q^{\mathrm{BM}}(\overline{\theta})$. This means $G^* = \underline{W}(\overline{\theta}, q^{\mathrm{OPT}}) > \underline{W}(\overline{\theta}, q^{\mathrm{BM}})$. But for all $\theta < \theta^\star = \overline{\theta}$, we have $q^{\mathrm{OPT}}(\theta) = q^{\mathrm{BM}}(\theta)$. Moreover, $\underline{W}(\theta, q^{\mathrm{OPT}}) = \underline{W}(\theta, q^{\mathrm{BM}})$ is continuous in θ on $\theta \leq \theta^\star$ as $q^{\mathrm{BM}}(\theta)$ is continuous on Θ. Hence, $\underline{W}(\theta, q^{\mathrm{OPT}})$ violates the constraint (6) in the left-neighborhood of $\overline{\theta}$, contradicting the robust optimality of q^{OPT}. ■

This completes the proof of Proposition 3. Q.E.D.

Proof of Proposition 4. *Part 1.* We want to show that $\theta^m = \overline{\theta}$ implies that $q^{\mathrm{BM}}(\overline{\theta}) \leq \mathrm{q}_\ell$. To see why, assume for contradiction, that $q^{\mathrm{BM}}(\overline{\theta}) > \mathrm{q}_\ell$. Then there exists an interval I including $\overline{\theta}$ such that q^{BM} is decreasing over I with $q^{\mathrm{BM}}(\theta) > \underline{P}^{-1}(\theta)$ for all $\theta \in I$. Thus, by Part B of Lemma 3, $\underline{W}(q^\star, \theta)$ is increasing over I, a contradiction to $\theta^m = \overline{\theta}$.

Part 2. Suppose there exists $\hat{\theta} \in \Theta$ such that $q^\star(\theta) > \underline{P}^{-1}(\theta)$ if $\theta < \hat{\theta}$ and $q^\star(\theta) \leq \underline{P}^{-1}(\theta)$ if $\theta \geq \hat{\theta}$. Lemma 3 then implies that $\underline{W}(\cdot, q^\star)$ is non-decreasing over $[\underline{\theta}, \hat{\theta}]$ and non-increasing over $[\hat{\theta}, \overline{\theta}]$. This property, along with the fact that $\underline{W}(\underline{\theta}, q^\star) \geq G^*$ and $\underline{W}(\overline{\theta}, q^\star) = G^*$, then implies that $\theta^m = \overline{\theta}$.

Part 3. The result follows from the following two lemmas.

Lemma 11 *Suppose that $\alpha = 0$, $\theta^m \in (\underline{\theta}, \theta^\star)$, and $F^\star$ is regular. Then, $q^\star(\theta^m) = \underline{P}^{-1}(\theta^m)$.*

Proof: From Lemma 8, we know that $q^\star(\theta^m) \geq \underline{P}^{-1}(\theta^m)$. Now suppose that $q^\star(\theta^m) > \underline{P}^{-1}(\theta^m)$. Because $\underline{P}^{-1}$ is continuous and $q^\star$ is non-increasing, there exists $\delta > 0$ such that, for any $\theta \in [\theta^m - \delta, \theta^m)$, $q^\star(\theta) > \underline{P}^{-1}(\theta)$. Since $q^\star(\theta) > \underline{P}^{-1}(\theta)$, we must have $q^\star(\theta) = q^{\text{BM}}(\theta)$ for all $\theta \in [\theta^m - \delta, \theta^m)$. Furthermore, because $z^\star$ is increasing, q^{BM} is decreasing over $(\theta^m - \delta, \theta^m]$. Part B of Lemma 3 (using $\alpha = 0$) then implies that $\underline{W}(\cdot, q^\star)$ is increasing over $(\theta^m - \delta, \theta^m]$ which contradicts the definition of θ^m. Hence, $q^\star(\theta^m) = \underline{P}^{-1}(\theta^m)$. ■

Lemma 12 *Suppose that $\alpha = 0$, $\theta^m \in (\underline{\theta}, \theta^\star)$, and $F^\star$ is regular. Then, for every robustly optimal mechanism $M^{\text{OPT}} = (q^{\text{OPT}}, u^{\text{OPT}})$, $q^{\text{OPT}}(\theta) = q^{\text{BM}}(\theta)$ for almost all $\theta \in [\underline{\theta}, \theta^m)$.*

Proof: Assume for contradiction that there exists a Lebesgue positive-measure set of types $I \subseteq [\underline{\theta}, \theta^m)$ such that $q^{\text{OPT}}(\theta) \neq q^{\text{BM}}(\theta)$ for all $\theta \in I$. By Lemma 10, we get $q^{\text{OPT}}(\theta) \leq q^{\text{BM}}(\theta)$ for all $\theta \in I$ (as q^{BM} is continuous and both q^{BM} and q^{OPT} are non-increasing). Then, let $\widetilde{M} = (\tilde{q}, \tilde{u})$ be the mechanism where the quantity schedule is given by

$$\tilde{q}(\theta) = \begin{cases} q^{\text{BM}}(\theta) & \text{if } \theta \in [\underline{\theta}, \theta^m] \\ q^{\text{OPT}}(\theta) & \text{otherwise} \end{cases}$$

and where the rents $\tilde{u}$ are given by the envelope formula together with $\tilde{u}(\overline{\theta}) = 0$. Below, we show that $\widetilde{M}$ yields a higher payoff to the government than M^{OPT} and $\widetilde{M} \in \mathcal{M}^{\text{SL}}$, contradicting the optimality of M^{OPT}. Because, for any θ, $q^{\text{BM}}(\theta)$ is the unique maximizer of $V^\star(\text{q}) - z^\star(\theta)\text{q}$, the objective function

$$\int_{\underline{\theta}}^{\overline{\theta}} \Big[V^\star(q(\theta)) - z^\star(\theta)q(\theta)\Big] F^\star(d\theta)$$

is strictly higher under $\widetilde{M}$ than under M^{OPT}. Next observe that, because $\tilde{q}$ is non-increasing and $\tilde{u}$ is defined by the envelope formula, $\widetilde{M}$ is IC and IR. We now show that $\tilde{q}$ satisfies the robustness constraints in (6). Clearly, this is true for any $\theta > \theta^m$. Thus consider $\theta \in [\underline{\theta}, \theta^m]$. Because for any $\theta \leq \theta^m$, $\tilde{q}(\theta) = q^{\text{BM}}(\theta) = q^\star(\theta)$,

$$\begin{aligned} \underline{W}(\theta, \tilde{q}) - \underline{W}(\theta, q^\star) = & \int_{\theta^m}^{\overline{\theta}} q^\star(y)dy - \int_{\theta^m}^{\overline{\theta}} q^{\text{OPT}}(y)dy \\ & \geq_{(a)} \Big[\underline{V}(q^{\text{OPT}}(\theta^m)) - \theta^m q^{\text{OPT}}(\theta^m)\Big] - \Big[\underline{V}(q^{\text{BM}}(\theta^m)) - \theta^m q^{\text{BM}}(\theta^m)\Big] \end{aligned}$$

$$+\int_{\theta^m}^{\overline{\theta}} q^\star(y)dy - \int_{\theta^m}^{\overline{\theta}} q^{\mathrm{OPT}}(y)dy$$
$$=_{(b)} \underline{W}(\theta^m, q^{\mathrm{OPT}}) - \underline{W}(\theta^m, q^\star)$$
$$\geq_{(c)} G^* - \underline{W}(\theta^m, q^\star)$$
$$\geq_{(d)} G^* - \underline{W}(\theta, q^\star),$$

Inequality (a) follows from the fact that $\underline{P}^{-1}(\theta^m)$ maximizes $\underline{V}(\mathrm{q}) - \theta^m\mathrm{q}$ over all q and $q^{\mathrm{BM}}(\theta^m) = \underline{P}^{-1}(\theta^m)$ (by Lemma 11). Equality (b) follows from the fact that $q^{\mathrm{BM}}(\theta) = q^\star(\theta)$. Inequality (c) follows from the fact that $M^{\mathrm{OPT}} \in \mathcal{M}^{\mathrm{SL}}$ which implies that $q^{\mathrm{OPT}}(y)$ satisfies the robustness constraints in (6). Inequality (d) follows from the definition of θ^m. Hence, $\underline{W}(\theta, \tilde{q}) \geq G^*$ also for all $\theta \in [\underline{\theta}, \theta^m]$. We conclude that $\widetilde{M} \in \mathcal{M}^{\mathrm{SL}}$ and yields a higher payoff to the government than M^{OPT} contradicting the optimality of M^{OPT}. ■

Part 4. The proof is in two steps. Both these steps use the fact that q^{BM} is decreasing, which follows from the regularity of $F^\star$ and $P^\star$ being decreasing.

Step 1. $\theta^m \neq \overline{\theta}$. Using the derivation the Baron-Myerson optimal quantity schedule for a regular $F^\star$, we have that

$$P^\star(q^{\mathrm{BM}}(\overline{\theta})) - \overline{\theta} = \frac{1}{f^\star(\overline{\theta})} < P^\star(\mathrm{q}_\ell) - \underline{P}(\mathrm{q}_\ell),$$

where the inequality follows from the assumption that $P^\star(\mathrm{q}) - \underline{P}(\mathrm{q}) > 1/f^\star(\overline{\theta})$ for all q. Using the fact that q_ℓ is such that $\overline{\theta} = \underline{P}(\mathrm{q}_\ell)$, we have that $q^{\mathrm{BM}}(\overline{\theta}) > \mathrm{q}_\ell = \underline{P}^{-1}(\overline{\theta})$. This implies that $q^\star(\theta) = q^{\mathrm{BM}}(\theta)$ for all θ. Because q^{BM} is decreasing and $\underline{P}$ is continuous, there exists $\hat{\theta} < \overline{\theta}$ such that $q^\star(\theta) = q^{\mathrm{BM}}(\theta) > \underline{P}^{-1}(\theta)$ for all $\theta \in [\hat{\theta}, \overline{\theta}]$. Part B of Lemma 3 then implies that the function $\underline{W}(\theta, q^{\mathrm{BM}})$ is increasing in θ over $[\hat{\theta}, \overline{\theta}]$. Hence,

$$\underline{W}(\hat{\theta}, q^\star) = \underline{W}(\hat{\theta}, q^{\mathrm{BM}}) < \underline{W}(\overline{\theta}, q^{\mathrm{BM}}) \leq \underline{W}(\overline{\theta}, q^\star).$$

The last inequality implies that $\theta^m < \overline{\theta}$.

Step 2. Because $\theta^m < \overline{\theta}$, Lemma 8 implies that $\theta^m < \theta^\star$. Now assume that $\theta^m \in (\underline{\theta}, \theta^\star)$. From Lemma 11, we then have that $\underline{P}(q^\star(\theta^m)) = \theta^m$. Therefore,

$$\underline{P}(q^\star(\theta^m)) = \theta^m = P^\star(q^\star(\theta^m)) - \frac{F^\star(\theta^m)}{f^\star(\theta^m)}.$$

This means that

$$P^\star(q^\star(\theta^m)) - \underline{P}(q^\star(\theta^m)) = \frac{F^\star(\theta^m)}{f^\star(\theta^m)} \leq \frac{1}{f^\star(\overline{\theta})},$$

where the inequality follows from the non-decreasingness of $F^\star(\theta)/f^\star(\theta)$. This is a contradiction to the assumption that $P^\star(\mathrm{q}) - \underline{P}(\mathrm{q}) > \frac{1}{f^\star(\overline{\theta})}$ for all q. Hence, $\theta^m = \underline{\theta}$. Q.E.D.

Proof of Proposition 5. We prove the result by showing that if $M = (q, u) \in \mathcal{M}^{\mathrm{SL}}$, then $q(\theta) = \underline{q}^{\mathrm{BM}}(\theta)$ for almost all $\theta \in \Theta$. The proposition then follows from this property together with the monotonicity of q and the continuity of $\underline{q}^{\mathrm{BM}}$.

First, we show that $G(\underline{M}^{\mathrm{BM}}) = W(\underline{M}^{\mathrm{BM}}; \underline{V}, \underline{F})$. To do so, observe that, when $\underline{F}$ is regular, $\underline{q}^{\mathrm{BM}}$ is such that, for all $\theta \in \Theta$,

$$\underline{q}^{\mathrm{BM}}(\theta) = \underline{P}^{-1}(\underline{z}(\theta)),$$

where, for all $\theta \in \Theta$, $\underline{z}(\theta) := \theta + (1-\alpha)\underline{F}(\theta)/\underline{f}(\theta)$. Thus, $\underline{q}^{\mathrm{BM}}(\theta) \leq \underline{P}^{-1}(\theta)$ for all θ, with the inequality strict for $\theta > \underline{\theta}$.[8] Part 1 of Lemma 3 then implies that $\underline{W}(\theta, \underline{M}^{\mathrm{BM}})$ is non-increasing in θ. Furthermore, because, for all $F \in \mathcal{F}$, $\underline{F} \succ_{FOSD} F$,

$$W(\underline{M}^{\mathrm{BM}}; \underline{V}, \underline{F}) \leq W(\underline{M}^{\mathrm{BM}}; \underline{V}, F).$$

Because, for any $V \in \mathcal{V}$ and any $F \in \mathcal{F}$, $W(\underline{M}^{\mathrm{BM}}; \underline{V}, F) \leq W(\underline{M}^{\mathrm{BM}}; V, F)$, we thus have that $W(\underline{M}^{\mathrm{BM}}; \underline{V}, \underline{F}) \leq W(\underline{M}^{\mathrm{BM}}; V, F)$. Thus, $G(\underline{M}^{\mathrm{BM}}) = W(\underline{M}^{\mathrm{BM}}; \underline{V}, \underline{F})$.

To establish that $M = (q, u) \in \mathcal{M}^{\mathrm{SL}}$ only if $q(\theta) = \underline{q}^{\mathrm{BM}}(\theta)$ for almost all $\theta \in \Theta$, it then suffices to show that $G(M) < G(\underline{M}^{\mathrm{BM}})$ for any IC and IR mechanism $M = (q, u) \in \mathcal{M}$ such that $q(\theta) \neq \underline{q}^{\mathrm{BM}}(\theta)$ over a subset of Θ of positive Lebesgue measure. Observe that, for any such mechanism,

$$G(M) \leq W(M; \underline{V}, \underline{F}) < W(\underline{M}^{\mathrm{BM}}; \underline{V}, \underline{F}),$$

where the second inequality follows from the fact that $\underline{q}^{\mathrm{BM}}$ is the unique maximizer (in the almost everywhere sense) of the function $W(M; \underline{V}, \underline{F})$.[9] Q.E.D.

Proof of Proposition 6. The proof is in two steps. Step 1 establishes that $M_s^\star \in \mathcal{M}^{\mathrm{SL}}$ whereas Step 2 establishes that, for any $M \in \mathcal{M}^{\mathrm{SL}}$, $W(M; V^\star, F^\star) \leq W(M_s^\star; V^\star, F^\star)$, meaning that $M_s^\star$ maximizes the government's payoff (under $(V^\star, F^\star)$) over the short list $\mathcal{M}^{\mathrm{SL}}$.

[8]This property holds even if $\underline{F}$ is not regular. In fact, any undominated mechanism $M = (q, u)$ is such that $q(\theta) \leq \underline{P}^{-1}(\theta)$ for all θ (Mishra and Patil, 2024).

[9]That is, for any $M = (q, u) \in \mathcal{M}$ such that $q(\theta) \neq \underline{q}^{\mathrm{BM}}(\theta)$ over a subset of Θ of positive Lebesgue measure, $W(M; \underline{V}, \underline{F}) < W(\underline{M}^{\mathrm{BM}}; \underline{V}, \underline{F})$.

Step 1. We fist establish that $M_s^\star \in \mathcal{M}^{\mathrm{SL}}$. Let

$$G_s^* := \int_{\theta_s}^{\bar{\theta}} \left(\underline{V}(\underline{q}^{\mathrm{BM}}(\theta)) - \underline{z}(\theta)\underline{q}^{\mathrm{BM}}(\theta) \right) \underline{F}(\mathrm{d}\theta).$$

We first establish that $G(M_s^\star) = G_s^*$. To see this, observe that, for any $(V, F) \in \mathcal{V} \times \mathcal{F}$,

$$W(M_s^\star; \underline{V}, \underline{F}) \leq_{(a)} W(M_s^\star; \underline{V}, F) \leq_{(b)} W(M_s^\star; V, F).$$

Inequality (a) follows from the fact that $\underline{F} \succ_{FOSD} F$ along with the fact that $\underline{W}(\cdot, M_s^\star)$ is non-increasing over Θ. The latter property in turn follows from Part 1 of Lemma 3 along with the fact that $q_s^\star(\theta) \leq \underline{P}^{-1}(\theta)$ for all $\theta \geq \theta_s$. Inequality (b) follows from the fact that $\underline{V}(\mathrm{q}) \leq V(\mathrm{q})$ for any q and $V \in \mathcal{V}$. Because

$$\begin{aligned} W(M_s^\star; \underline{V}, \underline{F}) &:= \int_{\theta_s}^{\overline{\theta}} \left[\underline{V}(\underline{q}^{\mathrm{BM}}(\theta)) - \theta \underline{q}^{\mathrm{BM}}(\theta) - (1-\alpha)\underline{u}^{\mathrm{BM}}(\theta) \right] \underline{F}(\mathrm{d}\theta) \\ &= \int_{\theta_s}^{\overline{\theta}} \left[\underline{V}(\underline{q}^{\mathrm{BM}}(\theta)) - \underline{z}(\theta)\underline{q}^{\mathrm{BM}}(\theta) \right] \underline{F}(\mathrm{d}\theta) = G_s^* \end{aligned}$$

we conclude that $G(M_s^\star) = G_s^*$. That $M_s^\star \in \mathcal{M}^{\mathrm{SL}}$ then follows from the fact that, for any $M = (q, u) \in \mathcal{M}$, $G(M) \leq G_s^*$, which, in turn, follows from the fact that

$$G(M) = \inf_{(V,F) \in \mathcal{V} \times \mathcal{F}} W(M; V, F) \leq W(M; \underline{V}, \underline{F}) \leq W(M_s^\star; \underline{V}, \underline{F}) = G_s^*,$$

where the second inequality follows from the fact that $M_s^\star$ maximizes $W(\cdot; \underline{V}, \underline{F})$ over $\mathcal{M}$.

Step 2. Arguments similar to those establishing Proposition 5 imply that if $M = (q, u) \in \mathcal{M}^{\mathrm{SL}}$, then $q(\theta) = \underline{q}^{\mathrm{BM}}(\theta)$ for almost all $\theta \in [\theta_s, \overline{\theta}]$. That, under the conjecture $(V^\star, F^\star)$, $M_s^\star$ maximizes $W(\cdot; V^\star, F^\star)$ over $\mathcal{M}^{\mathrm{SL}}$ then follows from this observation along with the fact that, for $\theta \in [\underline{\theta}, \theta_s)$,

$$q_s^\star(\theta) = \arg\max_{\mathrm{q} \geq \mathrm{q}_\ell^s} \left\{ V^\star(\mathrm{q}) - z^\star(\theta)\mathrm{q} \right\}.$$

Q.E.D.

Proof of Proposition 7. The proof is in two parts, each establishing the corresponding part in the proposition.

Part 1. For any $\theta \in (\underline{\theta}, \overline{\theta})$, let $n(\theta)$ be the largest $n > \overline{n}$ such that $\underline{\theta}_n \leq \theta < \underline{\theta}_{n+1}$. Existence of $n(\theta)$ is guaranteed because the sequence $(\underline{F}_n)$ is such that $\underline{\theta}_n \leq \underline{\theta}_{n+1} < \overline{\theta}$ and

$\lim_{n\to\infty} \underline{\theta}_n = \overline{\theta}$. For any $n \le n(\theta) - 1$, $\theta \in [\underline{\theta}_n, \overline{\theta}]$ and $\theta \in [\underline{\theta}_{n+1}, \overline{\theta}]$. Thus, by Proposition 6, $q_n^{\mathrm{OPT}}(\theta) = \underline{q}_n^{\mathrm{BM}}(\theta)$, and $q_{n+1}^{\mathrm{OPT}}(\theta) = \underline{q}_{n+1}^{\mathrm{BM}}(\theta)$. Condition (9) in turn implies that $\underline{q}_n^{\mathrm{BM}}(\theta) \le \underline{q}_{n+1}^{\mathrm{BM}}(\theta)$, that is, $q_n^{\mathrm{OPT}}(\theta)$ is non-decreasing in n for $n \le n(\theta) - 1$.

For any $n > n(\theta)$, $\theta < \underline{\theta}_n$, and therefore, $q_n^{\mathrm{OPT}}(\theta) = \max\{q^{\mathrm{BM}}(\theta), \underline{P}^{-1}(\underline{\theta}_n)\}$. The quantity $\underline{P}^{-1}(\underline{\theta}_n)$ is non-increasing in n because $\underline{\theta}_n \le \underline{\theta}_{n+1} < \overline{\theta}$ for every n. Consequently, $q_n^{\mathrm{OPT}}(\theta)$ is also non-increasing.

Part 2. To establish the second part of the proposition it suffices to exhibit a pair $j, k \in \mathbb{N}$, with $j < k$, such that $q_j^{\mathrm{OPT}}(\theta) > q_k^{\mathrm{OPT}}(\theta)$. To do so, consider the following two cases.

Case 1. Suppose $q^{\mathrm{BM}}(\theta) \ge \underline{P}^{-1}(\underline{\theta}_{n(\theta)+1})$. Then let $j = n(\theta)$ and $k = n(\theta) + 1$, and observe that

$$q_j^{\mathrm{OPT}}(\theta) = \underline{q}_j^{\mathrm{BM}}(\theta) = \underline{P}^{-1}\left(\theta + (1-\alpha)\frac{\underline{F}_j(\theta)}{\underline{f}_j(\theta)}\right) > \underline{P}^{-1}\left(\theta + (1-\alpha)\frac{F^\star(\theta)}{f^\star(\theta)}\right) = q^{\mathrm{BM}}(\theta) = q_k^{\mathrm{OPT}}(\theta),$$

where the inequality follows from (10).

Case 2. Suppose $q^{\mathrm{BM}}(\theta) < \underline{P}^{-1}(\underline{\theta}_{n(\theta)+1})$. Then let $j = n(\theta)+1$ and let k be such that $\underline{\theta}_k > \underline{\theta}_j$. Existence of such an k is ensured by the fact that $\underline{\theta}_n < \overline{\theta}$ for all n and $\lim_{n\to\infty} \underline{\theta}_n = \overline{\theta}$. Then

$$q_k^{\mathrm{OPT}}(\theta) = \max\{q^{\mathrm{BM}}(\theta), \underline{P}^{-1}(\underline{\theta}_k)\} < \underline{P}^{-1}(\underline{\theta}_j) = q_j^{\mathrm{OPT}}(\theta).$$

To see this, observe that $\underline{\theta}_k > \underline{\theta}_j$ implies that $\underline{P}^{-1}(\underline{\theta}_k) < \underline{P}^{-1}(\underline{\theta}_j)$. Hence, if $q_k^{\mathrm{OPT}}(\theta) = \underline{P}^{-1}(\underline{\theta}_k)$, then $q_k^{\mathrm{OPT}}(\theta) = \underline{P}^{-1}(\underline{\theta}_k) < \underline{P}^{-1}(\underline{\theta}_j) = q_j^{\mathrm{OPT}}(\theta)$. If, instead, $q_k^{\mathrm{OPT}}(\theta) = q^{\mathrm{BM}}(\theta)$, the result follows from the fact that, by assumption, $q^{\mathrm{BM}}(\theta) < \underline{P}^{-1}(\underline{\theta}_j)$. Q.E.D.

Proof of Proposition 8. *Part 1.* By definition of q_ℓ, we have that

$$\underline{P}_N(\mathrm{q}_\ell) = \underline{P}_N(\underline{P}^{-1}(\overline{\theta})) \ge \underline{P}(\underline{P}^{-1}(\overline{\theta})) = \overline{\theta}, \tag{30}$$

where the inequality follows from the fact that $\underline{P}_N(\mathrm{q}) \ge \underline{P}(\mathrm{q})$ for every q. Because $\underline{P}_N$ is decreasing and continuous, we have that $\mathrm{q}_\ell = \underline{P}_N{}^{-1}(\underline{P}_N(\mathrm{q}_\ell)) \le \underline{P}_N{}^{-1}(\overline{\theta}) = \mathrm{q}_\ell^N$. Hence $\mathrm{q}_\ell^N \ge \mathrm{q}_\ell$, with the inequality strict if, and only if, the inequality in (30) is strict, i.e., if and only if $\underline{P}_N(\mathrm{q}_\ell) > \underline{P}(\mathrm{q}_\ell)$.

The definition of $q^\star$ and $q_N^\star$ along with the monotonicity of these functions then implies that $\theta_N^\star \le \theta^\star$. That the inequality is strict when $\mathrm{q}_\ell^N > \mathrm{q}_\ell$, $\theta^\star < \overline{\theta}$ follows from the fact that, in this case, $q^{\mathrm{BM}}(\theta^\star) = \mathrm{q}_\ell < \mathrm{q}_\ell^N$. That q^{BM} is non-increasing along with the definition of $q^\star$ and $q_N^\star$ then imply that $\theta_N^\star < \theta^\star$. This completes the proof of part (a).

Part 2. Now assume that $\theta_N^\star \geq \theta^m$. Let

$$\Delta(\theta) := \underline{W}_N(\theta, q_N^\star) - \underline{W}(\theta, q^\star).$$

Note that, for any $\theta \leq \theta_N^\star$,

$$\begin{aligned}\Delta(\theta) &= \int_0^{q^{\text{BM}}(\theta)} \left[\underline{P}_N(z) - \underline{P}(z)\right] dz - \int_\theta^{\overline{\theta}} \left[q_N^\star(y) - q^\star(y)\right] dy \\ &= \int_0^{q^{\text{BM}}(\theta)} \left[\underline{P}_N(z) - \underline{P}(z)\right] dz - \int_{\theta_N^\star}^{\overline{\theta}} \left[q_N^\star(y) - q^\star(y)\right] dy,\end{aligned}$$

where both equalities follow from the fact that for $\theta \leq \theta_N^\star$, $q^\star(\theta) = q_N^\star(\theta) = q^{\text{BM}}(\theta)$. Because the second integral is independent of θ and because q^{BM} is decreasing, $\Delta(\theta)$ is decreasing over $[\underline{\theta}, \theta_N^\star]$. That $\theta^m \leq \theta_N^\star$, in turn implies that $\Delta(\theta)$ is decreasing over $[\underline{\theta}, \theta^m]$. Furthermore, for any $\theta \leq \theta^m$,

$$\underline{W}_N(\theta, q_N^\star) = \underline{W}(\theta, q^\star) + \Delta(\theta) \geq_{(a)} \underline{W}(\theta^m, q^\star) + \Delta(\theta^m) = \underline{W}_N(\theta^m, q_N^\star),$$

where inequality (a) follows from the fact that (i) $\underline{W}(\theta, q^\star) \geq \underline{W}(\theta^m, q^\star)$, which in turn follows from the definition of θ^m, and (ii) $\Delta(\theta) \geq \Delta(\theta^m)$, which in turn follows from the monotonicity of $\Delta(\theta)$ over $[\underline{\theta}, \theta^m]$. Thus, we have that $\theta_N^m \geq \theta^m$. Q.E.D

Proof of Lemma 5. For any $\widetilde{M} \in \widetilde{\mathcal{M}}$,

$$\begin{aligned}G(\widetilde{M}) &\leq_{(a)} \underline{V}(\underline{D}(p(\overline{\theta}))) - \overline{\theta}\underline{D}(p(\overline{\theta})) - (1-\alpha)\tilde{u}(\overline{\theta}, \underline{D}) \\ &\leq_{(b)} \underline{V}(\underline{D}(p(\overline{\theta}))) - \overline{\theta}\underline{D}(p(\overline{\theta})) \\ &\leq_{(c)} \underline{V}(\underline{D}(\overline{\theta})) - \overline{\theta}\underline{D}(\overline{\theta}) \\ &= G^*.\end{aligned}$$

Inequality (a) follows because the right-hand-side is just the expected welfare under distribution that puts probability one at $\overline{\theta}$ and when demand $\underline{D}$. Inequality (b) follows from the fact that $\tilde{u}(\overline{\theta}, \underline{D}) \geq 0$ as $\widetilde{M}$ is EPIR. Inequality (c) follows from the fact that

$$\overline{\theta} = \arg\max_{\text{p}} \left\{\underline{V}(\underline{D}(\text{p})) - \overline{\theta}\underline{D}(\text{p})\right\}.$$

Next, observe that G^* can be guaranteed by offering the constant-price mechanism $\underline{\widetilde{M}} := (p, t)$, where $p(\theta) = \overline{\theta}$ for all θ, and where t is such that, for all (θ, D), (11) holds with

$\tilde{u}(\overline{\theta}, D) = 0$. Then for all θ and all D, welfare is equal to

$$\begin{aligned}\widetilde{V}(\overline{\theta}; D) - \theta D(\overline{\theta}) - (1-\alpha)\int_{\theta}^{\overline{\theta}} D(\overline{\theta})dy &= \widetilde{V}(\overline{\theta}; D) - \overline{\theta}D(\overline{\theta}) + \alpha(\overline{\theta} - \theta)D(\overline{\theta}) \\ &\geq \widetilde{V}(\overline{\theta}; D) - \overline{\theta}D(\overline{\theta}) \\ &\geq \underline{V}(\underline{D}(\overline{\theta})) - \overline{\theta}\underline{D}(\overline{\theta}) = G^*.\end{aligned}$$

The last inequality follows from the fact that, when the cost is equal to $\overline{\theta}$, the price $p(\overline{\theta}) = \overline{\theta}$ maximizes total surplus $\widetilde{V}(\mathrm{p}; D) - \overline{\theta}D(\mathrm{p})$ for any D. The result then follows from the fact that total surplus under the surplus maximizing price is increasing in D. The above properties in turn imply that, for all F and D, $\widetilde{W}(\widetilde{\underline{M}}; V, F) \geq G^*$, which implies that $G(\widetilde{\underline{M}}) = G^*$.

Finally, that any $\widetilde{M} \in \widetilde{\mathcal{M}}$ for which $G(\widetilde{M}) = G^*$ is such that $p(\overline{\theta}) = \overline{\theta}$ and $\tilde{u}(\overline{\theta}, \underline{D}) = 0$ follows from the fact that Nature can always selects $D = \underline{D}$ and a distribution selecting $\theta = \overline{\theta}$ with probability one; when $\theta = \overline{\theta}$ and $D = \underline{D}$, the only way welfare can be made equal to G^* is by inducing efficient output by setting a price $p(\overline{\theta}) = \overline{\theta}$, and giving no rent to the seller, which amount to setting $\tilde{u}(\overline{\theta}, \underline{D}) = 0$. ■

Proof of Lemma 6. The proof is in two steps. Step 1 shows that, without loss of optimality, the government can restrict attention to price mechanisms that entail only mark-ups. Step 2 uses the result in Step 1 to establish the claim in the lemma.

Step 1. Suppose $\widetilde{M} \in \widetilde{\mathcal{M}}$ is such that $p(\theta) < \theta$ for some θ. There exists another price mechanism $\widetilde{M}^\dagger = (p^\dagger, t^\dagger) \in \widetilde{\mathcal{M}}$ with $p^\dagger(\theta) \geq \theta$ for all θ such that $\widetilde{W}(\widetilde{M}^\dagger; D, F) \geq \widetilde{W}(\widetilde{M}; D, F)$ for all $D \in \mathcal{D}$ and $F \in \mathcal{F}$, with the inequality strict if the subset of Θ for which $p(\theta) < \theta$ has strict positive measure under F.

To see this, let $\widetilde{M}^\dagger = (p^\dagger, t^\dagger)$ be the mechanism constructed from $\widetilde{M}$ by setting $p^\dagger(\theta) = \max\{p(\theta), \theta)\}$ for all $\theta \in \Theta$ and by setting $t^\dagger$ so that, for all $\theta \in \Theta$ and $D \in \mathcal{D}$,

$$\tilde{u}^\dagger(\theta, D) = \tilde{u}(\overline{\theta}, D) + \int_{\theta}^{\overline{\theta}} D(p^\dagger(y))dy,$$

where $\tilde{u}(\overline{\theta}, D)$ is type $\overline{\theta}$'s rent under the mechanism $\widetilde{M}$ when the demand is D. Clearly, $p^\dagger$ is non-decreasing and $\widetilde{M}^\dagger = (p^\dagger, t^\dagger) \in \widetilde{\mathcal{M}}$ meaning that $\widetilde{M}^\dagger$ is also EPIC and EPIR. Next observe that, for all $\theta \in \Theta$ and $D \in \mathcal{D}$,

$$\tilde{u}^{\dagger}(\theta, D) - \tilde{u}(\theta, D) = \int_{\theta}^{\overline{\theta}} \Big[D(p^{\dagger}(y)) - D(p(y))\Big] dy \leq 0,$$

where the inequality follows from the fact that $p^{\dagger}(\theta) \geq p(\theta)$ for all θ. The inequality is strict if there exists a subset of Θ of positive Lebesgue measure for which $p(\theta) < \theta$. Furthermore, for all $\theta \in \Theta$ and $D \in \mathcal{D}$,

$$\widetilde{V}(p^{\dagger}(\theta); D) - \theta D(p^{\dagger}(\theta)) \geq \widetilde{V}(p(\theta); D) - \theta D(p(\theta)),$$

where the inequality follows from the fact that $\widetilde{V}(\mathrm{p}; D) - \theta D(\mathrm{p})$ is single picked in p with a maximum at $\mathrm{p} = \theta$. It follows that, for any F and D, $\widetilde{W}(\widetilde{M}^{\dagger}; D, F) \geq \widetilde{W}(\widetilde{M}; D, F)$. We thus conclude that, to find robustly optimal price mechanisms, it suffices to restrict attention to price mechanisms w*ith markups* (which thus induce underproduction no matter the realized cost and demand).

Step 2. Equipped with the result in step 1, we now establish the result in the lemma.

Necessity. let $\widetilde{M}^{\mathrm{OPT}} = (p^{\mathrm{OPT}}, t^{\mathrm{OPT}})$ be a robustly optimal price mechanism. Because $\widetilde{M}^{\mathrm{OPT}}$ is EPIC, p^{OPT} must be non-decreasing. Because $\widetilde{M}^{\mathrm{OPT}} \in \widetilde{\mathcal{M}}^{\mathrm{SL}}$, Lemma 5 implies that $p^{\mathrm{OPT}}(\overline{\theta}) = \overline{\theta}$ and $\tilde{u}^{\mathrm{OPT}}(\overline{\theta}, \underline{D}) = 0$. Because $F^{\star}$ is increasing over Θ, the result in Step 1 implies that $p^{\mathrm{OPT}}(\theta) \geq \theta$ for almost all $\theta \in \Theta$. This last property along with the fact that p^{OPT} is non-decreasing implies that $p^{\mathrm{OPT}}(\theta) \geq \theta$ for all $\theta \in \Theta$. Since $\widetilde{M}^{\mathrm{OPT}}$ satisfies (14) and EPIR, equation (18) holds. Since the maximization of welfare is done under the conjecture $(F^{\star}, D^{\star})$, standard arguments imply that $\tilde{u}^{\mathrm{OPT}}(\overline{\theta}, D^{\star}) = 0$. Hence p^{OPT} must satisfy all the constraints in (**ROPT-P**). Next, recall that, given any mechanism $\widetilde{M} = (p, t) \in \widetilde{\mathcal{M}}^{\mathrm{SL}}$, welfare under the conjecture $(D^{\star}, F^{\star})$ is given by (15). Suppose there exists a price schedule $p^{\dagger}$ that also satisfies the constraints in (**ROPT-P**) and such that

$$\int_{\underline{\theta}}^{\overline{\theta}} \Big[V^{\star}(D^{\star}(p^{\dagger}(\theta))) - z^{\star}(\theta) D^{\star}(p^{\dagger}(\theta))\Big] F^{\star}(\mathrm{d}\theta) >$$
$$\int_{\underline{\theta}}^{\overline{\theta}} \Big[V^{\star}(D^{\star}(p^{\mathrm{OPT}}(\theta))) - z^{\star}(\theta) D^{\star}(p^{\mathrm{OPT}}(\theta))\Big] F^{\star}(\mathrm{d}\theta).$$

Let $t^{\dagger}$ be any transfer schedule such that the function $\tilde{u}^{\dagger}$ defined, for all $\theta \in \Theta$ and $D \in \mathcal{D}$, by $\tilde{u}^{\dagger}(\theta, D) := t^{\dagger}(\theta, D) - \theta D(p^{\dagger}(\theta))$ satisfies the following properties: (a) for all $D \in \mathcal{D}$ and

$\theta \in \Theta$,

$$\tilde{u}^\dagger(\theta, D) = \tilde{u}^\dagger(\overline{\theta}, D) + \int_{\theta}^{\overline{\theta}} D(p^\dagger(y))dy,$$

(b) for all $D \in \mathcal{D}$, $0 \leq (1-\alpha)\tilde{u}^\dagger(\overline{\theta}, D) \leq [\widetilde{V}(\overline{\theta}; D) - \overline{\theta}D(\overline{\theta}) - G^*]$, and (c) $\tilde{u}^\dagger(\overline{\theta}, \underline{D}) = \tilde{u}^\dagger(\overline{\theta}, D^\star) = 0$. Lemmas 4-5 imply that the mechanism $\widetilde{M}^\dagger = (p^\dagger, t^\dagger) \in \widetilde{\mathcal{M}}^{\mathrm{SL}}$ and yields the government a payoff under the conjecture $(D^\star, F^\star)$ strictly greater than $\widetilde{M}^{\mathrm{OPT}}$, contradicting the assumption that $\widetilde{M}^{\mathrm{OPT}}$ is robustly optimal. We conclude that p^{OPT} must solve program (**ROPT-P**).

Sufficiency. Suppose $\widetilde{M}^{\mathrm{OPT}} = (p^{\mathrm{OPT}}, t^{\mathrm{OPT}})$ satisfies the properties in the lemma. Lemmas 4 and 5 along with the result in Step 1 above imply that $\widetilde{M}^{\mathrm{OPT}} \in \widetilde{\mathcal{M}}^{\mathrm{SL}}$. The same lemmas, together with the fact that p^{OPT} solves program (**ROPT-P**) and the fact that, in any EPIC and EPIR mechanism, the government's payoff under the conjecture $(D^\star, F^\star)$ is given by (15), imply that the $\widetilde{M}^{\mathrm{OPT}}$ maximizes the government's objective (under the conjecture $(D^\star, F^\star)$) over $\widetilde{\mathcal{M}}^{\mathrm{SL}}$. Hence, $\widetilde{M}^{\mathrm{OPT}}$ is robustly optimal. ■

Proof of Proposition 9. Note that, for any $\theta \in \Theta$, the unique price that maximizes the integrand function in (**ROPT-P**) over $[0, \overline{\theta}]$ is $\min\{z^\star(\theta), \overline{\theta}\}$. Because $z^\star$ is increasing and, for any $\theta \in \Theta$, $z^\star(\theta) \geq \theta$ with $z^\star(\overline{\theta}) > \overline{\theta}$, the price function $p(\theta) = \min\{z^\star(\theta), \overline{\theta}\}$ satisfies all the constraints in (**ROPT-P**). Because $V^\star(D^\star(\mathrm{p})) - z^\star(\theta)D^\star(\mathrm{p})$ is quasi-concave in p, and attains a maximum at $z^\star(\theta)$, we conclude that the unique solution to program (**ROPT-P**) is the price function in (19). The proposition then follows from the above properties together with Lemma 6. Q.E.D.

Proof of Proposition 10. The proof is in two parts, each establishing the corresponding claim in the proposition.

Part (1). If $M^{\mathrm{OPT}} = M^\star$, then $q^{\mathrm{OPT}}(\theta) = \max\{q^{\mathrm{BM}}(\theta), \mathrm{q}_\ell\}$ for all θ, where $\mathrm{q}_\ell := \underline{P}^{-1}(\overline{\theta}) = \underline{D}(\overline{\theta})$ is the efficient quantity for cost $\overline{\theta}$ and demand $\underline{D}$. In this case, there exists $\theta^\star \leq \overline{\theta}$ such that $q^{\mathrm{OPT}}(\theta) = \mathrm{q}_\ell$ if $\theta \geq \theta^\star$, and $q^{\mathrm{OPT}}(\theta) = q^{\mathrm{BM}}(\theta)$ if $\theta < \theta^\star$. See Figure 11 for the illustration.

Under the unique robustly optimal price mechanism $\widetilde{M}^{\mathrm{OPT}} = (p^{\mathrm{OPT}}, u^{\mathrm{OPT}})$, the quantity the government expects to procure at the conjectured demand and cost θ is $D^\star(p^{\mathrm{OPT}}(\theta)) = \max\{q^{\mathrm{BM}}(\theta), \tilde{\mathrm{q}}_\ell\}$, where $\tilde{\mathrm{q}}_\ell := D^\star(\overline{\theta}) \geq \underline{D}(\overline{\theta}) = \mathrm{q}_\ell$. Thus, there exists $\tilde{\theta}^\star \leq \theta^\star$ such that

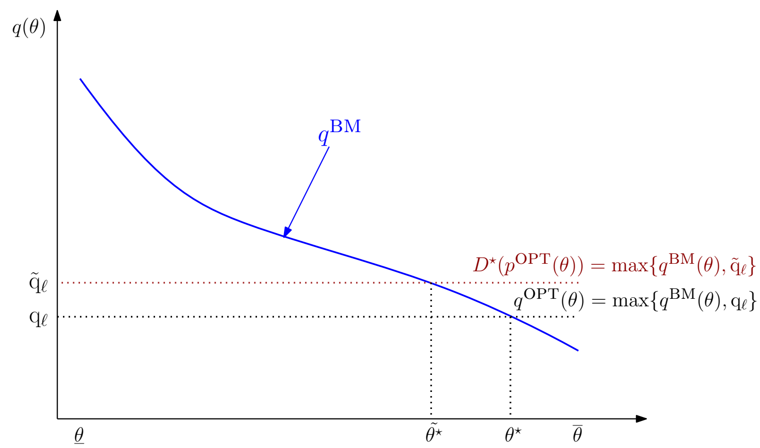

Figure 11: Illustration for the proof of Part (1) of Proposition 10

$D^{\star}(p^{\mathrm{OPT}}(\theta)) = \tilde{\mathrm{q}}_\ell$ if $\theta \geq \tilde{\theta}^{\star}$ and $D^{\star}(p^{\mathrm{OPT}}(\theta)) = q^{\mathrm{BM}}(\theta)$ if $\theta < \tilde{\theta}^{\star}$. See Figure 11 for the illustration.

For $\theta < \tilde{\theta}^{\star}$, we have that $D^{\star}(p^{\mathrm{OPT}}(\theta)) = q^{\mathrm{OPT}}(\theta) = q^{\mathrm{BM}}(\theta)$. However, for $\theta \geq \tilde{\theta}^{\star}$, we have that $D^{\star}(p^{\mathrm{OPT}}(\theta)) = \tilde{\mathrm{q}}_\ell \geq q^{\mathrm{OPT}}(\theta) \geq q^{\mathrm{BM}}(\theta)$. Because, for any θ, virtual surplus

$$V^{\star}(\mathrm{q}) - z^{\star}(\theta)\mathrm{q}$$

is strictly quasi-concave in q, reaching a maximum at $q^{\mathrm{BM}}(\theta)$, we thus have that, for any $\theta \geq \tilde{\theta}^{\star}$,

$$V^{\star}(D^{\star}(p^{\mathrm{OPT}}(\theta))) - z^{\star}(\theta)D^{\star}(p^{\mathrm{OPT}}(\theta)) \leq V^{\star}(q^{\mathrm{OPT}}(\theta)) - z^{\star}(\theta)q^{\mathrm{OPT}}(\theta).$$

Thus, we have $\widetilde{W}(\widetilde{M}^{\mathrm{OPT}}; D^{\star}, F^{\star}) \leq W(M^{\mathrm{OPT}}; V^{\star}, F^{\star})$, with the inequality strict if, and only if, $D^{\star}(\overline{\theta}) > \underline{D}(\overline{\theta})$.

Part (2). If $D^{\star}(\overline{\theta}) = \underline{D}(\overline{\theta})$, then $\mathrm{q}_\ell = \tilde{\mathrm{q}}_\ell$. In this case, the quantity that the government expects to procure under its conjecture $(D^{\star}, F^{\star})$ by running the robustly optimal price mechanism is $D^{\star}(p^{\mathrm{OPT}}(\theta)) = \max\{q^{\mathrm{BM}}(\theta), \mathrm{q}_\ell\} = q^{\star}(\theta)$, for all θ. This means that, by running the robustly optimal price mechanism $\widetilde{M}^{\mathrm{OPT}}$ the government obtains the same payoff as by running the Baron-Myerson-with-quantity-floor mechanism $M^{\star}$, i.e.,

$$\widetilde{W}(\widetilde{M}^{\mathrm{OPT}}; D^{\star}, F^{\star}) = W(M^{\star}; V^{\star}, F^{\star}). \tag{31}$$

As shown in Lemma 2, $M^{\star}$ is the solution to a relaxation of the full program yielding the robustly optimal quantity mechanism, implying that

$$W(M^{\star}; V^{\star}, F^{\star}) \geq W(M^{\mathrm{OPT}}; V^{\star}, F^{\star}). \tag{32}$$

When $M^{\text{OPT}} \neq M^{\star}$, the inequality in (32) is strict. Jointly, (31) and (32) imply that, when $M^{\text{OPT}} \neq M^{\star}$ and $D^{\star}(\bar{\theta}) = \underline{D}(\bar{\theta})$, price strictly dominates quantity. Q.E.D.

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