From Anticipations to Present Bias: 
A Theory of Forward-Looking Preferences 

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October 31, 2014

Abstract

How do future well-being and preferences affect the current well-being and preferences of forward-looking agents? Our theory explores this question, producing a new class of tractable models which capture and explain phenomena such as present bias, consumption interdependence, sign effects in discounting, and the desire to space out consumption. Agents manifest impatience toward the current period, but not necessarily toward the earlier of two future periods. The theory characterizes the well-known quasi-hyperbolic discounting model as the unique model of our class which does not display consumption interdependence. Finally, it provides a rigorous approach for analyzing the welfare of agents with time-inconsistent preferences.

Keywords: time inconsistency, forward-looking preferences, hyperbolic discounting, beta-delta discounting, anticipations, welfare criterion.

JEL Classification: D01, D60, D90

*The authors are grateful to Nageeb S. Ali, Nabil Al-Najjar, Doug Bernheim, Eddie Dekel, Drew Fudenberg, David Laibson, Bart Lipman, Emir Kamenica, Peter Klibanoff, Mark Machina, Paul Milgrom, David G. Pearce, Todd Sarver, Marciano Siniscalchi, Joel Sobel, and Asher Wolinsky, as well as seminar participants at UC San Diego, Stanford, Université de Montréal, RUD 2014, University of Chicago, Yale, NASMES 2014, SITE 2014, UCL, and WU St. Louis for useful comments. An early version of this project was developed by Strulovici and presented at Northwestern University and the 2011 SAET conference in Faro, Portugal under the title “Forward-Looking Behavior, Well-Being Representation, and Time Inconsistency.” Strulovici is grateful for financial support from the National Science Foundation and a Sloan Research Fellowship.
1 Introduction

Our well-being is largely influenced by anticipations about career prospects, long-run financial stability, life-changing events such as getting married or having a child, and less consequential ones such as attending a concert or going to the dentist. Economic agents who care about their future well-being thus recognize the importance of future anticipations on their current decisions. For instance, a worker choosing a level of savings for retirement values not only the resulting pension itself, but also the psychological comfort and sense of security that the pension will bring her throughout retirement: upon retiring—and in later years—she will anticipate a steady flow of consumption that will affect her well-being then, in a way that matters to her today. Similarly, a student choosing a career that involves some initial sacrifice (e.g., extreme work hours, low material comfort) and long-run intellectual and financial rewards knows that he will anticipate such rewards while he endures the sacrifice, and that this anticipation will do much to alleviate his pain and increase his well-being during that period, making bearable and perhaps even exciting what might otherwise be unbearable.

To what extent do these anticipations shape our preferences? To appreciate the answer, it is useful to recall some characteristics of the standard exponentially-discounted-utility (EDU) model, which fails to include such anticipations: i) How the agent trades off consumption in periods 1 and 9, say, is completely independent of his consumption in period 10, whether it is attending a great concert or enduring strenuous work throughout that period; ii) Immediate—i.e., actual—consumption is treated on par with anticipated—i.e., imagined—consumption; iii) Upon receiving ten open-date concert tickets (or when setting the dates of ten possibly heterogeneous events such as restaurants, games, and concerts), an EDU agent wants to attend the concert on the first ten available—possibly, consecutive—days; iv) The agent’s well-being at a given time is completely determined by his instantaneous consumption at that time.\footnote{In Samuelson’s (1937) words, the EDU model “involves the assumption that at every instant of time the individual’s satisfaction depends only upon the consumption at that time.” (p. 159) He considered this assumption to be “completely arbitrary.” We discuss other interpretations of the EDU model in Section 5.}

Were it not for its tractability, one may wonder about the popularity of the EDU model. More to the point, one may wonder whether there is a missing piece which could address the above shortcomings of that model in a tractable way. As it turns out, anticipations about future well-being constitute such a piece: including them suffices to get rid of all the undesirable characteristics mentioned above, and does so in an intuitive, economical, and tractable way. The resulting theory also provides a robust explanation for present
bias and delivers a new axiomatization for the quasi-hyperbolic \((\beta-\delta)\) model.

In our theory, agents anticipate their future well-being and realize that they will continue to do so in the future. The theory answers the following questions: 1) How do anticipations about future well-being shape preferences over consumption streams? 2) Can we characterize such preferences and represent them in a tractable model? 3) How can we perform welfare analysis with agents who have such preferences? 4) Is there a way to reconcile the EDU and \(\beta-\delta\) models with these anticipations?

In order to isolate and study the effects of purely forward-looking preferences, our theory sets aside backward-looking components of well-being, such as habits, satiation, addiction, and memories.\(^2\) Our formalization of ‘well-being’ is based on revealed preferences: by choosing at time 0 between consumption streams \(c = (c_0, c_1, \ldots)\) and \(c' = (c'_0, c'_1, \ldots)\), an agent reveals a relative ranking of the well-being provided at that time by these streams.\(^3\) Thus, any utility function \(U(c_0, c_1, \ldots)\) consistent with those choices is an ordinal transformation of the agent’s well-being. This formalization leads to a class of models of the form

\[
U(c_0, c_1, c_2, \ldots) = V(c_0, U_1, U_2, \ldots)
\]  

(1)

where \(U_t = U(c_t, c_{t+1}, \ldots)\) and \(V\) is a function aggregating the agent’s current consumption and future well-beings into a single well-being index in period 0. For example, if period 1 corresponds to the sacrifice period in the career choice example, the painful consumption \(c_1\) will be partly compensated, in the computation of \(U_1\), by the pleasant consumption stream \((c_2, c_3, \ldots)\) following it. As is well known, in the case of EDU we have \(U(c_0, c_1, \ldots) = u(c_0) + \delta U(c_1, c_2, \ldots)\). So, in the EDU model, well-being in period 0 cannot be represented as a function of future well-being beyond period 1.\(^4\)

Despite its generality, representation (1) delivers a first robust prediction: an agent who cares about her well-being at all future periods must exhibit present bias. Our theory thus provides a new and robust explanation for this well-known phenomenon (see Section 2.2.1). To obtain more predictions and make the model tractable, we follow the axiomatic approach used by Koopmans (1960) to derive the EDU model, with one main difference: our axioms are applied to future ‘well-beings’ rather than future consumption per se.\(^5\) In fact, the axioms are all stated in terms of preferences over consumption

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\(^2\)These are significant determinants of well-being and intertemporal preferences, which we hope to explore in subsequent work. The methodology of the present paper should prove helpful toward this objective.

\(^3\)This approach is standard, and Bernheim and Rangel (2009) argue that the only operationally meaningful notion of well-being is the preference relation revealed through choices.

\(^4\)Another way to see this is that any model in which \(U_0\) depends on \(U_1’s beyond \(U_1\) would exhibit time inconsistency.

\(^5\)This conceptual difference raises analytical challenges. Unlike the space of consumption streams, the
streams. For instance, an alternative interpretation of expression (1) is that an agent’s preference over consumption streams at time 0 is completely determined by his consumption at 0 and his preferences at future times. This dual interpretation between preferences and well-being, which bridges our decision theoretic axiomatization with our informal motivation in terms of anticipated well-being, is possible because in our theory an agent’s ‘well-being’ and his preference over streams are equivalent in an ordinal sense.

Our axioms—stationarity, separability, and monotonicity—specialize (1) to the following class of models:

$$U(c_0, c_1, c_2, \ldots) = u(c_0) + \sum_{t \geq 1} \alpha^t G(U(c_t, c_{t+1}, c_{t+2}, \ldots))$$

(2)

where $G(U_t)$ may be interpreted as the agent’s anticipatory utility at time 0 from well-being at time $t$, and $\alpha$ is his subjective discount factor. The axioms also impose specific parametric restrictions on $\alpha$, $u$, and $G$, which imply that the agent is impatient: he prefers receiving better consumption immediately than at any future date. These restrictions also imply that representation (2) is tractable and has a unique solution $U$, which satisfies a strong form of tail continuity.\(^6\)

When the function $G$ is linear, (2) turns out to be equivalent to the $\beta$-$\delta$ model. Our theory thus provides a new axiomatization of that model, which is easy to interpret in terms of anticipated well-being.

It is natural to think of $G$ as being concave: a higher well-being in period $t$ reduces the marginal utility from that well-being. In this case, (2) has several implications.\(^7\) First, it implies a preference for spacing out consumption. For example, an agent offered to attend twelve concerts—or entertaining events of different kinds—in the coming year may prefer to attend one per month. The agent may also prefer to delay pleasurable events, even if they are of different nature: given his attendance to a concert next week, the agent may prefer to eat in a starred restaurant in three, rather than two, weeks from now. Furthermore, increasing consumption at some period $t$ makes the agent more impatient. Intuitively, the agent cares less about increasing his future well-being, as his well-being reference is now higher, and thus values comparatively more immediate consumption. All these phenomena are instances of a deeper phenomenon, called consumption inter-

\(^6\)Precisely, for any $\varepsilon > 0$, there exists a horizon $T$ such that any two streams $c, c'$ that are identical up to time $T$ must give well-beings $U(c)$ and $U(c')$ within $\varepsilon$ of each other, regardless of the streams’ continuations. It is noteworthy that this strong tail-continuity is implied by the axioms, even though the only preference continuity explicitly assumed is the standard one.

\(^7\)For empirical support of these implications, see the survey by Frederick et al. (2002).
dependence: how the agent trades off consumption at two given dates depends, through well-being, on consumption at other dates too.

Our theory predicts that agents have time-inconsistent preferences: how an agent ranks two consumption streams depends on whether they start immediately or in the future. By providing a clear foundation for this property, the theory offers a rigorous approach for the welfare analysis of time-inconsistent agents, such as agents with $\beta - \delta$ preferences. For this type of time-inconsistent agents, in particular, we find that the most natural welfare criterion may be the usual, time-consistent criterion with discount factor $\delta$.

Representation (2) is incompatible with EDU. In fact, **EDU violates a single axiom of our theory**: the stationarity axiom. In Koopmans’ axiomatization of EDU, stationarity requires that the agent prefer $(c_0, c_1, c_2, \ldots)$ to $(c_0, c'_1, c'_2, \ldots)$ if and only if he prefers $(c_1, c_2, \ldots)$ to $(c'_1, c'_2, \ldots)$, i.e., the streams obtained after dropping $c_0$ and moving the continuation streams one period ahead. Koopmans’ stationarity thus compares immediate ‘real’ consumption with anticipated one: as $c_1$ is moved to time 0, it becomes real while $c_2$ remains anticipated. This axiom requires that the consumption trade-offs between $c_1$ and $c_2$ remain the same after this transformation. By contrast, we adopt the view that anticipated consumption is radically different from actual consumption, a view inspired by the fact that the former consumption is purely imagined, while the latter has specific physical and sensorial components.

To avoid confusion, our theory focuses on preferences over consumption streams, rather than intertemporal behavior. Of course, such preferences may be connected to intertemporal behavior in standard ways, such as the intrapersonal-equilibrium approach pioneered by Strotz (1955). In particular, an agent with the properties described in the present paper and who is sophisticated about his future preferences will value commitment and, in the absence of commitment, may exhibit time-inconsistent behavior. We find it useful to disentangle the two issues and discuss their connection in Section 5.

As suggested by the previous observations, our theory leads to a critique of the EDU model as a model of forward-looking preferences, also developed in Section 5.

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8A more recent analysis is provided by Laibson (1997) and Harris and Laibson (2001). Formally identical equilibria have been studied in the literature on intergenerational altruism, such as Phelps and Pollak (1968), Ray (1987), Bernheim and Ray (1989), and Pearce (2008).
2 Preference Representations

2.1 Preliminaries

We examine preferences over infinite consumption streams in a deterministic environment. At each time \( t \), the set of feasible consumption events is \( X \), a connected, separable, metric space. Time is discrete with infinite horizon: \( \mathbb{N} = \{0, 1, 2, \ldots\} \). For \( t \geq 0 \), the set of consumption streams starting at \( t \) is \( tC \subseteq X^\mathbb{N} \)—elements of this set will be denoted by \( t c = (c_t, c_{t+1}, \ldots) \). For each \( t \), the agent has a preference relation \( \succ^t \) with domain \( tC \).\(^9\) In principle, \( \succ^t \) and \( tC \) could vary over time. In this paper, however, they are time invariant.

**Assumption 1** (Time Invariance). For all \( t \geq 0 \), \( tC = C \) and \( \succ^t = \succ \).

The set \( C \) is endowed with the sup-norm: \( \|c - c'\|_C = \sup_t d(c_t, c'_t) \) where \( d \) is a bounded metric on \( X \).\(^{10}\) Given Assumption 1, for simplicity, hereafter we take the perspective of \( t = 0 \) and omit the time index of a preference unless it is required by the context.

**Remark 1.** For any \( c, c' \in C \), the expression \( c \succ^t c' \) means that at time \( t \) the agent always chooses \( c \) over \( c' \), if at \( t \) he has to commit to either stream for the entire future. This commitment assumption is essential for interpreting the agent’s choice as his ranking of streams.\(^{11}\) Otherwise, at \( t \) the agent may choose \( c \) because he likes \( c_t \) better than \( c'_t \) and anticipates abandoning \( c \) for some other stream in the future. In this case, at \( t \) the agent is not actually comparing \( c \) and \( c' \). Hence, his choice does not tell us anything about his preference over \( c \) and \( c' \). Moreover, without the commitment assumption, we cannot meaningfully examine whether preferences are time consistent.

A basic premise of this paper is that \( \succ \) has a utility representation. This is ensured by the following axioms (the symbols ‘\( \succsim \)’ and ‘\( \sim \)’ have the usual meaning).

**Axiom 1** (Weak Order). \( \succsim \) is a complete and transitive binary relation.

**Axiom 2** (Continuity). For all \( c \in C \), the sets \( \{c' \in C : c' \succsim c\} \) and \( \{c' \in C : c' \succ c\} \) are closed.

\(^9\)This paper continues to assume, as in the EDU model, that at each \( t \) preferences do not depend on past consumption. Relaxing this assumption is beyond the paper’s scope.

\(^{10}\)Assuming that \( d(\cdot, \cdot) \) is bounded is without loss of generality, as we can always replace it by \( \hat{d}(\cdot, \cdot) = d(\cdot, \cdot)/(1 + d(\cdot, \cdot)) \), which is another metric respecting the initial distances.

\(^{11}\)This assumption is also made, more or less explicitly, in Koopmans (1960, 1964, 1972), Hayashi (2003), Olea and Strzalecki (2014).
**Axiom 3** (Future Constant-Flow Dominance). For all $c \in C$, there exist $x,y \in X$ such that $(c_0,x,x,...) \not\succeq c \not\succeq (c_0,y,y,...)$.

Axioms 1 and 2 are standard. Axiom 3 captures the intuitive idea that, for any stream $c$, there are consumption events $x$ and $y$ that are sufficiently bad and good, so that facing $x$ (resp. $y$) forever in the future is worse (resp. better) than facing $1c$. These axioms lead to the following standard result.\(^{12}\)

**Theorem 1** (Utility Representation). Under axioms 1-3, there exists a continuous function $U : C \to \mathbb{R}$ such that $c \succ c'$ if and only if $U(c) > U(c')$.

**Remark 2.** In the rest of the paper, we will interpret $U(t_c)$ as the agent’s well-being (i.e., total utility) at time $t$ generated by stream $t_c$. We do so for the sake of conveying intuitions.

Since we are interested in forward-looking preferences, by assumption, well-being at $t$ depends on consumption at some period after $t$. It is also natural that well-being depends on immediate consumption.

**Axiom 4** (Non-triviality). There exist $x,x',\hat{x} \in X$ and $c,c',\hat{c} \in C$ such that $(x,\hat{c}) \succ (x',\hat{c})$ and $(\hat{x},c) \succ (\hat{x},c')$.

### 2.2 Forward-looking Preferences and Well-Being

We consider an agent who, at each $t$, is forward-looking in the sense that he cares about his future preferences, represented by well-being $U$. To formalize this idea, let $U$ be the range of $U$ and define

$$
\mathcal{F} = \{(f_1(c),f_2(c),\ldots) : f_t(c) = U(t_c) \text{ for } c \in C \text{ and } t > 0\}.
$$

Note that $\mathcal{F} \subset U^\mathbb{N}$, but in general $\mathcal{F}$ is not a Cartesian product (e.g., $\mathcal{F} \neq U^\mathbb{N}$) because well-being at $t$ depends on future consumption and hence on future well-being.

**Definition 1** (Well-Being Representation). Preference $\succ$ has a well-being representation if and only if

$$
U(c) = V(c_0,U(1c),U(2c),\ldots)
$$

for some function $V : X \times \mathcal{F} \to \mathbb{R}$ that is nonconstant in $c_0$ and $U(t_c)$ for some $t > 0$.

\(^{12}\)The proofs of the main results are in the Appendix A. All omitted proofs are in Appendix B (Online Appendix).
So, how the agent ranks streams \( c \) and \( c' \) at time 0 depends only on immediate consumption levels \( c_0 \) and \( c'_0 \), and on how he ranks continuation streams \( t_c \) and \( t_{c'} \) for at least some future period \( t \). In other words, the well-being from any \( c \) depends only on immediate consumption \( c_0 \) and, for at least some future \( t \), on the per-period well-being from \( t_c \). Well-being is therefore conceptually different from the immediate utility from a single consumption event: it captures an aggregate of sensorial pleasure from immediate consumption as well as purely mental satisfaction (or dissatisfaction) from future well-being. While such an aggregation seems difficult, it is somehow performed by any forward-looking agent who must choose current and future consumption. The definition is recursive, as one should expect: for a forward-looking agent, well-being today involves well-being in the future.

Axiom 5 is the key to obtaining representation (4). It captures a minimal property that seems natural for forward-looking preferences: given immediate consumption, if the agent anticipates being indifferent between two consumption streams at all future periods, he should also be indifferent at time 0.

**Axiom 5.** If \( c_t \sim t_{c'} \) for all \( t > 0 \), then \((c_0, 1c) \sim (c_0, 1c')\).

Axiom 5 rules out, for instance, the possibility that the agent prefers \( c \) to \( c' \) because, despite generating the same stream of immediate consumption and future well-being, they allocate future consumption differently over time.

**Theorem 2.** Axioms 4 and 5 hold if and only if \( \succ \) has a well-being representation.

**Proof.** Define \( \mathcal{F}_0 = X \times \mathcal{F} \) and let \( f_0(c) = c_0 \) and \( f(c) = (f_0, f_1, f_2, \ldots) \).

(⇒) First, we define equivalence classes on \( C \) as follows: \( c \) is equivalent to \( c' \) if \( f_t(c) = f_t(c') \) for all \( t \geq 0 \).\(^{13}\) Let \( C^* \) be the set of equivalence classes of \( C \), and let the function \( U^* \) be defined by \( U \) on \( C^* \). Then, the function \( f^* : C^* \to \mathcal{F} \), defined by \( f^*(c^*) = f(c) \) for \( c \) in the equivalence class \( c^* \), is by construction one-to-one and onto. Let \( (f^*)^{-1} \) denote its inverse and, for any \( f \in \mathcal{F}_0 \), define

\[
V(f) = U^*((f^*)^{-1}(f)).
\]

By Axiom 5, \( V \) is a well-defined function, and \( V(f(c)) = U(c) \) for every \( c \). By Axiom 4, \( V \) is nonconstant in \( f_0 \) and \( f_t \) for some \( t > 0 \).

\(^{13}\)In general, there may be several consumption streams in an equivalence class. For example, suppose that \( U(c) = c_0 + c_1 + c_2 + c_3 \), and let \( c = (1, 1, -1, 1, 1, -1, -1, \ldots) \) and \( c' = (1, -1, -1, 1, 1, -1, -1, 1, 1, \ldots) \).
Suppose that $V: F_0 \to \mathbb{R}$ is a function such that $V(f(c)) = U(c)$ and $V$ is nonconstant in $f_0$ and $f_t$ for some $t > 0$. Then, it is immediate to see that the implied preference satisfies Axioms 4 and 5.

We emphasize that Axiom 5 is weak: it requires that the agent be indifferent at 0 between two streams, only if he is indifferent between their continuations at all future dates, not just next one. Clearly, if at 0 the agent cares only about his well-being at 1—as under EDU—Axiom 5 holds. By allowing the current preference to depend on future preferences in a richer way, Axiom 5 is a key step of our approach to modeling forward-looking preferences differently from EDU (Koopmans (1960, 1964)).

As noted, EDU satisfies

$$U(c) = u(c_0) + \delta U(1c) = V(c_0, U(1c)).$$

When the preference at $t$ depends only on immediate consumption and the preference at $t+1$, we shall call the agent indirectly forward-looking. On the other hand, we shall call the agent directly forward-looking if, at all $t$, his preference depends directly on his preferences at all $s > t$.

2.2.1 Time (In)consistency and Present Bias

Time consistency is among the most prominent and studied properties of intertemporal preferences. This section highlights, within the general class (4), a tension between forward-looking preferences and time consistency.

Intuitively, time consistency requires that if a course of action is preferable tomorrow according to tomorrow’s preference, then it should remain preferable, for tomorrow, according to today’s preference. We define time consistency as follows (see, e.g., Siniscalchi (2011)). Recall that $\succ^t$ is the preference at time $t$.

**Definition 2** (Time Consistency). $1c \succ^1 1c'$ if and only if $(c_0, 1c) \succ^0 (c_0, 1c')$.

Proposition 1, below, shows that an agent who cares about his well-being beyond the immediate future cannot be time consistent. The purpose of this preliminary result is simply to identify and highlight a possible source of time inconsistency. This source corresponds to an intuitive, more general form of forward-looking preference than in

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14 Another possible interpretation is that an EDU agent fails to realize that, in the future, he will continue to care about future well-being.

15 Of course, one could also consider the case in which $V$ depends on $U(t, c)$ up to some finite $t > 1$.

16 Definition 2 looks similar to the stationarity axiom in Koopmans (1960, 1964). However, time consistency and stationarity are conceptually very different (see Section 2.3).
the EDU model: a preference that in each period depends on well-being beyond the subsequent period.\footnote{The purpose of Proposition 1 should not be misunderstood. It is well known that a preference is time consistent if and only if, at each \( t \), it has a specific recursive representation in which \( U(c) \) depends only on immediate consumption, \( c_0 \), and continuation utility \( U(1c) \).}

**Proposition 1.** Preference \( \succ \) satisfies time consistency if and only if

\[
V(c_0, U(1c), U(2c), \ldots) = V(c_0, U(1c))
\]

for all \( c \in C \), and \( V \) is strictly increasing in its second argument.

It is common to view time consistency of preferences as the norm and time inconsistency as an exception. Proposition 1 reverses this view. If we deem natural that an agent should care about his well-being beyond the immediate future, then we have to conclude that time inconsistency should be the norm, rather than the exception.

To see the intuition for Proposition 1, suppose that \( 1c \) corresponds to buying a ticket to go to Hawaii at time 2, and that, at 1, the agent slightly prefers not to buy the ticket. Now imagine that, at 0, he can choose whether to buy the ticket at 1. If at 0 the agent cares directly about his well-being beyond 1—i.e., when he will be at Hawaii—he should then strictly prefer to buy the ticket at 1. Intuitively, from the perspective of 1, the negative effect on immediate well-being of spending money for the ticket just offsets the positive effect of higher period-2 well-being. But at 0, since the agent cares *directly* about well-being beyond 1, he weighs more the positive effect of the Hawaii trip, thus strictly preferring to buy the ticket at 1. More generally, this mechanism can lead to situations in which, at 1, the agent wants to revise plans made at 0.

In general, time inconsistency can take different forms. Do directly forward-looking preferences imply any specific form? By Assumption 1, we already know that here time inconsistency cannot arise because preferences change over time. For example, it cannot be that at 0 the agent prefers higher consumption for every period, but at 1 he prefers lower consumption for every period. Indeed, directly forward-looking preferences imply a form of time inconsistency that involves a change in intertemporal trade-offs between the present or the future. In short, it implies present bias.

**Definition 3** (Present Bias). Let \( x, y, w, h \in X \) and \( c' \in C \). Suppose that \( (x, c) \succ (y, c) \) for all \( c \) and \( (z_0, \ldots, z_{t-1}, x, w, c') \sim (z_0, \ldots, z_{t-1}, y, h, c') \) for some \( t > 0 \), then \( (x, w, c') \succ (y, h, c') \).

Intuitively, this definition says the following. Suppose that, fixing consumption in all other periods, the agent always strictly prefers consumption \( x \) to \( y \). Also, suppose there
is a consumption $h$ (e.g., a trip to Hawaii) that can make him indifferent at 0 between getting $x$ or $y$ at some future $t$, provided $y$ is ‘compensated’ with $h$ at $t + 1$. Then, given the same choice at 0, a present-biased agent continues to strictly prefer $x$ to $y$, thus pursuing immediate gratification.

**Proposition 2.** If $V(c_0, U(1c), U(2c), \ldots)$ is strictly increasing in $U(tc)$ for all $t > 0$, then $\succ$ exhibits present bias.

As shown in the proof of Proposition 2, its conclusion continues to hold if in Definition 3 both $w$ and $h$ occur at some periods $s$ after $t + 1$.

Though perhaps surprising, this result has a simple intuition. Set $t = 1$ in Definition 3. To offset the better time-1 prospect of the stream with $x$ and make the agent indifferent at 0, the stream with $y$ at 1 must provide more future well-being. Thus, this stream relies more on future well-being to induce indifference at 0. As the streams are brought forward in time, they both lose some of their future well-being value. But this effect penalizes more the stream with $y$, which relied more on future well-being. As a result, the stream with $x$ is now strictly preferred.

### 2.3 Time-separable, Stationary Preferences

To obtain sharper results, we now refine the general representation $V$ in (4), by considering preferences that satisfy some form of time separability and stationarity. These properties will also yield tractability.

The first axioms captures the idea that $\succ$ is time separable—that is, separable in immediate consumption and future well-being, as well as across future well-being. Intuitively, this means that future well-being does not affect how the agent enjoys immediate consumption, nor does well-being at $t$ affect how he enjoys, at time 0, the well-being at another time $s$. Though plausible, these interdependences may work in different directions. Favoring anyone seems, at this stage, arbitrary. So the axiom rules them out. To state Axiom 6, let $\Pi$ consist of all unions of subsets of $\{\{1\}, \{2\}, \{3, 4, \ldots\}\}$.

**Axiom 6** (Immediate-Consumption and Well-Being Separability). *Fix any $\pi \in \Pi$. If $c, \hat{c}, c', \hat{c}' \in C$ satisfy*

(i) $tc \sim tc$ and $tc' \sim tc'$ for all $t \in \pi$,

(ii) $tc \sim tc'$ and $\hat{tc} \sim \hat{tc}'$ for all $t \in \mathbb{N} \setminus \pi$,

(iii) either $c_0 = \hat{c}_0$ and $\hat{c}_0 = \hat{c}'_0$, or $c_0 = c'_0$ and $\hat{c}_0 = \hat{c}'_0$,

*then $c \succ c'$ if and only if $\hat{c} \succ \hat{c}'$.*
To illustrate Axiom 6, consider $\pi = \{1\}$. Suppose that $c$ and $c'$ yield the same immediate consumption and well-being in all periods except at 1, and that the agent prefers $c$ to $c'$. If for both $c$ and $c'$ we do not change well-being at 1 but change immediate consumption and well-being in all other periods in the same way, thus obtaining $\hat{c}$ and $\hat{c}'$, according to the axiom the agent should prefer $\hat{c}$ to $\hat{c}'$. Axiom 6 is inspired by Debreu’s (1960) and Koopmans’ (1960) separability axioms. It differs in requiring that certain consumption streams be indifferent, rather than that certain consumption events be equal. This is because we want separability in well-being.

Axiom 7 captures some weak and natural monotonicity property of the preference. First, everything else equal, the agent is better off today if he anticipates being better off tomorrow. Second, if the agent prefers the initial part up to time $T$ of a stream $c$ to that of a stream $c'$ for any $T$, then he also prefers $c$ to $c'$.

**Axiom 7 (Monotonicity).** Let $c$ and $c'$ be any stream in $C$.

(i) If $c_0 = c'_0$, $1c \succsim 1c'$, and $c \sim t c'$ for all $t > 1$, then $c \succsim c'$; if in addition $1c \succ 1c'$, then $c \succ c'$.

(ii) If for any $T$ and continuation stream $c''$ we have $(c_0, c_1, \ldots, c_T, c'') \succsim (c'_0, c'_1, \ldots, c'_T, c'')$, then $c \succsim c'$.

Note that the EDU model satisfies Axioms 6 and 7.

**Two Notions of Stationarity**

Alternative stationarity axioms can be stated, leading to fundamentally different models. If we add Koopmans’ (1960) notion of stationarity to Axioms 5 and 6, we obtain the standard model of intertemporal preferences: a representation of the form $U(c) = \hat{V}(u(c_0), U(1c))$ with $\hat{V}$ strictly increasing in each argument. We discuss this notion at the end of the section and explain why it appears problematic to us.

Our new notion of stationarity takes the theory to a different path, ruling out the standard model. The notion is based on Axiom 8, which captures the idea that $\succ$ depends on future well-being in a stationary way. Intuitively, stationarity means that, at time 0, the agent does not change how he ranks consumption events, simply because they are moved to a subsequent date. In the general representation $V$, however, how the agent trades off at 0 well-being at, say, $t$ and $t + 1$ can depend on $t$. But this non-stationarity limits the model’s tractability and applicability across settings. Of course, requiring stationarity is reasonable only if postponing a consumption event does not change its nature. Since in this paper instantaneous consumption and future well-being
are conceptually different, Axiom 8 requires stationarity only with respect to future well-being.

**Axiom 8 (Well-Being Stationarity).** If \( c, c' \in C \) satisfy \( c_0 = c'_0 \) and \( 1c \sim 1c' \), then

\[
(c_0, 2c) \succ (c'_0, 2c') \iff c \succ c'.
\]

To gain intuition for Axiom 8, suppose \( c \) and \( c' \) differ only in well-being from period 2 onward. Then, if we drop consumption at 1 and shift both continuation streams back one period, the agent has to make the same well-being comparisons only starting one period earlier. Hence he should rank the new streams as he ranked \( c \) and \( c' \). Clearly, the standard EDU model violates Axiom 8: in that model, the premises of the axiom imply that \( c \sim c' \), but at the same time \( (c_0, 2c) \) and \( (c'_0, 2c') \) need not be indifferent.

These axioms lead to the representation in Theorem 3, one of the paper’s main results. Intuitively, by the theorem it is as if the agent derives an instantaneous utility \( u \) from immediate consumption, as usual, and a per-period anticipatory utility \( G \) from future well-being, which he discounts exponentially. Moreover, \( G \) is bounded, so future well-being can have only a limited impact on current well-being. Intuitively, the agent cannot become infinitely happy or unhappy just from imagining his future well-being. No bound applies, however, to the instantaneous utility.

**Theorem 3 (Additive Well-Being Representation).** Axioms 1-7 hold if and only if the function \( U \) may be chosen so that

\[
U(c) = u(c_0) + \sum_{t=1}^{\infty} \alpha^t G(U(tc)) \tag{5}
\]

where \( \alpha \in (0, 1) \), \( u : X \to \mathbb{R} \) and \( G : U \to \mathbb{R} \) are continuous, nonconstant functions, and \( G \) is strictly increasing and bounded. Moreover, if \( \hat{U} \), \( \hat{u} \), \( \hat{\alpha} \), and \( \hat{G} \) represent the same \( \succ \) as in (5), then \( \hat{\alpha} = \alpha \) and there exist \( a > 0 \) and \( b \in \mathbb{R} \) such that \( \hat{U}(c) = aU(c) + b \), \( \hat{u}(x) = au(x) + b \), and \( \hat{G}(\hat{U}) = aG(\hat{U} - b/a) \) for all \( c, x \), and \( \hat{U} \).

The theorem’s proof relies on known results in Debreu (1960) and Koopmans (1960). However, a complication arises in our setting. In general, the set \( \mathcal{F} \) of streams of future well-being induced by consumption streams in \( C \) (see (3)) is not a Cartesian product, as well-being at \( t \) depends on well-being after \( t \). To deal with this, roughly, the key is to show that (i) if we take a stream \( f \) in \( \mathcal{F} \), there is an open neighborhood of \( f \) which belongs to \( \mathcal{F} \) and has the structure of a Cartesian product, and (ii) it is possible to ‘cover’ \( \mathcal{F} \) with countably many of such neighborhoods which intersect with each other. Given (i) and (ii), we can obtain a preliminary additive representation on each neighborhood. Relying on
these representations’ uniqueness up to positive affine transformations, we can then ‘glue’ all of them into a single representation over the entire set $\mathcal{F}$.

One might wonder whether expression (5) is always well defined for any function $G$. By Theorem 1 and Axiom 4, $U$ is a nonconstant representation of $\succ$ with values in the interval $U \subset \mathbb{R}$ since $X$ is connected. Therefore, there always exist streams $c$ such that $U(c)$ is bounded. This implies joint restrictions on $\alpha$ and $G$. Proposition 3 identifies a sufficient (and almost necessary) restriction for (5) to be well defined. It also shows that the function $U$ in (5) is such that the effect on current well-being of changes in future consumption becomes arbitrarily small, if such changes occur sufficiently far in the future.

**Definition 4.** A function $U : C \rightarrow \mathbb{R}$ is $H$-continuous if, for every $\varepsilon > 0$, there exists a time $T(\varepsilon)$ such that the following holds: if $c, \tilde{c} \in C$ satisfy $c_t = \tilde{c}_t$ for $t \leq T(\varepsilon)$, then $|U(c) - U(\tilde{c})| < \varepsilon$.\(^{18}\)

**Proposition 3.**

(i) In representation (5), $U$ is $H$-continuous and for $\nu', \nu \in U$

$$|G(\nu') - G(\nu)| < \frac{1 - \alpha}{\alpha} |\nu' - \nu|.$$  

(ii) Suppose $G$ is strictly increasing, bounded, and $K$-Lipschitz continuous with $K < \frac{1 - \alpha}{\alpha}$, i.e., for all $\nu', \nu \in U$

$$|G(\nu') - G(\nu)| \leq K |\nu' - \nu|.$$  

Then, there exists a unique $H$-continuous function $U : C \rightarrow \mathbb{R}$ that solves (5).

This result helps to choose $G$ appropriately in applications. Moreover, it has several behavioral implications which we present shortly.

**Consumption Stationarity and the Standard Model**

Koopmans’ (1960) model of intertemporal preferences is based on the general representation $U(c) = \hat{V}(u(c_0), U(1c))$ with $\hat{V}$ strictly increasing in each argument. To obtain such a representation in the framework of the present paper, it is enough to add Koopmans’ stationarity axiom to Axioms 5 and 6. Note that Axiom 9 involves only a single preference (in particular, that at $t = 0$).

**Axiom 9** (Consumption Stationarity). $1c \succ^0 1c'$ if and only if $(c_0, 1c) \succ^0 (c_0, 1c')$.

\(^{18}\)This notion is similar to that of “continuity at infinity” of payoff functions in infinite-horizon games (see, e.g., Fudenberg and Tirole (1991)).
By time invariance (Assumption 1), Axiom 9 is mathematically equivalent to time consistency (Definition 2). Hence, together with Axiom 6, it is equivalent to \( U(c) = \hat{V}(u(c_0), U(1c)) \) by Proposition 1. Of course, as Koopmans (1960) showed, if we wanted to specialize \( \hat{V} \) to obtain the EDU representation, we would need stronger separability assumptions.

It is worth emphasizing that, despite its formal similarity with time consistency, Axiom 9 is conceptually very different. Indeed, Koopmans writes,

“[Stationarity] does not imply that, after one period has elapsed, the ordering then applicable to the ‘then’ future will be the same as that now applicable to the ‘present’ future. All postulates are concerned with only one ordering, namely that guiding decisions to be taken in the present. Any question of change or consistency of preferences as the time of choice changes is therefore extraneous to the present study.” (Koopmans et al. (1964), p. 85, emphasis in the original)

It is straightforward to construct examples of preferences \( \{\succ^t\}_{t=1}^\infty \) that satisfy Axiom 9 but are not time consistent, and vice versa.

3 Implications of the Additive Representation

This section focuses on the additive well-being representation of Theorem 3. We show that it implies a number of additional properties of the agent’s preference and it can be easily applied to study dynamic choice problems. We also show how a special case of representation (5) corresponds to the \( \beta-\delta \) model.

3.1 Impatience

Proposition 3 implies that a directly forward-looking agent is always impatient, even though his future well-being at each date depends on subsequent well-being and he correctly anticipates this.

**Definition 5** (Impatience). Let \( x, y \in X \) be such that \((x, c) \succ (y, c)\) for all \( c \in C \). Then \( \succ \) exhibits impatience if, for any \( t > 0 \), \( c^x \succ c^y \) where \( c^x_0 = x \), \( c^x_t = y \), \( c^y_0 = y \), \( c^y_t = x \), and \( c^z_s = c^z_s \) otherwise.

Impatience differs from present bias (Definition 3): impatience refers to a trade-off between achieving higher satisfaction at earlier rather than later periods; present bias refers
to how this trade-off changes when the earlier period occurs in the present rather than in the future.

**Corollary 1.** If axioms 1-7 hold, then $\succ$ exhibits impatience.

Representation (5) satisfies further properties in relation to present bias and impatience. These properties imply that it is enough to know how the agent resolves an intertemporal trade-off in the period right before it occurs, to know how he resolves it in all previous periods. First, suppose that at time $0$ the agent is indifferent between $c$ and $c'$ which involve, at some $t > 0$, the same trade-off as in Definition 3. Then, he is also indifferent between $sc$ and $sc'$ at all $s < t$. Second, at any $t > 0$, the agent will be impatient and will prefer anticipating at $t$ better future consumption as stated in Definition 5. But does he also want to anticipate better consumption at $t$ when considering this possibility before $t$? Because of time inconsistency, the answer can go either way. However, if at $t - 1$ the agent does (not) want to anticipate better consumption at $t$, then he also does (not) at all $s < t$. To see that impatience can fail with respect to future trade-offs, consider the following example.

**Example 1** (Violation of impatience in the future). $X = [0, 200]$ and, in representation (5), $\alpha = \frac{1}{2}$, $u(x) = x$, and $G(\nu) = \nu$ for $\nu \leq 100$ and $G(\nu) = 100$ otherwise.\(^{19}\) Let $x = 100$ and $y = 50$. Consider the streams $c = (0, 50, 100, 100, 0, 0, \ldots)$ and $c' = (0, 100, 100, 50, 0, 0, \ldots)$. Straightforward calculations yield $U(1c) = 125$ and $U(1c') = 162.5$; so at time 1 the agent is impatient according to our definition. However, $U(c) = 87.5$ and $U(c') = 81.25$; so, at time 0, he prefers not to anticipate at time 1 the better future consumption $x$.

### 3.2 A Bellman-type Equation for Dynamic Choice Problems

To see how to work with representation (5), consider the following consumption-saving problem. For expositional simplicity, we formulate it as a cake-eating problem—it should be clear that the method described here can be generalized to other Markovian decision problems. At time 0, the agent must commit to a stream $(c_0, c_1, \ldots) \in \mathbb{R}_+^N$ subject to the constraint $\sum_{t \geq 0} c_t \leq b$, where $b$ is the cake size. Let $C(b)$ be the set of all nonnegative consumption streams satisfying this constraint. Based on representation (5), the optimal utility is given by

$$U^*(b) = \sup_{\omega \leq b} \{ u(c_0) + \alpha W(b - c_0) \},$$

\(^{19}\)We choose $\alpha$ and $G$ in this way to keep the example simple. It is easy to see that $\alpha$ and $G$ can be slightly modified so as satisfy the properties stated in Theorem 3 and Proposition 3, without changing the point of the example.
where

\[ W(b') = \sup_{c' \in C(b')} \sum_{t=0}^{\infty} \alpha^t G(U_t(c')). \]

If we can solve for \( W \), we can then easily determine the optimal consumption plan. Note that, for any \( b \geq 0 \), we can express \( W(b) \) as

\[ W(b) = \sup_{c_0 \leq b} \left\{ \sup_{c' \in C(b-c_0)} \left( G \left( u(c_0) + \alpha \sum_{t=0}^{\infty} \alpha^t G(U_t(c')) \right) + \alpha \sum_{t=0}^{\infty} \alpha^t G(U_t(c')) \right) \right\}. \]

With increasing \( G \), this yields the following Bellman-type equation for \( W \):

\[ W(b) = \sup_{c_0 \leq b} \left\{ G(u(c_0) + \alpha W(b-c_0)) + \alpha W(b-c_0) \right\} \quad (7) \]

Given \( W \), the maximization in (6) determines the optimal \( c_0 \) and that in (7) determines \( c_t \) for all \( t > 0 \).

Equation (7) differs from usual Bellman equations mainly because the instantaneous-utility term is inside the function \( G \). Indeed, it reduces to a standard equation if \( G \) is linear. However, under minor regularity conditions on \( G \), (7) has a well-defined solution \( W \). To see this, define the operator \( J \) on the set \( B(\mathbb{R}_+) \) of bounded real-valued functions of \( \mathbb{R}_+ \) by

\[ J(W)(b) = \sup_{c_0 \leq b} \left\{ G(u(c_0) + \alpha W(b-c_0)) + \alpha W(b-c_0) \right\}. \]

Then, if \( G \) is bounded and \( K \)-Lipschitz continuous with \( K < (1-\alpha)/\alpha \), it is easy to show that \( J \) is a contraction and therefore has a unique fixed point. So equation (7) uniquely defines \( W \). It is straightforward to approximate numerically this fixed-point, which is just a univariate function, and the rate of convergence of numerical schemes is given as a function of the Lipschitz constant of \( G \).

When the agent cannot commit at 0, time inconsistency leads to an equilibrium problem, in which he chooses \( c_t \) at each \( t \). Existence and properties of Markovian equilibria in a similar setting—the ‘buffer-stock model,’ which includes stochastic shocks to the state (\( b \) here)—have been studied by Ray (1987), Bernheim and Ray (1989), Harris and Laibson (2001), and Quah and Strulovici (2013). Bernheim and Ray study a set of utility functions that includes those in Theorem 3, so their equilibrium analysis applies to the preferences studied here.

### 3.3 Intertemporal Rate of Utility Substitution

Representation (5) has interesting implications on how the agent trades off consumption across periods. By Axiom 6, separability holds between immediate consumption and
future well-being, as well as across future well-being. Nonetheless, the trade-off between consumption at 0 and \( t \) can depend on well-being—and hence consumption—before and after \( t \). This is because well-being at \( t \) affects well-being at all \( s < t \) and depends on well-being after \( t \).

To examine this, we consider the agent’s discount factor between 0 and \( t \). Of course, intertemporal consumption trade-offs also depend on the curvature of \( u \). To bypass this dependence, first note that by Theorem 3, given \( \alpha \) and \( G \), the preference is entirely driven by the instantaneous utility \( u \).

**Corollary 2 (\( u \)-Representation).** Given representation (5), there exists a nonconstant function \( \hat{U} : I_u^N \rightarrow \mathbb{R} \) (where \( I_u \) is \( u \)'s range) such that, for all \( c \in C \),

\[
U(c) = \hat{U}(u(c_0), u(c_1), \ldots).
\]

Relying on this result, given stream \( c \), define \( u_s = u(c_s) \) and the discount factor as

\[
d(t, c) = \frac{\partial \hat{U}(u_0, u_1, \ldots)/\partial u_t}{\partial \hat{U}(u_0, u_1, \ldots)/\partial u_0}.
\]

That is, \( d(t, c) \) is the marginal rate at which the agent substitutes instantaneous utility between 0 and \( t \). Note that in the EDU model \( d(t, c) = \delta^t \). For \( d(t, c) \) to be well defined, the derivatives in (8) must exist. This is always the case when \( G \) is differentiable.\(^{20}\)

**Proposition 4.** Suppose \( G \) in representation (5) is differentiable. Then, \( d(1, c) = \alpha G'(U(1c)) \) and, for \( t > 1 \),

\[
d(t, c) = \alpha^t G'(U(tc)) \left[ 1 + \sum_{\tau=1}^{t-1} G'(U(t-\tau c)) \prod_{s=1}^{\tau-1} (1 + G'(U(t-s c))) \right],
\]

where \( \prod_{s=1}^{\tau-1} (1 + G'(U(t-s c))) \equiv 1 \) if \( \tau = 1 \).

This formula has a simple explanation. Suppose \( u(c_t) \) rises by a small amount. This has two effects: (1) well-being rises at \( t \), which explains the term \( G'(U(tc)) \); consequently, (2) well-being rises for all \( \tau \) between period 1 and \( t \), which explains the summation in (9). Moreover, the rise in \( U(tc) \) affects \( U(t-\tau c) \) through all well-beings between \( t - \tau \) and \( t \), which explains the product in (9).

By Proposition 4, for general \( G \), the discount factor \( d(t, c) \) depends on consumption at \( t \) as well as on well-being before and after \( t \)—hence it depends on the entire stream \( c \) (see Section 3.5 for further discussion). In other words, what the agent expects to get in periods other than \( t \) creates a ‘reference effect’ (through anticipations) which affects the

\(^{20}\)Note that, being increasing, \( G \) is already differentiable at almost every point in \( U \).
trade-offs involving $c_t$. The only case in which $d(t, c)$ is independent of $c$ is when $G$ is linear. Surprisingly, in this case, the discount factor takes a very well-known form.

**Corollary 3.** Suppose $G(U) = \gamma U$ with $\gamma \in (0, \frac{1}{1+\alpha})$. Then, for all $t > 0$,

$$d(t, c) = \beta \delta^t,$$

where $\beta = \frac{\gamma}{1+\gamma}$ and $\delta = (1 + \gamma)^{-\alpha} < 1$.

**Proof.** By Proposition 4, the result is immediate for $t = 1, 2$. For $t > 2$,

$$d(t, c) = \alpha^t \gamma \left[ 1 + \gamma \sum_{\tau=1}^{t-1} (1 + \gamma)^{-\tau} \right] = \alpha^t \gamma (1 + \gamma)^{t-1}.$$

In the working-paper version of this paper (Galperti and Strulovici (2014)), we examine more general forms in which the agent may discount instantaneous utility and their relationship with how he cares about future well-being. We illustrate this relationship in the case of hyperbolic discounting, showing that it is enough that an agent cares slightly about well-being only a few periods beyond time 1, for him to behave as if he discounts future consumption hyperbolically.

### 3.4 Quasi-hyperbolic Discounting of Instantaneous Utility

Corollary 3 raises a natural question: Which properties of $\succ$ correspond to linearity of $G$ and hence to quasi-hyperbolic discounting? As noted, unless $G$ is linear, the trade-off between utility from consumption at 0 and at $t$ depends on well-being before and after $t$. This observation suggests Axiom 10.

**Axiom 10** (Trade-off Independence).

(i) $(c_0, c_1, 2c) \succ (c'_0, c'_1, 2c)$ if and only if $(c_0, c_1, 2c) \succ (c'_0, c'_1, 2c)$;

(ii) $(c_0, c_1, 2c) \succ (c'_0, c_1, 2c)$ if and only if $(c_0, c'_1, 2c) \succ (c'_0, c'_1, 2c)$.

Intuitively, condition (i) says that the trade-off between consumption at 0 and at 1 is independent of the continuation stream $2c$ and hence of future well-being. Condition (ii) says that the trade-off between consumption at 0 and after 1 is independent of consumption at 1 and hence of well-being at 1.

**Theorem 4** (‘Vividness’ Well-being Representation). Axiom 1-10 hold if and only if the function $U$ may be chosen so that

$$U(c) = u(c_0) + \sum_{t=1}^{\infty} \alpha^t \gamma U(\mu c),$$

(10)
where $\alpha \in (0, 1)$, $\gamma \in (0, \frac{1-\alpha}{\alpha})$, and $u : X \to \mathbb{R}$ is a continuous nonconstant function.

**Corollary 4** (Quasi-Hyperbolic Discounting). Axiom 1-10 hold if and only if there are $\beta, \delta \in (0, 1)$ and a continuous nonconstant function $u : X \to \mathbb{R}$ such that

$$U(c) = u(c_0) + \beta \sum_{t=1}^{\infty} \delta^t u(c_t).$$

**Proof.** By Theorem 4, for all $t$, $U(c)$ is a strictly increasing, linear function of $U(t, c)$, which is in turn a strictly increasing, linear function of $u(c_t)$. Hence, there exists a function $\kappa(t) : \mathbb{N} \setminus \{0\} \to \mathbb{R}_{++}$ such that

$$U(c) = u(c_0) + \sum_{t=1}^{\infty} \kappa(t) u(c_t).$$

Clearly, for all $t > 0$, $\kappa(t) = d(t, c)$ defined in (8). Corollary 3 implies the result. \qed

This result allows us to understand $\beta$-$\delta$ discounting of instantaneous utility in terms of simple properties of directly forward-looking preferences. First, a $\beta$-$\delta$ agent treats immediate consumption and future well-being as different entities. This explains the stark difference between discounting of future instantaneous utility from today’s perspective and from any future period’s perspective, which otherwise seems ad hoc and hard to justify. Second, a $\beta$-$\delta$ agent cares directly about well-being in all future periods, in a stationary way. This explain present bias: $\beta < 1$. Finally, he treats in a separable way immediate consumption and future well-being as well as well-being across future periods, and trades off consumption between any two periods in a way that does not depend on intermediate and future well-being. This explains additive time separability in instantaneous utility.

This explanation of quasi-hyperbolic discounting differs from the usual one, which views the agent as caring disproportionately about the present against any future period—if anything, directly forward-looking preferences capture the opposite. Moreover, Corollary 4 tightly links the degree of present bias, $\beta$, with the agent’s ex-ante marginal utility from future well-being, $\gamma$. We can also interpret $\gamma$ as the degree to which the agent finds future well-being ‘imaginable’ or ‘vivid.’ This interpretation relates to Böhm-Bawerk’s (1890) and Fisher’s (1930) idea that an agent’s current utility depends on immediate consumption as well as on his ability to imagine his future ‘wants.’

Corollary 4 has other implications. First, for any degree of present bias $\beta$, the agent discounts less instantaneous utility than future well-being in the long run: $\alpha = \delta(1 - \beta)$. So, for reasonable values of $\beta$, well-being discounting is much steeper than one might
think by looking only at consumption discounting. Second, Corollary 4 can explain why increasing vividness of future well-being can mitigate present bias.

**Corollary 5.** For any well-being discount factor $\alpha$, increasing vividness $\gamma$ mitigates present bias and long-run discounting of instantaneous utility: both $\beta$ and $\delta$ increase.

This corollary suggests the possibility of studying endogenous determination of present bias, in the spirit of how Becker and Mulligan (1997) study endogenous determination of impatience. In short, the ex-ante vividness of future well-being may depend on costly imagination effort by the agent and hence on his wealth, education, or even religion. Exploring this possibility is, however, beyond the scope of the present paper.

**Comparison with Other Axiomatizations of $\beta$-$\delta$ Discounting**

Existing axiomatizations of the $\beta$-$\delta$ model (see, e.g., Hayashi (2003); Olea and Strzalecki (2014)) differ from ours as follows. First, they continue to view the agent as caring about instantaneous utilities. Within this framework, they replace Koopmans’ (1960) stationarity (Axiom 9) with quasi-stationarity, namely stationarity from time 1 onward. However, quasi-stationarity seems difficult to justify when the agent evaluates streams based only on instantaneous utilities. If he views consumption in the same way in all periods, why should stationarity hold between tomorrow and the day after, but not between today and tomorrow? This issue does not arise with well-being stationarity (Axiom 8), for in our model tomorrow’s well-being is equivalent to well-being thereafter, but is essentially different from today’s consumption.

Second, to obtain the $\beta$-$\delta$ representation, Olea and Strzalecki (2014) need to ensure that current and future instantaneous utilities are cardinally equivalent. Their ingenious axioms permit useful experiments to identify and measure $\beta$ and $\delta$, but seem difficult to interpret. By contrast, the present paper starts by viewing the agent as caring about

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For example, using Laibson et al.’s (2007) estimates of $\beta = 0.7$ and $\delta = 0.95$, we get $\gamma = 2.33$ and $\alpha = 0.285$.

Vividness of the future well-being implied by today’s decisions may be influenced with specific ad campaigns. For example, consider the dramatic pictures and reminders printed on cigarette packs. It is hard to believe that such packaging is just meant to inform unaware customers of the consequences of smoking.

This seems, at least, the most natural interpretation of the second paper, which is based on the idea of annuity compensation: to avoid relying on assumptions of the specific form of $u$ to elicit $\delta$ and $\beta$ separately, the idea is to consider fix compensation levels and hence $u$’s and vary the time horizon at which they occur, so as to find exact points of indifference for the agent and hence infer the parameters of the model. The agent has different subjective views of the time distance between time 0 and 1 and between any two future periods. If he cares only about the $u$’s that he gets in each period, then it is possible to objectively space out these $u$’s in an appropriate way so as to get $\beta$ and $\delta$. 

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immediate consumption and future well-being (i.e., preferences). From this different viewpoint, it shows that $\beta$-$\delta$ discounting is tightly linked with a condition (Axiom 10) which has an intuitive interpretation: intertemporal consumption trade-offs do not depend on intermediate and future well-being.

Echenique et al. (2014) propose a different, interesting method to characterize several models of intertemporal choice, including the EDU and $\beta$-$\delta$ model. Their starting point is a data set consisting of an agent’s choices of finite consumption streams from standard budget sets and the prices of consumption at all dates that define such sets. They then identify versions of the Generalized Axiom of Revealed Preference that the data must satisfy to be consistent with the EDU and $\beta$-$\delta$ model, respectively. Using real data from the experiment in Andreoni and Sprenger (2012), they apply their axioms to classify subjects across models. They conclude that roughly only half of the 97 subjects are consistent with either EDU or $\beta$-$\delta$ discounting and about a third violate time separability in consumption.

3.5 Consumption Interdependence and Related Phenomena

Representation (10) is appealing, for it corresponds to a widely used and tractable model which captures well-documented behavioral phenomena related to present bias. Quasi-hyperbolic discounting, however, cannot capture other phenomena that appear as anomalies through the lenses of EDU. Instead, representation (5) can capture and, most importantly, can offer a general explanation for some of these anomalies.

According to some evidence, intertemporal preferences can exhibit consumption interdependences across dates, in contrast with both EDU and the $\beta$-$\delta$ model (again, see Frederick et al. (2002)). Clearly, a general non-separable representation like that in Theorem 1 allows for such interdependences, but offers no insight on their origin. By contrast, representation (5) identifies a specific source of interdependences: although the agent treats each period separately with regard to well-being, how he evaluates consumption at a future $t$ can depend on well-being, and hence consumption, before and after $t$. This can explain forward- and backward-looking interdependences that arise only in the future.\footnote{Such interdependences differ from other backward-looking interdependences, like habit formation, which this paper cannot capture.}

Such interdependences can create a preference for enjoying consumption events with similar instantaneous utility far apart in time.

Example 2 (Spacing out consumption). Let $X$, $u$, $\alpha$, and $G$ be as in Example 1. Consider
\(c = (50, 100, 50, 100, 0, 0, \ldots)\) and \(c' = (50, 100, 100, 50, 0, 0, \ldots)\). Simple calculations yield \(U(c) = 137.5\) and \(U(c') = 131.75\). Therefore, the agent prefers to space out consumption, alternating periods of high consumption with periods of low consumption.

The same interdependences can also create a preference for delaying future pleasurable consumption events.\(^{25}\)

**Example 3** (Delaying future pleasurable consumption). Let \(X\), \(u\), \(\alpha\), and \(G\) be as in Example 1. Suppose that the agent is getting consumption of 50 in all periods 0 through 3 and no consumption afterwards. Now we offer him to get an extra 50 units of consumption at one period between 1 and 3. Which period does the agent choose? Note that \(U(50, 50, 50, 0, \ldots) = 100\), therefore he does not benefit by the extra consumption at time 1. Also, \(U(50, 50, 0, \ldots) = 75\) and \(U(50, 100, 0, \ldots) = 100\). So the agent prefers to delay the extra consumption until time 3.

This desire for spacing out consumption and delaying future pleasurable events is consistent with some empirical evidence. For example, Frederick et al. (2002) discuss the following experiment (p. 364). There are five periods and two opportunities to dine at a fancy restaurant: French (F) and Lobster (L). Subjects are asked to rank consumption streams in two scenarios. In scenario 1, one stream features F at time 1 and the other features F at time 3; in all other periods, both streams involve dining at home. Scenario 2 coincides with scenario 1, except that both streams feature L at time 5. Presumably, dining at a fancy restaurant yields higher instantaneous utility than dining at home. According to the experiment, in scenario 1, most of the subjects prefers having F at 3. But in scenario 2 a significant fraction of them prefers having F at 1. This change is consistent with a concave \(G\)—perhaps the most natural case, since it simply means that the marginal value at 0 of well-being at any \(t > 0\) is decreasing. In scenario 1, an agent may prefer to delay F, for the improvement in well-being both at 2 and at 1 may offset discounting. But, when well-being at 5 is higher because of L, well-being at 3 is also higher. Consequently, the benefit of improving time-3 well-being by delaying F is lower than when L is absent; hence the agent prefers enjoying F at time 1.\(^{26}\)

Preferences with representation (5) can also lead the agent to prefer increasing to decreasing consumption streams (see Frederick et al. (2002)).

\(^{25}\)In infinite-horizon settings, by Corollary 1, the agent never wants to delay delightful events from the present. This can happen, however, in a finite-horizon version of representation as shown in the appendix of the working-paper version of this paper (Galperti and Strulovici (2014)).

\(^{26}\)Of course, another explanation of how people behave in this restaurant experiment is that they prefer not to eat similar food relatively close in time. A better experiment to test this paper’s predictions would replace option L with another enjoyable event, different in nature from F (e.g., a concert of the agent’s favorite band).
Example 4 (Preference for increasing consumption streams.). Let $X = [0, 200]$ and, in representation (5), $\alpha = 0.9$, $u(x) = x$, and $G(\nu) = 0.1 \times \nu$ for $\nu \leq 100$ and $G(\nu) = 10$ otherwise. There are only four periods: 0 to 4. Consider $c = (0, 82.9, 91, 100)$ and $c' = (0, 110, 92.8, 80)$. Some algebra yields $U(91, 100) = 100$, $U(82.9, 91, 100) = 100$, $U(92.8, 80) = 100$, and $U(110, 92.8, 80) = 125.48$. Therefore, $U(c) = 25.39$ and $U(c') = 22.94$. It follows that the agent prefers the increasing stream $c$, even though $c'$ gives a higher overall consumption: 282.8 against 273.9. Finally, note that an EDU agent with $\delta = 0.9$ and $u(x) = x$ strictly prefers $c'$ to $c$: their present discounted values are respectively 232.49 and 221.22.

Another anomaly that representation (5) can accommodate is discount factors which depend on the level of consumption itself (see Frederick et al. (2002)). By Proposition 4, this phenomenon would occur because how much an agent discounts instantaneous utility from $c_t$ depends on how much his ex-ante utility from well-being at $t$ (i.e., $G(U(c))$) responds to changes in $c_t$, which may depend on the level of $c_t$. More generally, $d(t, c)$ can depend on the entire stream $c$. This may capture Fisher’s (1930) idea that “[an agent’s] degree of impatience for, say, $100$ worth of this year’s income over $100$ worth of next year income depends upon the entire character of his [...] income stream pictured as beginning today and extending into the indefinite future.” (Ch. 4, §5) How $d(t, c)$ varies with $c$ ultimately depends on the properties of $G$. If $G'$ is decreasing (increasing), then the agent discounts more a stream yielding higher (lower) instantaneous utility in all periods. To state this formally, for any $c, c' \in C$, let $c \geq_u c'$ if and only if $u(c_t) \geq u(c'_t)$ for all $t \geq 0$.

Corollary 6. Let $d(t, c)$ be as in Proposition 4. For any $t > 0$, $c \geq_u c'$ implies $d(t, c) \leq d(t, c')$ if and only if $G'$ is decreasing. Conversely, $c \geq_u c'$ implies $d(t, c) \geq d(t, c')$ if and only if $G'$ is increasing.

This result shows a tight link between discounting and $G$’s curvature, which may be empirically tested. For instance, a concave $G$ implies that, after learning that he will lose his job in a year, a worker may become more patient—i.e., discount less. Similarly, after learning that she will receive a large bequest upon reaching adulthood, a minor may become less patient—i.e., discount more.

Finally, representation (5) can explain other empirical findings summarized by Frederick et al. (2002). The sign effect, for example, means that future gains are discounted more than future losses. When $G$ is concave, improving consumption at some time $t$ reduces the discount factor between 0 and $t$. Hence, from EDU’s viewpoint, it appears as if the
discount rate increases. Intuitively, the discount factor captures an indifference point in
the trade-off between instantaneous utility at 0 and at \( t \). If such a utility at \( t \) is higher,
this indifference occurs at a lower point, for the time-0 marginal value of well-being at \( t \)
is lower—in a sense, improving future well-being matters less.

4 Welfare Criteria and Normative Analysis

Models that allow for time-inconsistent preferences pose serious conceptual problems
when defining welfare criteria and addressing policy questions (see, e.g., Rubinstein
(2003); Bernheim and Rangel (2007, 2009)). Discussing \( \beta - \delta \) discounting, Rubinstein
(2003) notes,

“Policy questions were freely discussed in these models even though welfare assess-
ment is particularly tricky in the presence of time inconsistency. The literature
often assumed, though with some hesitation, that the welfare criterion is the utility
function with stationary discounting rate \( \delta \) (which is independent of \( \beta \)).” (p. 1208)

Another, perhaps more fundamental, issue is whether the existence of agents with time-
inconsistent preferences justifies some form of paternalistic intervention. An immediate
consequence of the present paper is to weaken the case for such interventions. If time
inconsistency were the result of some bounded rationality, one might be tempted to
argue that time-inconsistent agents can benefit from paternalistic interventions. This
argument, however, is not valid if time inconsistency is the result of directly forward-
looking preferences. In this case, why should a planner use welfare criteria other than
the agent’s ex-ante preference? If at 0 he is already fully taking into account his future
preferences over continuation streams, then the planner may simply adopt a ‘libertarian’
stance and use \( W^L(c) = U(c) \) to measure welfare.

From this paper’s viewpoint, a libertarian criterion seems even more appropriate for
time-inconsistent agents than for time-consistent ones. Indeed, under EDU, welfare is
usually and uncontroversially measured using the agent’s current preference. This prefer-
ence, however, takes into account the agent’s preference only in the next period, but not
in subsequent ones. By contrast, the \( \beta - \delta \) model implies that the current preference takes
into account, albeit in a simple way, preferences in all future periods—of course, this also
holds for more general, directly forward-looking preferences. In this view, the ex-ante
preference implied by the \( \beta - \delta \) model may seem a more reasonable welfare criterion than
that implied by EDU. For example, consider two policies, \( A \) and \( B \), inducing streams \( c^A \)

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and $c^B$ such that $1_t c^A \sim^1 1_t c^B$ but $1_t c^A \succ^t 1_t c^B$ for all $t > 1$. The criterion based on EDU implies that at $0$ $A$ is as desirable as $B$. Instead, the criterion based on the $\beta$-$\delta$ model implies that at $0$ $A$ is strictly more desirable. Thus, the second criterion favors what we may call long-run sustainability and may therefore seem more appealing.

Using the well-being representation in Theorem 4, we can also derive the well-established welfare criterion that the literature has so far assumed for the $\beta$-$\delta$ model. Recall that ex-ante choices of a $\beta$-$\delta$ agent reveal the corresponding parameters $\alpha$ and $\gamma$. We may interpret the factor $\alpha^t$ as the agent’s assessment of how likely it is that a policy under scrutiny will continue to matter at $t$ or his survival probability. This assessment may combine subjective aspects as well as objective information, which the agent may know better than the planner. Note that this interpretation of $\alpha$ parallels one of the usual interpretations of $\delta$ in the EDU model. Moreover, in this model, the instantaneous utility $u(c_t)$ is normally viewed as the agent’s welfare at each time $t$ and hence the planner aggregates instantaneous utilities using the weights $\delta^t$. Following the same logic, in the present model, the well-being $U(c_t)$ is viewed as the agent’s welfare at each $t$ and hence the planner may aggregate well-being across periods using the weights $\alpha^t$. Surprisingly, doing so is equivalent to aggregating his instantaneous utilities using the familiar weights $\delta^t$. Thus, a natural, focal specification of welfare weights for a time-inconsistent agent delivers a time-consistent planner.

**Proposition 5.** Suppose $U(c)$ can be represented as in Theorem 4 and Corollary 4, with corresponding parameters $(u, \alpha, \gamma)$ and $(u, \beta, \delta)$. Let $w : \mathbb{N} \to \mathbb{R}_+$ and define $W(c) = \sum_{t=0}^{\infty} w(t) U(t,c)$. Then,

$$W(c) = \sum_{t=0}^{\infty} \delta^t u(c_t)$$

if and only if $w(t) = \alpha^t$.

One direction of this result follows from an intuitive argument—the other is provided in Appendix A. Let $W(c) = \sum_{t=0}^{\infty} \alpha^t U(t,c)$. Using $\alpha$-$\gamma$ representation of $U(c)$ in Theorem 4, we have

$$W(c) = u(c_0) + (1 + \gamma) \alpha \sum_{t \geq 1} \alpha^{t-1} U(t,c) = u(c_0) + (1 + \gamma) \alpha W(1,c).$$

This shows why the planner is time consistent and $W(c)$ corresponds to the sum of instantaneous utilities exponentially discounted with factor $\hat{\delta} = (1 + \gamma)\alpha$. Intuitively, the planner is time consistent because she values well-being from time 1 onward in the same way as does the agent, but for the planner this value coincides with her ‘continuation utility’ $W(1,c)$. Of course, we know from Corollaries 3 and 4 that $\hat{\delta} = \delta$ in the $\beta$-$\delta$
version of $U(c)$. However, to see why this has to be the case for any $\hat{\delta}$, note that the $\alpha$-$\gamma$ representation satisfies

$$U(c) = u(c_0) + \gamma \alpha W (1c) = u(c_0) + \frac{\gamma \alpha}{\delta} \sum_{t>0} \hat{\delta}^t u(c_t).$$

It follows that $U(c)$ must have a quasi-hyperbolic representation in terms of instantaneous utilities, where the long-run discount factor coincides with the planner’s factor.

So, for the $\beta$-$\delta$ model, this paper suggests that the arguably most natural, welfare criteria are precisely those which have been used in practice. It also sheds light on their respective implications on how the planner weighs the agent’s well-being across periods.

Since time consistency may reveal that an agent is indirectly forward-looking, a natural question is then whether the planner should weigh more his well-being beyond the immediate future than does the agent himself and, if so, by how much. In the case of EDU—i.e., $U(c) = \sum_{t=0}^{\infty} \delta^t u(c_t)$—one might rely on $\delta$ to aggregate well-being across periods using the criterion $\hat{W}(c) = \sum_{t=0}^{\infty} \delta^t U(t,c)$ (see, e.g., Ray (2014)). Doing so, however, makes the planner time inconsistent. Indeed, one can show that $\hat{W}(c) = u(c_0) + \sum_{t=1}^{\infty} \delta^t (1 + t) u(c_t).$

5 Discussion and Related Literature

It is clearly beyond the scope of economics to provide an accurate description of agents’ well-being. Nonetheless, it is equally clear that agents’ future well-being does matter to them and influences their preferences. This paper has proposed a theory which takes into account this minimal and uncontroversial observation, with the following characteristics:

1. Well-being is modeled via a standard revealed-preference argument: agents rationally choose the streams that maximize their well-being.\(^{28}\)

2. Agents have time-inconsistent preferences of a specific form: they exhibit present bias.

3. The axioms of the theory deliver a class of tractable models, characterized by a novel ingredient: agents’ anticipatory utility $G$ from future well-being. These models’

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\(^{27}\)At first glance, EDU appears as the limit of the $\beta$-$\delta$ model as $\beta \to 1$. However, by Corollary 3 and Theorem 4, in the limit the corresponding $\alpha$ and $\gamma$ must take extreme, implausible values. Therefore, this paper suggests that it seems more appropriate to think of time-consistent and time-inconsistent agents as having radically different preferences.

\(^{28}\)This approach is standard. In fact, Bernheim and Rangel (2009) argue that it is the only operationally relevant approach.
qualitative properties are easily analyzed. Examples with a piecewise-linear $G$ have shown that it may be straightforward to compute agents’ well-being explicitly. In general, the analyst may have to solve a one-dimensional fixed-point problem to compute the agent’s optimal stream, as in the cake-eating illustration considered here or in similar consumption-saving problems. This procedure is standard and belongs to the toolkit of most economists. Moreover, parametric restrictions implied by the axioms ensure that the relevant operator is a contraction, guaranteeing that the procedure will be successful and will converge at a geometric rate.

4. Agents manifest impatience toward the current period: of two consumption events, they prefer to get the better one immediately than at any future date, regardless of consumption at all other dates. Impatience comes from the stationarity axiom, applied to our infinite-horizon setting. With a finite horizon, or without stationarity, agents with anticipatory utility from future well-being may prefer to delay pleasurable events or to precipitate painful ones. With stationarity, however, the discount effect must always dominate the anticipatory effect for preferences to be well defined, resulting in impatience.

5. Agents need *not* manifest impatience with respect to future dates: they may prefer to get the better of two consumption events at the later of two dates, and whether they are impatient depends on consumption at other dates. This feature distinguishes our general theory from all time-separable models in which the discount function is decreasing in time, such as the EDU and $\beta$-$\delta$ models.

6. More generally, agents may display consumption interdependence: consumption trade-offs between any two dates may depend on what is consumed at other dates. When the anticipatory utility function $G$ is strictly concave, consumption interdependence has several implications. Improving consumption at some future date makes the agent *more impatient*. This feature is consistent with the sign effect described by Frederick et al. (2002). More generally, changing the agent’s consumption stream over any subset of dates has the effect of changing the *reference* with respect to which he evaluates consumption trade-offs at other dates. Section 3.3 provides an explicit formula for the discount factor, showing how this factor depends on the entire consumption stream.

7. With concave $G$, agents also prefer to space out consumption: higher consumption at time $t$ increases well-being at time $t - 1$, making consumption at time $t - 1$ relatively less valuable compared to other dates.
8. When the consumption-interdependence channel is shut down, our theory reduces to the $\beta$-$\delta$ discounting model, providing a new axiomatization for that model.

9. Because the present theory includes a clear concept of well-being, it provides a rigorous foundation for welfare analysis. Our agents are rationally choosing their well-being maximizing stream, a property that undermines paternalistic approaches to welfare analysis aimed at correcting agents’ bounded rationality. However, an agent committing to a consumption stream at time 0 maximizes only his immediate well-being, possibly at the expense of his well-being at future dates.\footnote{Indeed, Roessler et al. (2014) show that committing to a dynamic plan can be valuable only when such a neglect of future decision makers arises.} A social planner may then wish to \textit{directly} take into account an agent’s well-being in future periods, above and beyond how this future well-being affects his current well-being. Section 4 shows that, for $\beta$-$\delta$ agents, a natural aggregation of these well-beings results in a \textit{time-consistent} welfare criterion, used on an ad-hoc basis in previous studies.\footnote{See, e.g., O’Donoghue and Rabin (1999); Della Vigna and Malmendier (2004).}

Among the connections discussed below between our theory and existing work, \textbf{three aspects are worth emphasizing}: 1) our notion of anticipatory utility, which incorporates future consumption through future well-being, 2) the difference between time-inconsistent preferences and time-inconsistent behavior, and 3) the notion that time consistency is a desirable feature of intertemporal choice. In addition, we re-examine the EDU model in the light of these aspects.

**Anticipatory Utility**

The importance of anticipations in agents’ well-being has been recognized by philosophers and early economists (Hume (1739), Bentham (1879), Jevons (1888), Böhm-Bawerk (1890), Nietzsche (1968)). More recently, Loewenstein (1987), Caplin and Leahy (2001), and Kőszegi (2010) have added to the EDU model a dimension called ‘anticipation utility,’ capturing feelings like excitement or anxiety, which the agent derives today from the current act itself of thinking about future events.

In our approach, future consumption enters an agent’s anticipatory utility through his future well-being. This property has two virtues. First, it leads to a more comprehensive approach to anticipatory utility, by recognizing that future consumption generally affects us today through our future well-being, as in the retirement and career examples of the introduction. Second, it allows us to aggregate many, possibly very different, consumption
events—such as strenuous physical work, vacationing in a pleasant resort, and rejoicing at the perspective of a future concert—into a single well-being index. For example, imagine an agent who must undergo a tooth removal one day before attending a great concert. Presumably, his well-being on the day of the dentist visit will be a bittersweet experience, combining tooth pain with pleasure for the imminence concert. How should this be factored in at earlier days? If those anticipations are treated separately, they become hard to track individually and may even require distinct discount rates for each type of consumption event. Our approach incorporates anticipatory utility from all consumption events and integrates it in a tractable way.31

In summary, (i) existing work has clearly recognized that anticipatory utility is an important part of current well-being, but through future consumption; (ii) in contrast to existing work, we model anticipations in terms of future well-being; (iii) we are not excluding the consumption-channel of anticipation, but aggregating it through the agent’s future well-being; (iv) this approach through well-being is more canonical and broader.

Anticipatory Utility in the EDU Model

The usual interpretation of the EDU model is typically devoid of anticipatory utility. This interpretation is consistent with Samuelson’s (1937) who wrote, as he introduced the model, that “it is completely arbitrary to assume that the individual behaves so as to maximize a [sum of the form \( \sum_{t=0}^{\infty} \delta^t u(c_t) \)]. This involves the assumption that at every instant of time the individual’s satisfaction depends only upon the consumption at that time.” (p.159).

There is another interpretation of the EDU model, which allows for anticipatory utility and is based the formula

\[
U(c_0, c_1, \ldots) = u(c_0) + \delta u(c_1) + \delta^2 u(c_2) + \cdots = u(c_0) + \delta U(c_1, c_2, \ldots). \tag{11}
\]

At time 0, the agent takes into account future consumption through his total utility at time 1, \( U(c_1, c_2, \ldots) \). Equivalently, his time-0 preference is completely determined by his time-0 consumption and time-1 preference over continuation streams. On the face of it, it seems difficult to explain why the agent should only consider his preference in the

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31One may be concerned that including well-being at several future times leads to ‘double counting’ of consumption. However, double counting (if any) is a consequence of a more fundamental point: the agent directly cares about his well-being at multiple future times. Moreover, despite this property, in our framework preferences and their utility representations are always well defined and lead to well-behaved choice problems. Finally, as noted, hyperbolic and \( \beta-\delta \) discounting models are special cases of directly forward-looking preferences. Yet, we usually do not view them as involving double counting.
next period. According to the revealed-preference approach, \( U(c_0, c_1, \ldots) \) is an ordinal representation of the agent’s well-being, and (11) implies that his well-being at 0 is just a function of his well-being at the next period: it cannot directly depend on his well-being at any future period. If, by contrast, \( U \) is not an ordinal representation of the agent’s preference, any welfare analysis based on his choices is severely weakened by the fact that those choices do not represent his well-being.

**Time-Inconsistent Preferences**

A large theoretical and empirical literature has followed Strotz’s (1955) seminal work on time-inconsistent preferences. Several explanations have been proposed for time inconsistency and, specifically, for present bias. In Akerlof (1991), present bias is based on a principle of cognitive psychology which says that agents unduly overweigh relatively more salient or tangible events, such as immediate consumption relative to future one. In Gul and Pesendorfer (2001), time inconsistency can arise from a general change in the agent’s preference over time. In Halevy (2008) and Saito (2011), present bias can result from a combination of two things: (1) the present is usually certain, whereas the future is uncertain; (2) the agent has a non-expected utility preference which is disproportionately sensitive to certainty as in Allais (1953) and Kahneman and Tversky (1979). In Kőszegi and Szeidl (2012), time inconsistency and present bias arise because, when facing a present decision, the agent focuses too much on its immediate consequences, but when considering that same decision ex ante, he is able to focus more on its overall consequences over time. Our theory provides a novel explanation for present bias, as part of a broader conceptual framework based on anticipatory utility. In our theory, present bias may have nothing to do with an ‘excessive’ taste for the present.

One of the most popular models of time-inconsistent preferences is the \( \beta-\delta \) discounting model, introduced by Phelps and Pollak (1968) to analyze economies populated by ‘imperfectly altruistic’ generations. Laibson (1997) changes the conceptual content of that model by applying it to individual decision-making. He justifies using that model with its ability “to capture the qualitative properties” of the empirically supported, hyperbolic-discounting model (see references in Laibson (1997)). In Phelps and Pollack, the first generation cares discretely more about its consumption than that of future generations—consisting, after all, of unborn strangers—in a uniform way. It is unclear, however, why a single agent may care significantly more about his immediate consumption than his own future consumption in a uniform way. Existing axiomatizations of the \( \beta-\delta \) model in settings with a single agent (see, e.g., Hayashi (2003); Olea and Strzalecki (2014)) do
not address this question. They continue to view the agent as caring about instantaneous utilities and replace Koopmans’ (1960) stationarity with quasi-stationarity—i.e., stationarity from tomorrow onward—without seeking more primitive causes of why stationarity fails. By contrast, in this paper $\beta$-$\delta$ discounting emerges as the manifestation of a conceptually different view of what determines preferences over consumption streams (immediate consumption and future well-being) and by invoking a natural stationarity notion involving only well-being. Moreover, we show that the specific $\beta$-$\delta$ formula is tightly linked with an intuitive property of intertemporal consumption trade-offs, which is easier to interpret than the axioms used in the literature to obtain that formula (see Section 3.4).

Since Phelps and Pollak (1968), a literature on intergenerational altruism has developed, which contains formal similarities with some aspects of our theory. In particular, Saez-Marti and Weibull (2005) and Fels and Zeckhauser (2008) derive the mathematical equivalence between the $\beta$-$\delta$ formula and expression (5) with linear $G$. They do not provide an axiomatic foundation of either representation and do not address the conceptual issues at the heart of the present paper. Bergstrom (1999) studies systems of utility functions that include altruism towards others, focusing on the infinite regress that they may generate. In his study of hedonistic altruism and welfare, Ray (2014) examines welfare criteria that aggregate well-being of altruistic, time-consistent, generations and that are formally similar to those in Section 4 (see also Bernheim (1989)).

**Time-Inconsistent Behavior**

Our theory focuses on characterizing the agent’s preference at time 0, assuming that he compares consumption streams as if he were able to commit to them.

Without commitment, the agent faces a succession of choices, and it becomes of obvious importance to distinguish time inconsistency of *preferences* from time inconsistency of actual *choices*. Needless to say, these two concepts should be kept separate. As Strotz (1955) already suggested, there seem to be at least two reasons why time-inconsistent preferences need not result in time-inconsistent choices.

32 Fully aware of his inconsistency, an agent may anticipate his future inability to carry out certain decisions. Hence, on the one hand, he may commit to them in advance (Strotz’s ‘strategy of pre-commitment’).  

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32 Time inconsistency is sometimes regarded as unimportant for economic analysis (see, e.g., Mulligan (1996)). One argument in support of this view is that a time-inconsistent agent would be subject to money-pump schemes which would quickly lead him to complete immiseration. Anticipating this, the agent would never trade in markets. Consequently, there would be no reason to allow for time-inconsistent agents in economic models. Laibson and Yariv (2004) provide a counterargument.
On the other hand, he may consider only current decisions which rely on future courses of action that he can implement over time (Strotz’s ‘strategy of consistent planning’). So, depending on the economic environment, it seems perfectly conceivable to observe agents with time-inconsistent preferences make time-consistent choices.

This does not imply, of course, that we could never distinguish consistent from inconsistent agents—only the latter, for instance, value commitment ex ante. Moreover, the sort of intra-personal equilibrium that a time-inconsistent agent may engage in with his future selves may have different properties from the consumption path chosen by a time consistent agent.\footnote{See Harris and Laibson (2001).} For example, there may exist multiple equilibria, and this multiplicity may lead to choices that vary discontinuously in parameters of the model. The analysis of intertemporal choice is obviously completely different if the agent has time-inconsistent preferences, and one cannot simply focus on time-consistent agents on the ground that this analysis may lead to time-consistent behavior.\footnote{Pearce’s (2008) work on nonpaternalistic sympathy has a similar flavor. That paper considers a cake-eating model with finitely many generations. It assumes that each generation’s well-being depends on its consumption as well as on the well-being of all other or only future generations, and focuses on equilibrium analysis, proving general inefficiency results. Other related papers include Ray (1987) and Bernheim and Ray (1989).}

**The ‘Virtue’ of Time Consistency**

One argument sometimes given in support of the EDU model is that it is time consistent and hence tractable: optimal consumption streams can be computed by standard recursive methods—in particular, the Bellman equation. In our view, tractability is the only truly compelling argument in favor of the standard model. We wish to make a clear distinction between this consideration of expediency, for the modeler, and the agent’s actual preferences. One thing is to use a time-consistent model for simplicity; another is to claim that the agent himself chooses a time-consistent choice rule for simplicity. The latter argument would require a theory of what it means for the agent to choose ‘simple’ streams. Moreover, this approach would confuse the agent’s actual preference over streams and his preference for simplicity.

As noted, to meaningfully elicit preferences over consumption streams from how the agent chooses among them, we must assume that he is committing to them. But if there is commitment, it is unclear why time consistency is a ‘virtue’ of EDU. Without commitment, time consistency of the EDU model may become appealing to the agent. But without commitment, the agent obviously cannot commit to obeying that model. Instead, one has to consider intra-personal equilibria, in which the agent is making plans...
under the concern that he will revise them in the future. This possibility, however, adulterates the exercise of eliciting his underlying preference and says nothing about whether EDU is a good model of such a preference.

We hope that our theory clarifies that consumption interdependence, present bias, and other phenomena incompatible with EDU are perfectly consistent with full rationality. Experiments that reveal such behaviors are perfectly consistent with utility-maximizing agents. Unlike EDU agents, however, these agents exhibit the realistic feature of experiencing anticipatory utility in a coherent way. For such models, there is an equivalent of the Hamilton-Jacobi-Bellman equation (recall (7)). Moreover, as noted in the previous section, an agent with time-inconsistent preferences may, in some settings, exhibit time-consistent behavior. Finally, a social planner who aggregates in a natural way the agent’s well-being at all periods will have a time-consistent welfare function.
A Appendix: Proofs of the Main Results

A.1 Proof of Proposition 2

Let $U(c) = V(c_0, U(1c), U(2c), \ldots)$ where $V$ is strictly increasing in $U(tc)$ for all $t > 0$. By definition, $(x, c) > (y, c)$ means that $U(x, c) > U(y, c)$. Hence, for all $0 \leq s \leq t$,

$$U(s z_t, x, c) > U(s z_t, y, c),$$

where, for $s < t$, $s z_t = (z_s, \ldots, z_t)$ and $t z_t = z_t$. This follows by induction. For $s = t$,

$$U(t z_t, x, c) = V(t z_t, U(x, c), U(c), \ldots) > V(t z_t, U(y, c), U(c), \ldots) = U(t z_t, y, c).$$

Now suppose that the claim holds for $r + 1 \leq s \leq t$, with $0 \leq r < t$. Then

$$U(r z_t, x, c) = V(z_r, U(r z_t, x, c), \ldots, U(t z_t, x, c), U(x, c), \ldots) > V(z_r, U(r z_t, y, c), \ldots, U(t z_t, y, c), U(y, c), \ldots) = U(r z_t, y, c).$$

By definition, $(0 z_t, x, c') \sim (0 z_t, y, h, c')$ means that

$$V(z_0, U(1 z_t, x, w, c'), \ldots, U(t z_t, x, w, c'), U(x, w, c'), U(w, c'), \ldots) = V(z_0, U(1 z_t, y, h, c'), \ldots, U(t z_t, y, h, c'), U(y, h, c'), U(h, c'), \ldots).$$

Since $U(0 z_t, x, c) > U(0 z_t, y, c)$ for all $c$,

$$V(z_0, U(1 z_t, y, w, c'), \ldots, U(t z_t, y, w, c'), U(y, w, c'), U(w, c'), \ldots) < V(z_0, U(1 z_t, y, h, c'), \ldots, U(t z_t, y, h, c'), U(y, h, c'), U(h, c'), \ldots).$$

This implies that $U(h, c') > U(w, c')$. Otherwise, $U(y, h, c') \leq U(y, w, c')$ and, by induction, $U(s z_t, y, h, c') \leq U(s z_t, y, w, c')$ for all $0 \leq s \leq t$, which is a contradiction.

Finally, we must have $U(x, w, c') > U(y, h, c')$. Otherwise, again by induction, for all $0 \leq s \leq t$

$$U(s z_t, y, h, c') > U(s z_t, x, w, c'),$$

which contradicts $(0 z_t, x, w, c') \sim (0 z_t, y, h, c')$.

Suppose that we replace condition $(0 z_t, x, w, c') \sim (0 z_t, y, h, c')$ with $(0 z_t, x, t+2 z_s, w, c') \sim (0 z_t, y, t+2 z_s, h, c')$ where $s \geq t + 2$. By the same argument as before, $(0 z_t, y, t+2 z_s, w, c') \sim (0 z_t, y, t+2 z_s, h, c')$ and so $(h, c') > (w, c')$. If not, by induction $(t z_s, w, c') \sim (t z_s, h, c')$ for all $0 \leq \tau \leq s$ (where $z_{t+1} = y$). Then, we must have $(x, t+2 z_s, w, c') > (y, t+2 z_s, h, c')$. If not, since $(t z_s, h, c') > (t z_s, w, c')$ for all $t + 2 \leq \tau \leq s$, we would have $(0 z_t, y, t+2 z_s, h, c') > (0 z_t, y, t+2 z_s, w, c')$.


\[(0z_t, x, t+2z_s, w, c').\]

### A.2 Proof of Theorem 3

By the definition of \( F \) in (3), note that \( F \) need not be a Cartesian product, as \( f_t \) depends on \( f_s \) for \( s > t \). Letting \( t f = (f_t, f_{t+1}, \ldots) \), we can denote elements in \( F \) by \((f_1, f_2, \ldots, f_{t-1}, f_t)\). On \( F_0 = X \times F \) (where \( f_0(c) = c_0 \)), if the primitive \( \succ \) induces a \( \succ^* \) with representation \( V : F_0 \to \mathbb{R} \); by Theorem 2, \( \succ^* \) is well defined. If \( F_0 \) were a Cartesian product, we could mimic the steps in Debreu (1960) (Theorem 3) and Koopmans (1972) on the domain \( F_0 \) to prove our theorem. However, this is not possible. We will then proceed as follows. In step 1, we show that \( \succ^* \) satisfies the essentiality and strong separability properties at the heart of Debreu’s (1960) Theorem 3. In step 2, we show that the ranking of streams \((f_0, f_1, f_2, 3f) \in F_0 \) depends only on a function of \( 3f \); so we can restrict attention to a four dimensional space. In step 3, we show that this space is a Cartesian product ‘locally;’ so we can apply Debreu’s result to obtain an additive representation ‘locally.’ Since additive representations are unique up to positive, affine transformations, we can extend uniquely the additive representation to the entire \( F_0 \). In step 4, we show that this representation takes the form in our Theorem 3.

**Step 1.** Lemma 1 says that, if \((f_0, f_1, f_2, 3f) \succ^* (f'_0, f'_1, f'_2, 3f')\), then changing the common components of \((f_0, f_1, f_2, 3f)\) and \((f'_0, f'_1, f'_2, 3f')\) in the same way leaves the ranking under \( \succ^* \) unchanged.

**Lemma 1.** Fix any nonempty subset \( \pi \) of \( \{0, 1, 2, 3\} \). Then

\[
(f_0, f_1, f_2, 3f) \succ^* (f'_0, f'_1, f'_2, 3f') \iff (\hat{f}_0, \hat{f}_1, \hat{f}_2, 3\hat{f}) \succ^* (\hat{f}'_0, \hat{f}'_1, \hat{f}'_2, 3\hat{f}'),
\]

where \( f_t = \hat{f}_t, f'_t = \hat{f}'_t, 3f = 3\hat{f} \), and \( 3f' = 3\hat{f}' \) if \( t \) or \( 3 \) are in \( \pi \), and \( f_t = f'_t, \hat{f}_t = \hat{f}'_t, 3f = 3f' \), and \( 3\hat{f} = 3\hat{f}' \) if \( t \) or \( 3 \) are not in \( \pi \).

**Proof.** Recall that \( t c \sim \_t c' \) is equivalent to \( f_t = f'_t \). Then, by Axiom 6, for any \( \pi \)

\[
V(f_0, f_1, f_2, 3f) > V(f'_0, f'_1, f'_2, 3f') \iff V(\hat{f}_0, \hat{f}_1, \hat{f}_2, 3\hat{f}) > V(\hat{f}'_0, \hat{f}'_1, \hat{f}'_2, 3\hat{f}').
\]

Using Lemma 1 with \( \pi = \{0\} \) and \( \pi = \{1, 2, 3\} \), we obtain the following.

**Lemma 2.** The function \( V : F_0 \to \mathbb{R} \) can be written in the form

\[
V(f) = W(u(f_0), d(1f)), \quad (12)
\]

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where \( u : X \to \mathcal{I}_u \subset \mathbb{R} \) and \( d : \mathcal{F} \to \mathcal{D} \subset \mathbb{R} \). \( W \) is jointly continuous in its two arguments and strictly increasing in each of them, \( u \) is continuous, and \( \mathcal{I}_u \) and \( \mathcal{D} \) are non-degenerate intervals.

**Proof.** Consider \( >^* \) on \( X \times \mathcal{F} \) and Lemma 1 with \( \pi = \{0\} \) and \( \pi = \{1, 2, 3\} \). By an argument similar to that in Section 5 of Koopmans' (1960), for any \( f \) we can write \( V(f) = W(u(f_0), d(1f)) \), where \( u(f_0) = V(f_0, 1\hat{f}) \) for some \( 1\hat{f} \in \mathcal{F} \) and \( d(1f) = V(f'_0, 1f) \) for some \( f'_0 \in X \). Recall that \( V(f(c)) = U(c) \) for all \( c \in C \). Hence, the continuity property of \( U \) implies continuity of \( u \). By Axiom 4, neither \( u \) nor \( d \) can be constant. Since \( X \) is connected, \( u \) takes all values in a connected interval \( \mathcal{I}_u \subset \mathbb{R} \). Since \( d(1f(c)) = U(f'_0, 1c) \), \( U \) is continuous, and \( X \) is connected, \( d \) takes all values in a connected interval \( \mathcal{D} \subset \mathbb{R} \). By definition of \( u \) and Lemma 1 with \( \pi = \{0\} \), \( W \) must be strictly increasing in its first argument on \( \mathcal{I}_u \). Similarly, by definition of \( d \) and Lemma 1 with \( \pi = \{1, 2, 3\} \), \( W \) must be strictly increasing in its second argument on \( \mathcal{D} \). Given \( \hat{c}, U(\cdot, 1\hat{c}) \) takes values in an interval. Then the strictly increasing \( W(\cdot, 1f(\hat{c})) \) also takes values in an interval and hence must be continuous in its first argument on \( \mathcal{I}_u \). By a similar argument, \( W \) must be continuous in its second argument, and hence jointly continuous on \( \mathcal{I}_u \times \mathcal{D} \).

Hereafter, let \( \overline{u} = \sup \mathcal{I}_u \) and \( \underline{u} = \inf \mathcal{I}_u \). Also note that the function \( d \) in Lemma 2 defines a ranking on \( \mathcal{F} \).

**Lemma 3.** There exist \( x, y, z, x', y', z' \in X \) and \( c \in C \) such that (i) \( (z, c) \succ (z', c) \), (ii) \( (y, z, c) \sim (y', z, c) \), and (iii) \( (x, y, z, c) \sim (x', y', z', c) \).

**Proof.** By Axiom 4, there exist \( z, z' \in X \) and \( c \in C \) such that \( (z, c) \succ (z', c) \). Using representation (12), we have \( u(z) > u(z') \). Now, consider \( (y', z, c) \) and \( (y', z', c) \) where \( y = z \) and \( y' = z' \).

**Case 1:** \( (y', z', c) \succ (y', z, c) \). Since \( \mathcal{I}_u \) is connected, we can modify \( y \) to \( y'' \in X \) so that \( u(y'') \) takes any value in \( [u(y'), u(y)] \). By Axiom 2, there exists \( y'' \) such that \( (y'', z', c) \sim (y', z, c) \); moreover, we must have \( u(y'') > u(y') \). Now consider \( (x, y'', z', c) \) and \( (x', y', z, c) \) where \( x = z \) and \( x' = z' \). By Axiom 7(i), \( (x', z, c) \succ (x', z', c) \); so, by Axiom 8, \( (x', y', z, c) \succ (x', y', z', c) \).

**Case 1.1:** \( (x, y'', z', c) \succ (x', y', z, c) \). Since we can modify \( x \) to \( x'' \in X \) so that \( u(x'') \) takes any value in \( [u(x'), u(x)] \), by Axiom 2, there exists \( x'' \) such that \( (x'', y'', z', c) \sim (x', y', z, c) \).

**Case 1.2:** \( (x, y'', z', c) \prec (x', y', z, c) \). We can modify \( z \) and \( y'' \) to \( \tilde{y}, \tilde{z} \in X \) so that \( u(\tilde{z}) \) and \( u(\tilde{y}) \) take any value in \( [u(z'), u(z)] \) and \( [u(y'), u(y'')] \). Moreover, we can do so maintaining \( (y', z', c) \sim (y', z, c) \) by Axiom 2. Since \( (x, y', z', c) \succ (x', y', z', c) \), by Axiom 2, there exist \( \tilde{y} \) and \( \tilde{z} \) such that \( (x, \tilde{y}, z', c) \sim (x, y', \tilde{z}, c) \). Finally, we must have \( u(\tilde{z}) > u(z') \), so \( (z, c) \succ (z', c) \).

**Case 2:** \( (y, z', c) \prec (y', z, c) \). We can modify \( z \) to \( \hat{z} \in X \) so that \( u(\hat{z}) \) takes any value in \( [u(z'), u(z)] \). Since \( (y, z', c) \succ (y', z', c) \), by Axiom 2 there exists \( \hat{z} \) such that \( (y, z', c) \sim (y', \hat{z}, c) \),
and by Axiom 7(i) we must have \((\hat{z}, c) \succ (z', c)\) and hence \(u(\hat{z}) > u(z')\). Now consider \((x, y, z', c)\) and \((x', y', \hat{z}, c)\) where \(x = z\) and \(x' = z'\).

**Case 2.1:** \((x, y, z', c) \succeq (x', y', \hat{z}, c)\). We can modify \(x\) to \(\hat{x}\) so that \(u(\hat{x})\) takes any value in 
\([u(x'), u(x)]\). By Axiom 7(i), \((x', z', c) < (x', \hat{z}, c)\); so, by Axiom 8, \((x', y, z', c) < (x', y', \hat{z}, c)\). Then, by Axiom 2 there exists \(\hat{x}\) such that \((\hat{x}, y, z', c) \sim (x', y', \hat{z}, c)\).

**Case 2.2:** \((x, y, z', c) < (x', y', \hat{z}, c)\). We can modify \(y\) and \(\hat{z}\) to \(\hat{y}\) and \(\hat{z}'\) so that \(u(\hat{y})\) and \(u(\hat{z}')\) take any value in 
\([u(y'), u(y)]\) and \([u(z'), u(\hat{z})]\). Moreover, we can do so maintaining \((\hat{y}, z', c) \sim (y', z', c)\) by Axiom 2. Since \((x, y', z', c) \succeq (x', y', z', c)\), by Axiom 2 there exists \(\hat{y}\) and \(\hat{z}'\) such that \((x, \hat{y}, z', c) \sim (x', y', z', c)\).

Hereafter, for \(t \in \{0, 1, 2, 3\}\), we will refer to the factor \(t\) of \(\mathcal{F}_0\) as the component of position \(t + 1\) in the representation \((f_0, f_1, f_2, 3f)\) of every \(f \in \mathcal{F}_0\) (e.g., the factor 2 is the third component of every \((f_0, f_1, f_2, 3f) \in \mathcal{F}_0\)).

**Definition 6** (Debreu (1960)). For \(t \in \{0, 1, 2\}\), if \(f \succ f'\) for some \(f, f' \in \mathcal{F}_0\) with \(f_s = f'_s\) for all \(s \neq t\), then the factor \(t\) of \(\mathcal{F}_0\) is called essential for \(\succ^*\). If \((f_0, f_1, f_2, 3f_0) \succ^* (f'_0, f'_1, f'_2, 3f'_0)\) for some \(f, f' \in \mathcal{F}_0\) with \(f_s = f'_s\) for \(s = 0, 1, 2\), then the factor 3 is called essential for \(\succ^*\).

**Lemma 4.** For all \(t \in \{0, 1, 2, 3\}\), the factor \(t\) of \(\mathcal{F}_0\) is essential.

**Proof.** By Axiom 4, the factor 0 is essential. Using the streams in Lemma 3, let \(1c = (x, y, z, c)\) and \(1c' = (x', y', z', c)\) and consider the corresponding \(f\) and \(f'\) in \(\mathcal{F}_0\) with any \(f_0 = f'_0\). We have \(f_1 = f'_1, f_2 = f'_2, f_4 > f'_4\), and \(f_t = f'_t\) for all \(t > 3\). By Axiom 7(i), \((f_0, f_1, f_2, 3f) \succ^* (f'_0, f'_1, f'_2, 3f'_0)\) and \((f_0, f_1, f_2, 3f) \succ^* (f'_0, f'_1, f'_2, 3f'_0)\). So the factors 2 and 3 are essential.

**Step 2.** By Lemma 1 with \(\pi = \{2, 3\}\), \(\succ^*\) also satisfies the following property:

\[
(f_0, f_1, 2f) \succ^* (f_0, f_1, 2f') \iff (f_0, \hat{f}_1, 2f) \succ^* (f_0, \hat{f}_1, 2f').
\]

Define \(\hat{Q} = \{(f_1(c), d(2f(c))) : c \in C\}\). Note that \(\hat{Q} \subset U \times D\), but it need not be a Cartesian product because the value of \(d\) affects that of \(f_1\). 

**Lemma 5.** There exists a continuous function \(\hat{V} : X \times \hat{Q} \to \mathbb{R}\) such that, for all \(f \in \mathcal{F}_0\),

\[
\hat{V}(f) = \hat{V}(f_0, f_1, d(2f)), \tag{13}
\]

where \(d\) is the function defined in Lemma 2. For any \(f_1, f'_1, d',\) and \(d''\) we have the following:

- (5.i) if \((f_1, d'')\) and \((f'_1, d'')\) are in \(\hat{Q}\), \(\hat{V}(f_0, f_1, d'') > \hat{V}(f_0, f'_1, d'')\) iff \(f_1 > f'_1\);
- (5.ii) if \((f_1, d')\) and \((f_1, d'')\) are in \(\hat{Q}\), \(\hat{V}(f_0, f_1, d') > \hat{V}(f_0, f_1, d'')\) iff \(d' > d''\).
Proof. Recall that \( d(\cdot) \) defines a ranking on \( \mathcal{F} \) and that \( 2f \in \mathcal{F} \). For any \((f_0, f_1, 2f)\) and \((f_0', f_1', 2f')\) such that both \((f_0, f_1, 2f)\) and \((f_0', f_1, 2f)\) are in \( \mathcal{F}_0 \), by Lemma 1 with \( \pi = \{2, 3\} \), \( V(f_0, f_1, 2f) \geq V(f_0, f_1', 2f') \) iff \( V(f_0', f_1', 2f') \geq V(f_0', f_1', 2f') \). Moreover, for any \((f_1, 2f)\) and \((f_1, 2f')\) in \( \mathcal{F} \), by Axiom 8, \( V(f_0, f_1, 2f) \geq V(f_0, f_1, 2f') \) iff \( W(u(f_0), d(2f)) \geq W(u(f_0), d(2f')) \), and therefore iff \( d(2f) \geq d(2f') \). So, the ranking of \((f_0, f_1, 2f)\) and \((f_0, f_1, 2f')\) depends only on the value of \( d(\cdot) \). Now, for any \( f \in \mathcal{F}_0 \), set
\[
\tilde{V}(f_0, f_1, d(2f)) = V(f_0, f_1, 2f).
\]
The previous argument implies property (5.i).

\( \tilde{V} \) is well defined for the following reasons. First, if \((f_0, f_1, 2f)\) and \((f_0', f_1', 2f')\) are such that \( f_t = f_t' \) for \( t = 0, 1 \) and \( d(2f) = d(2f') \), then \( V(f_0, f_1, 2f) = V(f_0', f_1', 2f') \) again by Axiom 8. Second, if \((f_0, f_1, 2f)\) and \((f_0', f_1', 2f')\) are such that either \((f_0, f_1, 2f) \notin \mathcal{F}_0 \) or \((f_0', f_1', 2f') \notin \mathcal{F}_0 \), then \((f_0, f_1) \neq (f_0', f_1') \). So, even if \( d(2f) = d(2f') \), \( \tilde{V}(f_0, f_1, d(2f)) \) can be different from \( \tilde{V}(f_0', f_1', d(2f')) \).

Consider now \((f_1, d', f_2', \tilde{d}'') \in \tilde{Q} \). There exist \( c, c' \in C \), such that \( f_t = f_t(c) \) and \( f_t' = f_t(c') \) for \( t = 0, 1 \), and \( d(2f(c)) = d(2f(c')) = d'' \). By Lemma 2, without loss, we can assume that \( 2c = 2c' \) so that \( 2f(c) = 2f(c') = 2f'' \). By Axiom 7(i), then \( V(f_0, f_1, 2f'') > V(f_0', f_1', 2f'') \) iff \( f_1 > f_1' \), and property (5.ii) follows from (13).

Finally, \( \tilde{V} \) is continuous for the following reasons. For any \((f_1, d) \in \tilde{Q} \), \( \tilde{V}(\cdot, f_1, d) = U(\cdot, c) \) for any \( c \) such that \( f_1 = f_1(c) \) and \( d = d(2f(c)) \). Hence, the continuity property of \( U \) implies that \( \tilde{V} \) is continuous in its first argument. Given any \( f_0 \) and \( d \in D \), \( \tilde{V}(f_0, \cdot, d) = U(f_0, \cdot, 2c) \) for some \( c \) such that \( f_0(c) = f_0 \) and \( d = d(2f(c)) \). Hence, \( \tilde{V} \) must take value in a connected interval and, being strictly increasing, it must be continuous in its second argument given \( f_0 \) and \( d \). By a similar argument, for any \((f_0, f_1)\), \( \tilde{V}(f_0, f_1, \cdot) \) must take values in a connected interval and, being strictly increasing, it must be continuous in its last argument. It follows that \( \tilde{V} \) must be continuous on the connected set \( X \times \tilde{Q} \).

Now define \( Q = \{ (f_1(c), f_2(c), d(3f(c))) : c \in C \} \). By an argument similar to that in the proof of Lemma 5, using Lemma 1 with \( \pi = \{3\} \), we obtain the following.

**Lemma 6.** There exists a continuous function \( \nabla : X \times Q \to \mathbb{R} \) such that, for all \( f \in \mathcal{F}_0 \),
\[
V(f) = \nabla(f_0, f_1, f_2, d(3f)), \tag{14}
\]
where \( d \) is the function defined in Lemma 2. Moreover, for any \( f_1, f_1', f_2, f_2', d', \) and \( d'' \) we have the following:
(6.i) if \((f_1, f_2, d'), (f_1', f_2, d') \in Q \), then \( \nabla(f_0, f_1, f_2, d') > \nabla(f_0, f_1', f_2, d') \) iff \( f_1 > f_1' \);
(6.ii) if \((f_1, f_2, d')\), \((f_1, f'_2, d')\) \(\in Q\), then \(V(f_0, f_1, f_2, d') > V(f_0, f_1, f'_2, d')\) iff \(f_2 > f'_2\);
(6.iii) if \((f_1, f_2, d')\), \((f_1, f_2, d'')\) \(\in Q\), then \(V(f_0, f_1, f_2, d') > V(f_0, f_1, f_2, d'')\) iff \(d' > d''\).

Hereafter, for any \(c \in C\), let \(d_3(c) = d(3f(c))\). Also, we say that \(c \in C\) induces \((f_0, f_1, f_2, d_3) \in X \times Q\) if \(f_t(c) = f_t\) for \(t = 0, 1, 2\) and \(d_3(c) = d_3\). Note that the function \(V\) defines a preference \(\succ\) on \(X \times Q\); moreover, by definition, for \(c, c' \in C\)

\[(f_0(c), f_1(c), f_2(c), d_3(c)) \succ (f_0(c'), f_1(c'), f_2(c'), d_3(c')) \iff f(c) \succ^* f(c').\]

**Lemma 7.** The preference \(\succ\) satisfies the following property (see Definition 4 in Debreu (1960)). Fix any nonempty subset \(\pi\) of \(\{0, 1, 2, 3\}\). Then

\[(f_0, f_1, f_2, d_3) \succ (f_0', f_1', f_2', d_3') \iff (\hat{f}_0, \hat{f}_1, \hat{f}_2, \hat{d}_3) \succ (\hat{f}_0', \hat{f}_1', \hat{f}_2', \hat{d}_3'),\]

where \(f_t = \hat{f}_t, f'_t = \hat{f}'_t, d_3 = \hat{d}_3\), and \(d'_3 = \hat{d}'_3\) if \(t\) or \(3\) are in \(\pi\), and \(f_t = f'_t\), \(\hat{f}_t = \hat{f}'_t, d_3 = d'_3\), and \(\hat{d}_3 = \hat{d}'_3\) if \(t\) or \(3\) are not in \(\pi\).

**Proof.** Given \(\pi\), let \(\pi^c\) be its complement. If \(3 \in \pi^c\), then there exist \(c, c', \hat{c}, \hat{c}' \in C\) such that, for \(t = 0, 1, 2\), \(f_t = f_t(c)\), \(f'_t = f_t(c')\), \(\hat{f}_t = f_t(\hat{c})\), \(\hat{f}'_t = f_t(\hat{c}')\), \(d(3f(c)) = d(3f(c'))\), and \(d(3f(\hat{c})) = d(3f(\hat{c}'))\). Then, by Lemma 2, \(f_2(c) = f_2(c_0, c_1, c_2, 3c')\) and \(f_2(\hat{c}) = f_2(\hat{c}_0, \hat{c}_1, \hat{c}_2, 3\hat{c}')\). Similarly, by Lemma 5, \(f_1(c) = f_1(c_0, c_1, c_2, 3c')\) and \(f_1(\hat{c}) = f_1(\hat{c}_0, \hat{c}_1, \hat{c}_2, 3\hat{c}')\). Therefore, we can take \(3c = 3c'\) and \(3\hat{c} = 3\hat{c}'\), so that \(3f = 3f'\) and \(3\hat{f} = 3\hat{f}'\). It follows from Lemma 1, that

\[V(f_0, f_1, f_2, 3f) > V(f_0', f_1', f_2', 3f') \iff V(\hat{f}_0, \hat{f}_1, \hat{f}_2, 3\hat{f}) > V(\hat{f}_0', \hat{f}_1', \hat{f}_2', 3\hat{f}').\]

Hence, by (14), the result follows.

Suppose \(3 \in \pi\). Again, there exist \(c, c', \hat{c}, \hat{c}' \in C\), each inducing the respective element of \(X \times Q\) — in particular, \(d(3f(c)) = d(3f(\hat{c}))\) and \(d(3f(c')) = d(3f(\hat{c}'))\). Then, by Lemma 2, \(f_2(c) = f_2(c_0, c_1, c_2, 3c)\) and \(f_2(c') = f_2(c_0', c_1', c_2', 3c')\). Similarly, by Lemma 5, \(f_1(c) = f_1(c_0, c_1, c_2, 3c)\) and \(f_1(\hat{c}) = f_1(\hat{c}_0, \hat{c}_1, \hat{c}_2, 3\hat{c}')\). Therefore, we can take \(3c = 3\hat{c}\) and \(3c' = 3\hat{c}'\), so that \(3f = 3\hat{f}\) and \(3f' = 3\hat{f}'\). It follows again from Lemma 1, that

\[V(f_0, f_1, f_2, 3f) > V(f_0', f_1', f_2', 3f') \iff V(\hat{f}_0, \hat{f}_1, \hat{f}_2, 3\hat{f}) > V(\hat{f}_0', \hat{f}_1', \hat{f}_2', 3\hat{f}').\]

Hence, by (14), the result follows.

**Step 3:** Let \(\mathcal{O}\) be the set of vectors \((f_1(c), f_2(c), d(3f(c)))\) for \(c \in C\), such that \(u < u(c) < \bar{u}\)

---

35Recall that by Lemma 6, if \((f_0, f_1, f_2, \bar{f})\) and \((f_0, f_1, f_2, \bar{f}')\) are in \(F_0\) and \(d(\bar{f}) = d(\bar{f}')\), then \(V(f_0, f_1, f_2, \bar{f}) = V(f_0, f_1, f_2, \bar{f}')\).
for \( t = 1, 2 \) and \( d(3f(c)) \in \text{int}\mathcal{D} \). It is straightforward to check that \( \mathcal{O} \) is nonempty and that \( \mathcal{Q} \) is included in the closure of \( \mathcal{O} \).\(^{36}\)

**Lemma 8.** For any \((f_1, f_2, d_3) \in \mathcal{O}\), there exists \( \eta > 0 \) such that the rectangle

\[
R(f_1, f_2, d_3; \eta) = (f_1 - \eta, f_1 + \eta) \times (f_2 - \eta, f_2 + \eta) \times (d_3 - \eta, d_3 + \eta)
\]

lies in \( \mathcal{O} \).

**Proof.** Fix \((f_1, f_2, d_3) \in \mathcal{O}\) and, for the inducing \( c \), let \( u_t = u(c_t) \) for \( t = 1, 2 \). Since \( d_3 \in \text{int}\mathcal{D} \), there is an interval \((\underline{d_3}, \overline{d_3}) \subset \mathcal{D}\) containing \( d_3 \). Since \( \underline{u} < u_2 < \overline{u} \), given \( d_3 \), there is an interval \((\underline{f}_1(d_3), \overline{f}_1(d_3)) \subset \mathcal{U}\), containing \( f_2 \) and spanned by \( u_2 \in \text{int}\mathcal{U}_u \). Let \( \eta' > 0 \) be such that \([d_3 - \eta', d_3 + \eta'] \subset (\underline{d_3}, \overline{d_3})\). By the properties of \( W \) in Lemma 2, there exists \( \eta > 0 \) such that \( f_2(\underline{d}_3) < f_2(\overline{d}_3 + \eta') < f_2 \) and \( f_1(\overline{d}_3) > f_1(\overline{d}_3 - \eta') > f_1 \). Hence, for all \( \eta'' \in [d_3 - \eta', d_3 + \eta'] \), all \( f_2' \in [f_2 -\varepsilon(\eta'), f_2 +\varepsilon(\eta')] \) are achievable by changing only \( u_2 \), where \( \varepsilon(\eta') = \min\{f_2 - f_2(\overline{d}_3 + \eta'), f_1(\overline{d}_3 - \eta') - f_2\} \). Since \( \underline{u} < u_1 < \overline{u} \), given \( f_2 \) and \( d_3 \), there is an interval \((\underline{f}_1(d_3), \overline{f}_1(d_3)) \subset \mathcal{U}\), containing \( f_1 \) and spanned by \( u_1 \in \text{int}\mathcal{U}_u \). By the properties of \( V \) in Lemma 5, there exist \( \eta'' > 0 \) and \( \varepsilon'' > 0 \) such that \([d_3 - \eta'', d_3 + \eta''] \subset (\underline{d}_3, \overline{d}_3), [f_2 -\varepsilon'', f_2 +\varepsilon''] \subset (f_2(\overline{d}_3), f_1(\overline{d}_3)),\) and \( f_1(d_3) < f_1(f_2 +\varepsilon''d_3 + \eta'') < f_1(\overline{d}_3\overline{d}_3) > f_1(\overline{d}_3 - \varepsilon'', d_3 - \eta'') > f_1 \). Hence, for all \( (\eta', \eta'') \in [f_2 -\varepsilon'', f_2 +\varepsilon''] \times [d_3 -\eta'', d_3 +\eta''] \), all \( f_1'' \in [f_1 -\delta(\varepsilon'', \eta''), f_1 +\delta(\varepsilon'', \eta'')] \) are achievable by changing only \( u_1 \), where \( \delta(\varepsilon'', \eta'') = \min\{f_1 - f_1(\overline{d}_3 + \eta''), f_1(\overline{d}_3 - \eta'') - f_1\} \). Let \( \hat{\eta} = \min\{\eta', \eta''\}, \varepsilon = \min\{\varepsilon(\hat{\eta}), \varepsilon''\}, \) and \( \delta = \delta(\varepsilon, \hat{\eta}) \). Noting that \( \varepsilon(\hat{\eta}) \geq \varepsilon(\eta') \) and letting \( \eta = \min\{\hat{\eta}, \varepsilon, \delta\} \), we have that all \((f_1', f_2', d_3') \) in \( R(f_1, f_2, d_3; \eta) \) are induced by some \( c \in C \) and belong to \( \mathcal{O} \).

\[\square\]

**Lemma 9.** \( \mathcal{O} \) is connected.

**Proof.** We will show that \( \mathcal{O} \) is path connected and hence connected. Take any \((f_1', f_2', d_3'), (f_1'', f_2'', d_3'') \in \mathcal{O}\) with inducing streams \( c', c'' \in C \). By definition, \( u(c_t') \in \text{int}\mathcal{U}_u \) for \( t = 1, 2 \) and \( d_3', d_3'' \in \text{int}\mathcal{D} \). Since \( \mathcal{D} \) is an interval, we can vary consumption from \( t = 3 \) onward, creating a path from \( 3c' \) to \( 3c'' \) so as to cover the interval between \( d_3' \) and \( d_3'' \). Along this path \( d_3 \) remains in \( \text{int}\mathcal{D} \); moreover, by Lemma 2, \( f_2 \) varies covering an interval between \( f_2' \) and \( f_2(c_0, c_1', c_2, 3c'') \), and by Lemma 5, \( f_1 \) varies covering an interval between \( f_1' \) and \( f_1(c_0, c_1', c_2, 3c'') \). Since \( c_1' \) and

\(^{36}\)To see that \( \mathcal{O} \neq \emptyset \), consider any constant \( c' \in C \) such that \( \underline{u} < u(c_0) < \overline{u} \). By changing \( c_0 \), so that \( u(c_3) \) varies continuously in an open interval around \( u(c_3') \), by continuity of \( U \) we can continuously span an open interval around \( f_3(c') \). By Axiom 7(ii), this variation in \( c_3 \) leads to variations in \( f_2(c), \) which must span an open interval around \( f_2(c') \), again by continuity of \( U \). Since we are not changing \( c_2 \), by Lemma 2, \( d_3(c) \) must change in an open interval around \( d_3(c') \). Finally, by Lemma 5, \( f_1(c) \) must also vary continuously in an open interval around \( f_1(c') \). To see that \( \mathcal{Q} \subset \text{cl}\mathcal{O} \), notice that any point of \( \mathcal{Q} \) induced by some \( c \in C \) can be approximated, by slightly modifying \( c \), by a \( c' \) such that \( d_3(c') \in \text{int}\mathcal{D} \) and \( u(c') \in \text{int}\mathcal{U}_u \) for \( t = 1, 2 \), i.e., a point in \( \mathcal{O} \).
$c'_2$ are unchanged, all $(f_1, f_2, d_3)$ along the path are in $\mathcal{O}$. Now fix $3c = 3c''$ and vary $c_2$ to create a path from $c'_2$ to $c''_2$ so as to cover the interval between $u(c'_2)$ and $u(c''_2)$. Along this path $u(c_2)$ remains in $\text{int} \mathcal{I}_a$; moreover, by Lemma 2, $f_2$ varies covering the interval between $f_2(c'_0, c'_1, c'_2, 3c'')$ and $f_2(c'_0, c'_1, c''_2, 3c'')$, and by Lemma 5, $f_1$ varies covering an interval between $f_1(c'_0, c'_1, c'_{2''}, 3c'')$ and $f_1(c'_0, c''_1, c''_2, 3c'')$. Since $c'_1$ is unchanged, again all $(f_1, f_2, d_3)$ along this second path are in $\mathcal{O}$. Finally, fix $2c = 2c''$ and vary $c_1$ to create a path from $c'_1$ to $c''_1$ so as to cover the interval between $u(c'_1)$ and $u(c''_1)$. Along this path $u(c_1)$ remains in $\text{int} \mathcal{I}_a$; moreover, by Lemma 2, $f_1$ varies covering the interval between $f_1(c'_0, c'_1, c'_{2''}, 3c'')$ and $f_1(c'_0, c''_1, c''_2, 3c'')$. Since $c'_2$ is unchanged, again all $(f_1, f_2, d_3)$ along this third path are in $\mathcal{O}$. The three paths together form a connected path from $(f'_1, f'_2, d'_3)$ to $(f''_1, f''_2, d''_3)$ which never leaves $\mathcal{O}$.

We are now ready to obtain an additive representation of $\succ$, relying on Debreu (1960).

**Lemma 10.** The preference $\succ$ over $X \times \mathbb{Q}$ can be represented by an additive function

$$V^0(f_0, f_1, f_2, d_3) = \hat{u}(f_0) + a(f_1) + b(f_2) + \zeta(d_3),$$

where $\hat{u}$, $a$, $b$, and $\zeta$ are continuous, and $a$, $b$, $\zeta$ are strictly increasing on $\mathcal{U}$.

**Proof.** We first show that $\succ$ has an additive representation over $X \times \mathcal{O}$. By continuity, we then extend this representation to $X \times \mathbb{Q}$.

The set $\mathcal{O}$ may be expressed as a countable union of open rectangles $\{R^i\}_{i \in \mathbb{N}}$ of the form in Lemma 8, and such that for any $j$ there is an $i < j$ such that $R^i \cap R^j \neq \emptyset$. To construct $\{R^i\}_{i \in \mathbb{N}}$, proceed as follows. Let $\{\overline{B}^n\}_{n=1}^\infty$ be the sequence of closed balls of radius $n$ centered at the origin in $\mathbb{R}^3$. Then, let

$$\mathcal{K}^n = \{o \in \mathcal{O} \mid o \in \overline{B}^n, B^{1/n}(o) \subset \mathcal{O}\},$$

where $B^{1/n}(o)$ is the open ball of radius $1/n$ centered at point $o$. For each $n$, $\mathcal{K}^n$ is compact\(^{37}\) and the increasing sequence $\{\mathcal{K}^n\}_{n=1}^\infty$ converges to $\mathcal{O}$. So, each $\mathcal{K}^n$ can be covered by finitely many rectangles of the form in Lemma 8. Since $\mathcal{K}^n \subset \mathcal{K}^{n+1}$, when moving from $\mathcal{K}^n$ to $\mathcal{K}^{n+1}$, one can cover $\mathcal{K}^{n+1}$ by simply adding rectangles to those used to cover $\mathcal{K}^n$. Without loss, any added rectangle contains a point with rational coordinates not contained in other rectangles, so that the list of rectangles needed to cover $\mathcal{O}$, denoted by $\{R^i\}_{i \in \mathbb{N}}$, is countable. Finally, since $\mathcal{O}$ is connected, each $R^j$ must intersect at least another $R^i$. For simplicity, we can relabel the rectangles so that, for each $j$, we have $R^j \cap R^i \neq \emptyset$ for some $i < j$.

\(^{37}\mathcal{K}^n\) is clearly bounded. Consider any sequence $\{o^m\} \subset \mathcal{K}^n$ converging to $o'$. Since $\overline{B}^n$ is closed, $o' \in \overline{B}^n$. There remains to show that $B^{1/n}(o') \subset \mathcal{O}$. Let $o''$ be any point such that $||o' - o''|| = r < 1/n$.

Then $||o'' - o^m|| \leq r + ||o' - o^m||$. So, for $m$ large enough, $o'' \in B^{1/n}(o^m)$ and hence $o'' \in \mathcal{O}$. 

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For any $\mathcal{R}^i$, Lemmas 4 and 7 guarantee that the hypotheses of Debreu’s (1960) Theorem 3 are satisfied on $X \times \mathcal{R}^i$. Therefore, $\succ$ may be expressed over each $X \times \mathcal{R}^i$ as

$$V^i(f_0, f_1, f_2, d_3) = \hat{a}^i(f_0) + a^i(f_1) + b^i(f_2) + \zeta^i(d_3),$$

for functions $\hat{a}^i, a^i, b^i$, and $\zeta^i$ that are continuous and, except for $\hat{a}^i$, strictly increasing by the properties of $\bar{V}$ which induces $\succ$.

By construction, $\mathcal{R}^0$ and $\mathcal{R}^1$ have a nonempty open intersection. Over $X \times (\mathcal{R}^0 \cap \mathcal{R}^1)$ representations $V^0$ and $V^1$ must be positive affine transformations of each other (Debreu’s (1960) Theorem 3). So there exist constants $\rho > 0$ and $\chi \in \mathbb{R}$ such that, on $X \times (\mathcal{R}^0 \cap \mathcal{R}^1)$,

$$\hat{u}^0(f_0) = \rho \hat{u}^1(f_0) + \chi, \quad a^0(f_1) = \rho a^1(f_1), \quad b^0(f_2) = \rho b^1(f_2), \quad \zeta^0(d_3) = \rho \zeta^1(d_3).$$

Using these conditions, we can extend $\hat{u}^0, a^0, b^0$, and $\zeta^0$ to the set $X \times (\mathcal{R}^0 \cup \mathcal{R}^1)$. Indeed, each function $a^0, b^0$, and $\zeta^0$ is defined on $\mathcal{R}_k^i$ which denotes the projection of $\mathcal{R}^i$ on the $k$th dimension of $Q$. Consider $a^0$. By extending $a^0$ over $\mathcal{R}_1^i \setminus \mathcal{R}_0^i$ using $a^1$, the resulting function $a^0$ is well defined and continuous on $\mathcal{R}_1^i \cup \mathcal{R}_1^i$. By a similar reasoning for $b^0$ and $\zeta^0$, the function $V^0$ can be extended to $X \times (\mathcal{R}_0^i \cup \mathcal{R}_1^i) \times (\mathcal{R}_0^j \cup \mathcal{R}_2^j) \times (\mathcal{R}_0^k \cup \mathcal{R}_3^k)$. Since this product includes $X \times \mathcal{R}^0 \cup \mathcal{R}^1$, the function $V^0$ is, in particular, well defined and continuous on it.

Finally, since for each $j > 0$ we have $\mathcal{R}_j^i \cap \mathcal{R}_i^j \neq \emptyset$ for some $i < j$, we can extend by induction representation $V^0$ from $X \times \mathcal{R}^0$ to $X \times (\cup_{i \in \mathbb{N}} \mathcal{R}^i) = X \times \mathcal{O}$, in countably many steps. Notice that the functions $a, b$, and $\zeta$ (henceforth omit the superscript ‘0’) entering the formula of $V^0$ are defined, through the induction, over the respective projections of $\mathcal{O}$.

Since any point of $X \times \mathcal{O}$ is contained in $X \times \mathcal{R}^i$ for some $i \in \mathbb{N}$, $V^0$ and its components $\hat{u}, a, b$, and $\zeta$ are continuous over $X \times \mathcal{O}$. Moreover, $V^0$ represents $\succ$ on $X \times \mathcal{O}$. To see this, we need to check that for any $(f_0', f_1', f_2', d_3')$ and $(f_0'', f_1'', f_2'', d_3'')$ in $X \times \mathcal{O}$, $V^0(f_0', f_1', f_2', d_3') > V^0(f_0'', f_1'', f_2'', d_3'')$ if and only if $(f_0', f_1', f_2', d_3') \succ (f_0'', f_1'', f_2'', d_3'')$. Note that $(f_1', f_2', d_3')$ and $(f_1'', f_2'', d_3'')$ must both belong to some $\mathcal{K}^n$ in the previous construction. Since $V^0$ represents $\succ$ on $X \times \mathcal{K}^n$, it ranks $(f_0', f_1', f_2', d_3')$ and $(f_0'', f_1'', f_2'', d_3'')$ correctly, which proves the claim.

It remains to show that $V^0$ can be extended to the entire domain $X \times Q$, additively, and that it represents $\succ$ over this domain. We first show that $V^0$ can be extended to a continuous function over $X \times Q$. Recall that $\bar{V}$ is continuous and represents $\succ$ over $X \times Q$—and hence over $X \times \mathcal{O}$. So, there exists a strictly increasing map $\phi: Y \to Y^0$ such that $V^0 = \phi \circ \bar{V}$, where $Y^0$ and $Y$ are the ranges of $V^0$ and $\bar{V}$ on $X \times \mathcal{O}$. $Y^0$ and $Y$ are intervals of $\mathbb{R}$ because $X \times \mathcal{O}$ is connected and $V^0$ and $\bar{V}$ are continuous over this domain. Since $\phi$ is strictly increasing, it must be continuous on its domain, otherwise it would not cover $Y^0$. Let $\overline{Y}$ be the range of $\bar{V}$ over
$X \times Q$. Since $X \times Q \subset cl(X \times O)$ and $\overline{V}$ is continuous, $\overline{Y}$ contains at most two more points than $Y$ (its boundaries), and this may occur only when the relevant boundaries are finite. One can extend $\phi$ to these points, whenever applicable, by taking the limit of $\phi$: for example, if $\bar{y}$ denotes the upper bound of $\overline{Y}$ and $\overline{y} \notin Y$, one may define $\phi(\bar{y})$ as $\lim_{y \uparrow \bar{y}} \phi(y)$. Finally, we can extend $V^0$ to $X \times Q$ by letting $V^0 = \phi \circ \overline{V}$ over this domain. By construction, $V^0$ is continuous as the composition of continuous functions.

Next, we show that this extension of $V^0$ to $X \times Q$ still obeys the additive representation obtained on $X \times O$ in terms of $\hat{u}, a, b$ and $\zeta$. We first show that $a, b$, and $\zeta$ can be extended on the relevant projections of $Q$ (not just of $O$). Since $O$ is connected and $Q \subset clO$, the extension is only needed (possibly) at the two boundaries of $D$ for $\zeta$, and at the boundaries of $U$ for $a$ and $b$; these extensions are necessary only if these boundaries are achieved by some $(f_1, f_2, d_3) \in Q$.

To extend $\zeta$, suppose that there is an $(f_1, f_2, d_3) \in Q$ such that $d_3$ is the upper bound of $D$—the other case follows similarly. Without loss, we can choose $f_1, f_2 \in intU$. By perturbing $c_3$, we can then construct a sequence $\{(f_1^n, f_2^n, d_3^n)\}$ such that $f_1^n$ and $f_2^n$ are in some compact $K \subset intU$ and $d_3^n \in intD$ for all $n$, and $d_3^n \to d_3$. By construction, each $(f_1^n, f_2^n, d_3^n) \in O$. Fixing some $f_0$, the sum $\bar{\hat{u}}(f_0) + a(f_1^n) + b(f_2^n) + \zeta(d_3^n)$ is well defined and equal to $V^0(f_0, f_1^n, f_2^n, d_3^n)$ for each $n$. Moreover, possibly moving to subsequences, $f_1^n \to \hat{f}_1$ and $f_2^n \to \hat{f}_2$ for some $\hat{f}_1, \hat{f}_2 \in K$. Since $a$ and $b$ are continuous over $K$, $a(f_1^n)$ and $b(f_2^n)$ converge on these subsequences. Therefore, $\zeta(d_3^n)$ is well defined as the difference $V^0(f_0, \hat{f}_1, \hat{f}_2, d_3) - \bar{\hat{u}}(f_0) - a(\hat{f}_1) - b(\hat{f}_2)$, because $V^0$ has already been extended to $(f_0, \hat{f}_1, \hat{f}_2, d_3)$. Moreover, since $V^0$ was extended continuously over $X \times Q$, $\zeta$ must also be continuous at $d_3$.

We can similarly extend $\bar{b}$ to the boundary of $U$, whenever needed. To see this, take any $(f_1, f_2, d_3) \in Q$ such that $f_2$ lies at a boundary of $U$, say $\bar{b} = \bar{\bar{b}}$—again, the other case follows similarly. Moreover, we can choose $c_1$ in the inducing stream $c \in C$ so that $f_1 \in intU$. By perturbing $c_2$, we can build a sequence $\{(f_1^n, f_2^n, d_3^n)\}$ such that $f_1^n$ is in a compact $K \subset intU$ and $f_2^n \in intU$ for all $n$, and $f_2^n \to \overline{\bar{\overline{f}}}$.

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40One can show that for $\bar{y} \in \overline{Y} \setminus Y$, $\lim_{y \uparrow \bar{y}} \phi(y)$ must be finite. Suppose not: First, there exist i) $\bar{s} = (f_0, \hat{f}_1, \hat{f}_2, d) \in X \times Q$ such that $\bar{V}(\bar{s}) = \bar{y}$, which means that the agent prefers $\bar{s}$ to any other stream; and ii) a sequence $s^n = (f_0^n, f_1^n, f_2^n, d^n) \in X \times O$ that converges to $\bar{s}$, and such that $V^0(s^n)$ diverges to $+\infty$. Because $V^0$ is additive, this means that there must be at least one sequence, among $\bar{\bar{u}}(f_0^n), a(f_1^n), b(f_2^n)$, and $\zeta(d^n)$, with a subsequence diverging to $+\infty$. For example, suppose that $d^n$ is such that $\zeta(d^n)$ diverges to $+\infty$. Then, for any stream $c$ such that $d(3f(c)) = \bar{d}$, we have $\bar{V}(f_0(c), f_1(c), f_2(c), \bar{d}(c)) = \bar{y}$. Indeed, fix any $c_0$ and $c_0$’s such that $\bar{u} < a(c_0) < \bar{u}$ for $t \in 1, 2$. Choosing the sequence of continuation streams ($c_0, \ldots$) corresponding to the sequence of $d^n$ converging to $\bar{d}$, $V^0$ evaluated at those streams (and the fixed $c_0, c_1, c_2$) must diverge to $+\infty$. This implies that $\bar{V}$ converges to $\bar{y}$ for that sequence. By continuity of $\bar{V}$ over its entire domain, this implies that when choosing ($c_3, \ldots$) such that $d(3f(c)) = \bar{d}$, we have $\bar{V}(f_0(c), f_1(c), f_2(c), d(3f(c))) = \bar{y}$, regardless of the values of $c_0, c_1, c_2$. This, however, violates the fact that preferences are strictly increasing in $u(c_0)$ (Lemma 2), a contradiction. A similar contradiction can be derived if instead $\bar{\bar{u}}(f_0^n)$, or $a(f_1^n)$, or $b(f_2^n)$ has a subsequence diverging to $+\infty$. This shows that necessarily $Y_0$ is bounded above whenever $\bar{y} \in \overline{Y} \setminus Y$. By a similar argument for the lower bound, we conclude that $\phi$ is bounded at any boundary for which it needs to be extended.
some \( \hat{f}_1 \in K \), we obtain a well define limit for \( V^0, a, \) and \( \zeta, \) from which we can obtain the value of \( b(\nu) \). The argument for \( a \) is identical.

In conclusion, the function \( \hat{u}(\cdot) + a(\cdot) + b(\cdot) + \zeta(\cdot) \) is equal to \( V^0 \) over the entire set \( X \times Q \), and represents \( \succeq \) over this domain.

\( \square \)

**Step 4**: By Lemma 1 with \( \pi = \{1, 2, 3\} \), for any \( f_0 \in X \), the induced preference \( \succ^*_0 \) on \( F \) is independent of \( f_0 \). By Lemma 10, we can conclude that \( \succ^*_0 \) has a representation

\[
V^*_0(f_1, f_2, 3f) = a(f_1) + b(f_2) + \zeta(d(3f)).
\]

(15)

Note that Axiom 8 holds for any \( f_0 \). So if \( f_1 = f_1', (f_1, f_2, 3f) \succeq^*_0 (f_1', f_2', 3f') \) iff \((f_2, f_3, 4f) \succeq^*_0 (f_2', f_3', 4f')\).

**Lemma 11.** There exist \( \alpha > 0, \xi \in \mathbb{R} \), and \( G: U \to \mathbb{R} \) continuous and strictly increasing such that, for any finite \( T \geq 2 \) any \( f \in F \),

\[
V^*_0(f) = \sum_{t=1}^{T} \alpha^t G(f_1) + \alpha^T \tilde{d}(T+1f) + \xi \sum_{t=0}^{T-2} \alpha^t.
\]

(16)

**Proof.** Consider again \( R^0 \) in the proof of Lemma 10. By definition of a rectangle, if \((f_1, f_2, 3f)\) and \((f_1', f_2', 3f')\) are such that \((f_1, f_2, d(3f)), (f_1', f_2', d(3f')) \in R^0 \), then all \( \hat{f}_1 \in R^0 \) are feasible with both \((f_2, 3f)\) and \((f_2', 3f')\). By the stationarity property of \( \succ^*_0 \), we have

\[
a(\hat{f}_1) + b(f_2) + \zeta(d(3f)) \geq a(\hat{f}_1) + b(f_2') + \zeta(d(3f'))
\]

iff

\[
a(f_2) + b(f_3) + \zeta(d(4f)) \geq a(f_2') + b(f_3') + \zeta(d(4f')).
\]

Hence, since additive representations are unique up to positive affine transformations, for all \((f_2, 3f)\) such that \((f_1, f_2, d(3f)) \in R^0 \),

\[
\alpha(a(f_2) + b(f_3) + \zeta(d(4f))) + \xi = b(f_2) + \zeta(d(3f))
\]

(17)

for some \( \alpha > 0 \) and \( \xi \in \mathbb{R} \).

The argument used for \( R^0 \) can be equivalently applied to any \( R^j \) in the covering \( \{R^j\}_{i \in \mathbb{N}} \) of \( O \). Moreover, since for each \( j \geq 0 \) we have \( R^j \cap R^i \neq \emptyset \) for some \( i < j \), it is clear that the \( \alpha \) in (17) must be the same for all \( f \in F \) such that \((f_1, f_2, d(3f)) \in O \). That (17) must hold for all \( f \in F \) is implied by the following two observations. First, if \( c \in C \) induces \((f_1, f_2, d_3) \in O \), it imposes no restriction on \( d(4f(c)) \), which can take any value in \( D \)—hence \( 4f \) can take any value in \( F \). To see this, recall that for \( f \in F \) we defined \( d(f) = V(f_0, f) \) for some \( f_0 \in X \), and \( V(f_0, f) = \)
\( \hat{V}(f_0, f_1, d(2f)) \) by Lemma 5. So, since \( \hat{V} \) is strictly increasing in its second and third argument, the condition \( d_3(c) \in \text{int} \mathcal{D} \) only implies \( f_3(c) \in \text{int} \mathcal{U} \), but \( d(4f(c)) \) can be at the boundary of \( \mathcal{D} \). Therefore, (17) already holds for any value of \( 4f \in \mathcal{F} \). Second, suppose that \( f \) is such that \((f_1, f_2, d(3f))\) is at boundary of \( \mathcal{Q} \). Take a sequence \( \{f^n\} \) such that \((f_1^n, f_2^n, d(3f^n)) \in \mathcal{O} \) for all \( n \) and converges to \((f_1, f_2, d(3f))\). The sequence can be chosen so that \( 4f \) is fixed; perturbing only \( c_1, c_2, \) and \( c_3 \) is enough to guarantee that we are in \( \mathcal{O} \). Now recall that the functions \( a, b, \) and \( \zeta \) are continuous by Lemma 10. Then, the right-hand side of (17) converges, as do the first two terms of the left-hand side. The last term is constant and equal to \( \zeta(d(4f)) \), so it converges trivially. Therefore (17) holds everywhere.

We conclude that, for all \( f \in \mathcal{F} \),

\[
\hat{V}^*_0(f_1, f_2, 3f) = a(f_1) + \zeta + \alpha \hat{V}^*_0(f_2, f_3, 4f).
\]

Therefore, using this condition recursively and (15), for any \( f \in \mathcal{F} \) and finite \( T > 2 \), we have

\[
\hat{V}^*_0(f_1, f_2, 3f) = \sum_{t=0}^{T-1} \alpha^t a(f_{t+1}) + \alpha^{T-1} (b(f_{T+1}) + \zeta(d(f_{T+2}))) + \zeta \sum_{t=0}^{T-2} \alpha^t.
\]

The result then follows by defining \( G = \alpha^{-1}a \) and \( \bar{d}(\cdot) = \alpha^{-1}(b(\cdot) + \zeta(d(\cdot))) \).

By Lemma 11, for any finite \( T \geq 2 \), we can represent \( > \) for streams \( c \) as

\[
\hat{U}(c) = \hat{u}(c_0) + \sum_{t=1}^{T} \alpha^t G(U(c_t)) + \alpha^T \bar{d}(U(T+1c), U(T+2c), \ldots) + \zeta \sum_{t=0}^{T-2} \alpha^t. \tag{18}
\]

The next two technical lemmas will be useful to complete the proof of our theorem.

**Lemma 12.** For any constant streams \( c, c' \in C \), \( c > c' \) iff \( \hat{u}(c_0) > \hat{u}(c'_0) \).

**Proof.** Suppose \( \hat{u}(x) > \hat{u}(y) \) and consider \( c = (x, x, \ldots) \) and \( \hat{c} = (x, y, \ldots) \). For any \( t \geq 0 \) and \( c'' \in C \), define \( c' = (c_0, \ldots, c_t, c'') \) and \( \hat{c}' = (\hat{c}_0, \ldots, \hat{c}_t, c'' \ldots) \). For \( t = 0 \), we have \( \hat{U}(c') = \hat{U}(\hat{c}') \). For any \( t > 0 \), using (18), we first have \( \hat{U}(c') \geq \hat{U}(\hat{c}') \). Then, using again (18) backward recursively and monotonicity of \( G \), we conclude that \( \hat{U}(c') \geq \hat{U}(\hat{c}') \). Since this is true for any \( t \geq 0 \) and \( c'' \in C \), Axiom 7(ii) implies \( c > c' \). Now note that, again by (18), \( \hat{c} > (y, y, \ldots) \). Hence, by Axiom 1, \( c > (y, y, \ldots) \).

Now suppose \( \hat{u}(x) = \hat{u}(y) \) and consider \( c = (x, x, \ldots) \) and \( \hat{c} = (y, y, \ldots) \). For any \( t \) and \( c'' \in C \), define \( c' \) and \( \hat{c}' \) as before. Using again (18) backward recursively and the fact that \( G \) is a function, we conclude that \( \hat{U}(c') = \hat{U}(\hat{c}') \). Since this is true for any \( t \) and \( c'' \in C \), Axiom 7(ii) implies \( c \sim c' \).
Lemma 13. For any \( c \in C \), there exists \( x \in X \) such that \( c \sim (x, x, \ldots) \).

Proof. By Lemma 19 in Appendix B, for any \( c \in C \), there exists \( y \in X \) such that \( c \sim (c_0, y, y, \ldots) \). Suppose \( (c_0, y, y, \ldots) \not> (y, y, \ldots) \). If \( (c_0, y, y, \ldots) \not> (y, y, \ldots) \), then \( \hat{u}(c_0) > \hat{u}(y) \). Let \( \hat{c} = (c_0, c_0, \ldots) \) and \( \hat{c} = (c_0, y, y, \ldots) \). For any \( t \geq 0 \) and any \( \epsilon'' \in C \), consider \( \hat{c} = (\hat{c}_0, \ldots, \hat{c}_t, \epsilon'') \) and \( \hat{c} = (\hat{c}_0, \ldots, \hat{c}_t, \epsilon'') \). We have \( \hat{c} \succ \hat{c} \). Indeed, for \( t = 0 \), \( \hat{c} = \hat{c} \). For \( t > 0 \), we can proceed using (18). Since \( \hat{u}(c_0) > \hat{u}(y) \), \( U(\hat{c}) > U(\hat{c}') \). For \( s < t \), since \( \hat{u}(\hat{c}_s) \geq \hat{u}(\hat{c}_s') \) and \( G \) is strictly increasing, we have \( U(s \hat{c}) \geq U(s \hat{c}') \). By Axiom 7(ii), we then have \( \hat{c} \succ \hat{c} \) and hence \( (c_0, c_0, \ldots) \succ \hat{c} \succ (y, y, \ldots) \). Since \( X \) is connected, by Axiom 2, there exists \( x \in X \) such that \( (x, x, \ldots) \sim c \). The case \( (c_0, y, y, \ldots) \not< (y, y, \ldots) \) follows similarly.

We can now prove that \( \alpha < 1 \).

Lemma 14. \( \alpha < 1 \).

Proof. Consider consumption streams that are constant from \( t = 3 \) onward. Then \( f_t \) is constant for \( t \geq 3 \). So we can write \( d(4f) = d(4f) = e(f_3) \) in (17) and thus obtain

\[
(1 - \alpha)e(f_3) = ab(f_3) + \alpha a(f_2) - b(f_2) + \xi.
\]

First, note that \( f_3 > f_3' \) implies \( e(f_3) > e(f_3') \). By Lemma 12, \( f_3 > f_3' \) implies \( u(c_3) > u(c_3') \). Define \( c = (c_3, c_3, \ldots) \) and \( c' = (c_3, c_3', c_3', \ldots) \). Replicating the argument in the proof of Lemma 13, we have \( c \succ (c_3, c_3', c_3', \ldots) \). Moreover, by Axiom 7(ii), \( (c_3, c_3', c_3', \ldots) \succ c' \). Then, by Axiom 1 and Lemma 2, \( W(u(c_3), d(f_3, f_3, \ldots)) > W(u(c_3), d(f_3', f_3', \ldots)) \), which holds if \( d(f_3, f_3, \ldots) > d(f_3', f_3', \ldots) \).

Second, we can find \( \hat{c}, \hat{c} \in C \), constant from \( t = 3 \) onward, such that \( f_2(\hat{c}) = f_2(\hat{c})' \) and \( f_3(\hat{c}) > f_3(\hat{c})' \). Consider \( x, y \in X \) with \( u(x) > u(y) \) and the streams \( (x, y, y, \ldots) \) and \( (y, x, x, \ldots) \). By the previous argument based on Axiom 7(ii), \( (x, x, x, \ldots) \succ (x, y, y, \ldots) \). If \( (x, y, y, \ldots) \succ (y, x, x, \ldots) \), then by (18) and continuity of \( \hat{u} \) there exists \( z \in X \) such that \( (x, y, y, \ldots) \sim (z, x, x, \ldots) \). In this case, let \( \hat{c} = (c_0, c_1, z, x, x, \ldots) \). If \( (x, y, y, \ldots) \prec (y, x, x, \ldots) \), then by Axiom 2 there exists \( w \in X \) such that \( (x, y, y, \ldots) \prec (y, w, w, \ldots) \). Moreover, \( u(w) > u(y) \). Otherwise, since \( (y, y, y, \ldots) \succ (y, w, w, \ldots) \) for \( u(y) \geq u(w) \) (again by the same argument as before), we would have \( (x, y, y, \ldots) \succ (y, w, w, \ldots) \) by (18) and Axiom 1. In this case, let \( \hat{c} = (c_0, c_1, x, y, x, \ldots) \). Finally, let \( \hat{c} = (c_0, c_1, x, y, y, \ldots) \).

To conclude the proof, note that for \( \hat{c} \in \{\hat{c}, \hat{c}\} \), \( (1 - \alpha)e(f_3(c)) = ab(f_3(c)) + \xi' \) for some constant \( \xi' \). Since \( b \) and \( e \) are strictly increasing, we must have \( \alpha < 1 \).
Note that, if \( c \) is constant from any \( T \geq 3 \) onward, by Lemma 14 and (17)
\[
\tilde{d}(U(tc), U(t+c), \ldots) = \frac{\alpha}{1-\alpha} G(U(tc)) + \frac{\xi}{\alpha(1-\alpha)}.
\]
So, for eventually constant streams, we can write
\[
U(c) = \hat{u}(c_0) + \sum_{t=1}^{T} \alpha^t G(U(tc)) + \frac{\alpha^{T+1}}{1-\alpha} G(U(t+1c)) + \frac{1 + \alpha(1-\alpha^{T-1})}{\alpha(1-\alpha)} \xi.
\]

**Lemma 15.** \( G \) is bounded on \( U \).

**Proof.** By Axiom 3, \( V^*_0 \) is finite for all \( c \in C \). Suppose that \( G \) is unbounded above—the other case follows similarly. Then, for each \( r \in \mathbb{R}_{++} \), there must be a stream \( c^r \) with utility \( U^r \) such that \( G^r \equiv G(U^r) \geq r \). Moreover, for \( r > r' \), we can choose \( c^r \) and \( c^{r'} \) so that \( G^r > G^{r'} \), relying on continuity of \( G \) and connectedness of \( U \). By Lemma 13, for each \( r \) we can also let \( c' \) be constant. As a preliminary observation, note the following: given \( r' > r \), a stream \( c \) that equals \( c' \) for the first \( k \) periods and \( c'' \) forever after must satisfy \( G(U(c)) \geq r \). This is because, by definition, \( U(tc) > U(tc') \) for \( t \geq k \); then, by monotonicity of \( G \) and using (19) backward recursively, we have \( U(tc) > U(tc') \) for \( 0 \leq t < k \).

Now construct stream \( \hat{c} \) as follows. For some \( M > 1 \) and each \( t > 0 \), consider the constant stream \( c^{(M/\alpha)^t} \) with the property \( \alpha^t G^{(M/\alpha)^t} \geq M^t \). Then, let \( \hat{c}_0 \) be such that \( u < \hat{u}(\hat{c}_0) < \bar{u} \) and, for each \( t > 1 \), let \( \hat{c}_t = c^{(M/\alpha)^t}_t \). Now, for any \( T > 0 \), let \( c^T \) be equal to \( \hat{c} \) up to \( T \) and to \( c^{(M/\alpha)^T} \) thereafter. Using (19), we have
\[
\overline{U}(c^T) = \hat{u}(c_0) + \sum_{t=1}^{T-1} \alpha^t G(U(tc)) + \frac{\alpha^T}{1-\alpha} G^{(M/\alpha)^T} + \frac{1 + \alpha(1-\alpha^{T-2})}{\alpha(1-\alpha)} \xi
\]
\[
\geq \hat{u}(c_0) + \sum_{t=1}^{T-1} M_t + \frac{1}{1-\alpha} M^T + \frac{1 + \alpha(1-\alpha^{T-2})}{\alpha(1-\alpha)} \xi,
\]
where the inequality follows by recursively applying our preliminary observation. Note that the lower bound on \( \overline{U}(c^T) \) goes to \( +\infty \) as \( T \to \infty \).

Now fix any \( T \) and \( c^T \). To simplify notation, let \( \bar{c} = c^T \). Using Axiom 7(ii), we have \( \overline{U}(\bar{c}) \geq \overline{U}(c) \). To see this, consider any \( t \geq 0 \) and \( c'' \in C \), and let \( \bar{c} = (\bar{c}_0, \ldots, \bar{c}_t, c'') \) and \( \bar{c}' = (\bar{c}_0, \ldots, \bar{c}_t, c'') \). For \( t \leq T \), we have \( \bar{c}' \sim \bar{c} \) because the two streams are identical. For \( t > T \), we first have that \( u(\bar{c}_s) > u(\bar{c}_s) \) for \( T < s \leq t \) by Lemma 12. Hence, \( \overline{U}(\bar{c}_t, c'') > \overline{U}(\bar{c}_t, c'') \). Second, using again monotonicity of \( G \) and (16) recursively, we conclude \( \overline{U}(\bar{c}'') \geq \overline{U}(\bar{c}'') \). By Axiom 7(ii), we then have the claimed property.

It follows that, for any \( T \), \( \overline{U}(\bar{c}) \geq \overline{U}(c^T) \) and hence, since \( \hat{u}(\hat{c}_0) \) is bounded by assumption, \( V^*_0(f(\hat{c})) \) must be infinite, violating Axiom 3.
Lemma 16. For any $c \in C$, $U(c) = \hat{u}(c_0) + \sum_{t=1}^{\infty} \alpha^t G(U(tc))$.

Proof. Again by Axiom 3, $V^*_{0}$ is finite for all $c \in C$. Using (16) for any finite $T$ and observing that $tf$ can take any value in $F$, we conclude that the function $\hat{d}$ must be finite because $G$ is bounded. The result then follows by letting $T \rightarrow \infty$, relying on $\alpha < 1$ and ignoring the additive constant.

To conclude, both functions $U$ and $\bar{U}$ represent $\succ$ over $C$. So, they are strictly increasing transformations of one another. Letting $\bar{G}$ denote the function of $\bar{U}$ such that $\bar{G}(\bar{U}(c)) = G(U(c))$ for all $c$, we obtain representation (5). For uniqueness, note that the additive form of $\bar{U}$ is unique up to positive affine transformations, i.e., $\bar{U} = \rho U + \chi$ for $\rho > 0$ and $\chi \in \mathbb{R}$. So,

$$\bar{U}(c) = \rho \hat{u}(c_0) + \chi + \sum_{t=1}^{\infty} \alpha^t \rho G(U(tc)) = \rho \hat{u}(c_0) + \chi + \sum_{t=1}^{\infty} \alpha^t \rho G \left( \frac{\bar{U}(c) - \chi}{\rho} \right).$$

A.3 Proof of Proposition 3

Part (i). Take $\nu', \nu \in U$. By definition, there exist $c', c \in C$ such that $U(c') = \nu'$ and $U(c) = \nu$. By Lemma 13, we can take $c' = (x, x, \ldots)$ and $c = (y, y, \ldots)$ for some $x, y \in X$. Suppose $u(x) > u(y)$. Then, by Lemma 12, $U(x, \ldots) > U(y, \ldots)$. By representation (5),

$$U(x) - \frac{\alpha}{1-\alpha} G(U(x)) > U(y) - \frac{\alpha}{1-\alpha} G(U(y)).$$

Rearranging, we get that for any $\nu' > \nu$ in $U$

$$G(\nu') - G(\nu) < \frac{1-\alpha}{\alpha} (\nu' - \nu).$$

Lemma 17. For any $\varepsilon > 0$, there exists a constant $K \in (\frac{1-\alpha}{2\alpha}, \frac{1-\alpha}{\alpha})$ such that, for all $\nu' > \nu$ in $U$,

$$G(\nu') - G(\nu) \leq \max\{K(\nu' - \nu), \varepsilon\} \quad (20)$$

Proof. See Appendix B (Online Appendix).

To show that $U$ is $H$-continuous, consider any $c, \tilde{c} \in C$ and define $c^T = (c_0, c_1, \ldots, c_T, c)$ and $\tilde{c}^T = (c_0, c_1, \ldots, c_T, \tilde{c})$. Using Lemma 17, we will show that for any $\varepsilon > 0$, there exists $T$ such that

$$|U(c^T) - U(\tilde{c}^T)| < \frac{2\alpha \varepsilon}{1-\alpha}. \quad (21)$$
To do so, let \( M = \frac{\alpha}{1-\alpha} 2 \sup_{u \in U} |G(U)| \) and \( \delta = (1 + K) \alpha \). Since \( K < (1 - \alpha)/\alpha \), we have \( \delta < 1 \). Let \( T \) denote the first time such that \( KM \delta^T < \varepsilon \). Note that for all \( t < T \), we have \( \max\{KM \delta^t, \varepsilon \} = KM \delta^t \).

We first show that for all \( t < T \), we have \( |u(c^t) - u(\tilde{c}^t)| \leq M \delta^t \). The proof works by induction. For \( t = 0 \), we have \( c_0^t = \tilde{c}_0^t \), so

\[
|u(c^0) - u(\tilde{c}^0)| = \sum_{s=1}^{\infty} \alpha^s |G(u(\tilde{c}^0)) - G(u(c^0))| \leq M
\]

Suppose the claim holds for \( t < T - 1 \), we will show it holds for \( t + 1 \). We have

\[
|u(c^{t+1}) - u(\tilde{c}^{t+1})| \leq \alpha |G(u(1_1 c^{t+1})) - G(u(1_1 \tilde{c}^{t+1}))| + \alpha \sum_{s=1}^{\infty} \alpha^s |G(u(s+1 c^{t+1})) - G(u(s+1 \tilde{c}^{t+1}))|.
\]

By the induction hypothesis, the sum in (22) is bounded above by \( M \delta^t \). And because \( t < T - 1 \), we have \( KM \delta^t \geq \varepsilon \). Therefore,

\[
|u(c) - u(c')| \leq \alpha KM \delta^t + \alpha M \delta^t \leq M \delta^{t+1},
\]

which shows the claim.

Finally, for \( t = T \), (22) still applies, but this time the first term is bounded by \( \alpha \varepsilon \), because \( KM \delta^T < \varepsilon \). This implies that

\[
|u(c) - u(c')| \leq \alpha \varepsilon + \alpha M \delta^T \leq \alpha \varepsilon + \alpha \varepsilon / K = \delta \varepsilon / K.
\]

Since \( \delta < 1 \) and \( K > (1 - \alpha)/2 \alpha \), (21) follows.

Part (ii). Let \( C(M) \) be the set of consumption streams such that \( |u(c_t)| \leq M \) for all \( t \), and \( B(M) \) be the space of bounded real-valued functions with domain \( C(M) \). Endowed with the sup norm \( \|u\|_\infty = \sup_{c \in C(M)} |u(c)| \), \( B(M) \) is a complete metric space. Let \( \mathcal{J} \) be the operator on \( B(M) \) defined by

\[
\mathcal{J}(U)(c) = u(c_0) + \sum_{t=1}^{\infty} \alpha^t G(U(t,c)).
\]

By construction, \( \mathcal{J}(U) \) is bounded over \( C(M) \), as \( u \) is bounded by \( M \) and \( U \) is bounded over \( C(M) \). Moreover, since \( G \) is \( K \)-Lipschitz continuous with \( K < (1 - \alpha)/\alpha \), \( \mathcal{J} \) must be a contraction, as is easily checked. So, \( \mathcal{J} \) has a unique fixed point; call it \( U_M \). As \( M \) increases, the domain of \( U_M \) increases. However, for any \( M, N \), uniqueness of the fixed point guarantees that \( U_M \) and \( U_N \) coincide on the intersection of their domains. Thus, we obtain a unique solution \( U^* \) to (5) over \( C(B) = \cup_M C(M) \).
Let $\mathcal{H}$ be the set of $H$-continuous functions. To verify that $U^* \in \mathcal{H}$, it suffices to show that (a) $\mathcal{J}$ maps $\mathcal{H}$ onto itself, and (b) $\mathcal{H}$ is closed under the sup norm. Indeed, this will guarantee that $\mathcal{J}$’s fixed-point belongs to $\mathcal{H}$. To show (a), take any $U \in \mathcal{H}$ and $\varepsilon > 0$. Since $\alpha < 1$ and $G$ is bounded, there is $T > 0$ such that $\frac{\alpha^T 2\hat{G}}{1-\alpha} < \varepsilon / 2$, where $G = \sup_{\nu \in \mathcal{U}} |G(\nu)|$. Moreover, since $U \in \mathcal{H}$, there exists $N$ such that $|U(c) - U(\tilde{c})| < \varepsilon / 2$ whenever $c_t = \tilde{c}_t$ for all $t \leq N$. For any $c$ and $\tilde{c}$,

$$|\mathcal{J}(U)(c) - \mathcal{J}(U)(\tilde{c})| \leq \sum_{t=1}^{\infty} \alpha^t |G(U(tc)) - G(U(t\tilde{c}))| \leq K \sum_{t=1}^{T-1} \alpha^t |U(tc) - U(t\tilde{c})| + T \frac{2\hat{G}}{1-\alpha},$$

where $K$ is the Lipschitz constant of $G$. The first term is less than $\frac{K\alpha}{(1-\alpha)^T} \max_{t \leq T-1} |U(tc) - U(t\tilde{c})|$. Now suppose that $c_t = \tilde{c}_t$ for all $t \leq N' = N + T$. This implies that $(tc)_t = (t\tilde{c})_t$ for all $t \leq T$ and $t' \leq N$, because $c$ is truncating at most $T$ elements of $c$, and $c$ and $\tilde{c}$ were identical up to time $T + N$, by construction. By definition of $N$, we have $|U(tc) - U(t\tilde{c})| < \varepsilon / 2$ for all $t \leq T$ and, hence, $|\mathcal{J}(U)(c) - \mathcal{J}(U)(\tilde{c})| < \varepsilon$. Setting $T(\varepsilon) = N'$ shows that $\mathcal{J}(U)$ satisfies $H$-continuity.

To prove (b), consider a sequence $\{U^m\}$ in $\mathcal{H}$ that converges to some limit $U$ in the sup norm. Now fix $\varepsilon > 0$. There is $m$ such that $\|U^m - U\|_\infty < \varepsilon / 3$. Since $U^m \in \mathcal{H}$, there is $N$ such that $|U^m(c) - U^m(\tilde{c})| < \varepsilon / 3$ whenever $c_t = \tilde{c}_t$ for all $t \leq N$. Thus, for such $c, \tilde{c}$,

$$|U(c) - U(\tilde{c})| \leq |U(c) - U^m(c)| + |U^m(c) - U^m(\tilde{c})| + |U^m(\tilde{c}) - U(\tilde{c})| < \varepsilon,$$

which shows that $U \in \mathcal{H}$.

To extend the definition of $U^*$ from $C(B)$ to $C$, for any $c \in C \setminus C(B)$, consider any sequence $\{c^n\}$ in $C(B)$ such that $c^n_t = c_t$ for all $t \leq n$, and let $U^*(c) = \lim_{n \to +\infty} U^*(c^n)$. This limit is well-defined and independent of the chosen sequence. To see this, note that, for any such sequence $\{c^n\}$ and any $\varepsilon > 0$, $H$-continuity of $U^*$ implies that there is $T$ such that $|U^*(c) - U^*(\tilde{c})| < \varepsilon$ whenever $c_t = \tilde{c}_t$ for all $t \leq T$. Hence, $|U^*(c^n) - U^*(\tilde{c}^n)| < \varepsilon$ for all $n, m \geq T$, since the consumption levels of $c^n$ and $\tilde{c}^n$ coincide up to $\min\{n, m\}$. So, $\{U^*(c^n)\}$ forms a Cauchy sequence in $\mathbb{R}$ and thus converges. Moreover, the limit is independent of the chosen sequence, as for any $\varepsilon > 0$, $|U^*(c^n) - U^*(\tilde{c}^n)| < \varepsilon$ for $n$ large enough and sequences $\{c^n\}$ and $\{\tilde{c}^n\}$ of the type constructed above.

The limit $U$ thus defined satisfies representation (5). Since $U^*$ is a fixed point of $\mathcal{J}$ on $C(B)$ and $c^n$ belongs to $C(B)$, for each $n$

$$U^*(c^n) = u(c^n_0) + \sum_{t=1}^{\infty} \alpha^t G(U^*(\tilde{c}^n_t))$$

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The left-hand side converges to $U^*(c)$. Moreover, for each $t$, $U^*(tc^α)$ converges to $U^*(tc)$ (which is similarly well defined). Since $G$ is continuous, $G(U^*(tc^α))$ converges to $G(U^*(tc))$ for each $t$. Since $α < 1$ and $G$ is bounded, by the dominated convergence theorem, the right-hand side converges to $u(c_0) + \sum_{t=1}^{∞} α^tG(U^*(tc))$, which proves that (5) holds for all $c \in C$.

Finally, there is a unique $H$-continuous extension of $U^*$ from $C(B)$ to $C$ that solves (5). To see this, let $U$ be any $H$-continuous solution to (5). Since $U$ is a fixed point of $J$ and the fixed point is unique on $C(B)$, $U$ must coincide with $U^*$ on $C(B)$. Take any $c \in C \setminus C(B)$ and $ε > 0$. By $H$-continuity of $U$ and $U^*$, both $|U(c) − U(\bar{c})|$ and $|U^*(c) − U^*(\bar{c})|$ are less than $ε/2$ for some $\bar{c} \in C(B)$ equal to $c$ for all $t$ up to a large $N$. Since $U$ and $U^*$ must be equal at $\bar{c}$, $|U(c) − U^*(c)| < ε$. Since $ε$ was arbitrary, $U(c) = U^*(c)$ for all $c$, establishing uniqueness.

### A.4 Proof of Theorem 4

Using Axiom 10 and Theorem 3, we also have

\[
(c_0, c_1, 2c) \succ (c_0, c_1, 2c′) \iff (\hat{c}_0, c_1, 2c′) \succ (\hat{c}_0, c_1, 2c′) \tag{23}
\]

\[
(c_0, c_1, 2c) \succ (c_0, c_1, 2c′) \iff (\hat{c}_0, c_1, 2c′) \succ (\hat{c}_0, c_1, 2c′) \tag{24}
\]

\[
(c_0, c_1, 2c) \succ (c_0, c_1, 2c′) \iff (\hat{c}_0, c_1, 2c′) \succ (\hat{c}_0, c_1, 2c′) \tag{25}
\]

\[
(c_0, c_1, 2c) \succ (c_0, c_1, 2c′) \iff (\hat{c}_0, c_1, 2c′) \succ (\hat{c}_0, c_1, 2c′) \tag{26}
\]

By Debreu’s (1960) Theorem 3, conditions (23)-(26) and (i)-(ii) in Axiom 10 imply that $\succ$ can be represented by

\[
w_0(c_0) + w_1(c_1) + w_2(2c),
\]

for some continuous and nonconstant functions $w_0$, $w_1$, and $w_2$. By Theorem 3, $\succ$ is also represented by

\[
u(c_0) + αG(u(c_1) + g(2c)) + αg(2c),
\]

where $g(2c) = \sum_{t=2}^{∞} α^{t-1}G(U(tc))$. It follows that

\[
u(c_0) + αG(u(c_1) + g(2c)) + αg(2c) = ξ [w_0(c_0) + w_1(c_1) + w_2(2c)] + χ,
\]

for some $ξ > 0$ and $χ \in \mathbb{R}$. This implies that

\[
αG(u(c_1) + g(2c)) + αg(2c) = ξ [w_1(c_1) + w_2(2c)],
\]

and therefore $G$ must be affine. Since $G$ must be increasing, without loss of generality let $G(U) = γU$ with $γ > 0$. Finally, by Proposition 3, $γ < \frac{1−α}{α}$.
A.5 Proof of Proposition 4

By assumption, for all \( t \),
\[
U(t,c) = u(c_t) + \sum_{\tau=t+1}^{\infty} \beta \delta^{\tau-t} u(c_\tau),
\]
where \( 0 < \beta = \frac{\gamma}{1+\gamma} < 1 \), \( 0 < \delta = (1+\gamma)\alpha < 1 \), \( 0 < \alpha < 1 \).

For the ‘if part’ see the main text. For the ‘only if’ part, using (27), we get
\[
\sum_{t=0}^{\infty} w(t)U(t,c) = w(0)u(c_0) + \sum_{t=1}^{\infty} u(c_t) \left[w(t) + \beta \delta^t \left( \sum_{\tau=0}^{t-1} \frac{w(\tau)}{\delta^\tau} \right) \right].
\]

By assumption, \( \sum_{t=0}^{\infty} w(t)U(t,c) = \sum_{t=0}^{\infty} \delta^t u(c_t) \). So the coefficients of \( u(c_t) \) must match for all \( t \). For \( t = 0 \), \( w(0) = 1 \). Then, for \( t = 1 \), \( w(1) = (1-\beta)\delta = \alpha \). Now suppose \( w(t) = \alpha^t \) for all \( t = 0, \ldots, \tau \). Then,
\[
w(\tau+1) = \delta^{\tau+1} - \beta \delta^{\tau+1} \frac{1 - \alpha^{\tau+1}}{1 - \frac{\alpha}{\delta}} = \alpha^{\tau+1}.
\]

Hence, by induction, \( w(t) = \alpha^t \) for all \( t \).

References


B Online Appendix: Omitted Proofs

(TO APPEAR ONLY AS ONLINE MATERIAL)

B.1 Proof of Theorem 1

The proof follows and generalizes that of Diamond (1965), and is based on the following lemmas.

Lemma 18 (Debreu (1954)). Let $C$ be a completely ordered set and $Z = (z_0, z_1, ...)$ be a countable subset of $C$. If for every $c, c' \in C$ such that $c \prec c'$, there is $z \in Z$ such that $c \preceq z \preceq c'$, then there exists on $C$ a real, order-preserving function, continuous in any natural topology.\footnote{A natural topology is one under which Axiom 2 holds for that topology.}

Lemma 19. For any $c \in C$, there exists $x \in X$ such that $c \sim (c_0, x, x, ...)$.

Proof. Given $c$, let $D_c = \{(c_0, y, y, ...): y \in X\}$, $A = \{d \in D_c : d \succeq c\}$, and $B = \{d \in D_c : d \succeq c\}$. By Axiom 1, $A \cup B = D_c$; by Axiom 2, $A$ and $B$ are closed; by Axiom 3, $A$ and $B$ are nonempty. Moreover, $D_c$ is connected. Indeed, for any continuous function $\phi : D_c \to \{0, 1\}$, the function $\tilde{\phi} : X \to \{0, 1\}$ defined by $\tilde{\phi}(x) = \phi(c_0, x, x, ...)$ is also continuous. Connectedness of $X$ implies that $\tilde{\phi}$ is constant and, hence, that $\phi$ is constant, showing connectedness of $D_c$. This implies that $A \cap B \neq \emptyset$.

\[\square\]

To conclude the proof of Theorem 1, let $Z_0$ be a countable dense subset of $X$, which exists since $X$ is separable, and let $Z$ be the subset of $C$ consisting of streams $(x, y, y, ...)$ with $x, y \in Z_0$. Lemma 19 implies that $Z$ satisfies the hypothesis of Lemma 18, which yields the result. Indeed, by Lemma 19 there are $x, y \in X$ such that $(c_0, x, x, ...) \sim c \prec c' \sim (c_0', y, y, ...).$ Consider the set $E \subset X^2$ consisting of $(z, w)$ such that $(c_0, x, x, ...) \prec (z, w, w, ...)$, $(c_0', y, y, ...)$, $E$ is nonempty by connectedness of $X$ and open by Axiom 2. Since $Z$ is dense in $X^2$, $E$ must contain an element of $Z$.

B.2 Proof of Proposition 1

Suppose that $V(c_0, U(1c), U(2c), ...) = V(c_0, U(1c))$ for all $c \in C$ and $V$ is strictly increasing in $U(1c)$. By Assumption 1, if $1c \sim 1c'$, then\footnote{This step would be meaningless if $U(1c)$ represented how the agent evaluates consumption streams starting at 1 from the perspective of 0, but not necessarily how he evaluates such streams from the perspective of 1. This observation applies to the rest of the proof.} $U(1c) = U(1c')$ and, since $V$ is a function, $V(c_0, U(1c)) = V(c_0, U(1c'))$; hence $(c_0, 1c \sim 0 (c_0, 1c').$ If $1c \succ 1c'$, then $U(1c) > U(1c')$ and, since $V$ is strictly increasing in its second argument, $V(c_0, U(1c)) > V(c_0, U(1c'))$; hence $(c_0, 1c \succ 0 (c_0, 1c').$
Suppose $1c \sim 1 c'$ implies $(c_0, 1c) \sim^0 (c_0, 1c')$. Then, for any $(U(1c), U(2c), \ldots)$ and $(U(1c'), U(2c'), \ldots)$ such that $U(1c) = U(1c')$,

$$V(c_0, U(1c), U(2c), \ldots) = V(c_0, U(1c'), U(2c'), \ldots).$$

So $V$ can depend only on its first two arguments. Suppose $1 c \succ 1 c'$ implies $(c_0, 1c) \succ^0 (c_0, 1c')$. Then, $U(1c) > U(1c')$. Moreover, it must be that $V(c_0, U(1c)) > V(c_0, U(1c'))$; that is, $V$ must be strictly increasing in its second argument.

### B.3 Proof of Corollary 1

By Theorem 3, $\succ$ can be represented by

$$U(c) = u(c_0) + \sum_{t=1}^{\infty} \alpha^t G(U(tc)).$$

Since $(x, c) \succ (y, c)$, $u(x) = u(y) + \overline{u}$ for some $\overline{u} > 0$. Hence, for any $t > 0$, $U(tc) - U(cy)$ equals $\overline{u} - \sum_{s=1}^{t} \alpha^s \Delta G_s$, where $\Delta G_s$ is defined recursively as follows: for $s = t$,

$$\Delta G_t = G(U(tc^y)) - G(U(tc^y) - \overline{u}),$$

otherwise

$$\Delta G_s = G(U_s(c^y)) - G \left( U_s(c^y) - \sum_{k=1}^{t-s} \alpha^k \Delta G_{s+k} \right).$$

By Proposition 3, $\Delta G_t < \frac{1-\alpha}{\alpha} \overline{u}$ and

$$\Delta G_{t-1} = G(U_{t-1}(t-1c^y)) - G(U_{t-1}(t-1c^y) - \alpha \Delta G_t)$$

$$< (1-\alpha) \Delta G_t < \frac{(1-\alpha)^2}{\alpha} \overline{u}.$$

Now, suppose that, for all $k$ such that $s < k \leq t - 1$, $\Delta G_k < \frac{(1-\alpha)^2}{\alpha} \overline{u}$. It follows that

$$\Delta G_s < \frac{1-\alpha}{\alpha} \left[ \sum_{\tau=1}^{t-s} \alpha^\tau \Delta G_{s+\tau} \right] < \frac{1-\alpha}{\alpha} \left[ \sum_{\tau=1}^{t-s-1} \alpha^\tau \frac{(1-\alpha)^2}{\alpha} + \alpha^{t-s} \frac{(1-\alpha)}{\alpha} \right] \overline{u}$$

$$= \frac{(1-\alpha)^2}{\alpha} \left[ \sum_{\tau=0}^{t-s-2} \alpha^\tau (1-\alpha) + \alpha^{t-s-1} \right] \overline{u} = \frac{(1-\alpha)^2}{\alpha} \overline{u}.$$

Therefore,

$$\sum_{s=1}^{t} \alpha^s \Delta G_s < \pi \left[ \alpha^t \frac{1-\alpha}{\alpha} + \sum_{s=1}^{t-1} \alpha^s \frac{(1-\alpha)^2}{\alpha} \right] = \pi (1-\alpha).$$
We conclude that $U(c^x) - U(c^y) > \alpha \bar{u} > 0$.

### B.4 Proof of Corollary 2

By representation (5), $U$ clearly depends on $c_0$ only through $u_0 = u(c_0)$. This implies that $U(c)$—and hence also $U(c)$ (from (5))—depends on $c_1$ only through $u_1 = u(c_1)$. By induction, $U(c)$ depends on $(c_0, \ldots, c_t)$ only through $(u_0, \ldots, u_t)$, for each $t$. There remains to establish the result at infinity: If $c$ and $\tilde{c}$ are two streams such that $u(c_t) = u(\tilde{c}_t)$ for all $t$, we need to show that $U(c) = U(\tilde{c})$. From the previous step, assume without loss of generality that $c_t = \tilde{c}_t$ for all $t \leq T$, where $T$ is any large, finite constant. Since $U$ is $H$-continuous, we can choose $T$ so that $|U(c') - U(\tilde{c}')| < \varepsilon$ for all $c', \tilde{c}'$ that coincide up to $T$. Since $c$ and $\tilde{c}$ satisfy this property, $|U(c) - U(\tilde{c})| < \varepsilon$, and since $\varepsilon$ was arbitrary, $U(c) = U(\tilde{c})$. This shows that the sequence $\{u_t = u(c_t)\}_{t=0}^\infty$ of period-utility levels entirely determines the value of $U(c)$, proving the result.

### B.5 Proof of Proposition 4

Consider representation (5) in Theorem 3. For every $c \in C$, we have sequences $\{u_s\}_{s=0}^\infty$ and $\{U_s\}_{s=0}^\infty$, where $u_s = u(c_s)$ and $U_s = \hat{U}(u_s, u_{s+1}, \ldots)$. Using the notation,

$$d(t, c) = \frac{\partial U_0 / \partial u_t}{\partial U_0 / \partial u_0}.$$ 

Note that $\frac{\partial U_s}{\partial u_s} = 1$ for all $s \geq 0$. Since $G$ is differentiable, we have

$$\frac{\partial U_0}{\partial u_t} = \sum_{\tau=0}^{t-1} \alpha^{t-\tau} G'(U_{t-\tau}) \frac{\partial U_{t-\tau}}{\partial u_t}.$$ 

More generally, for $1 \leq \tau \leq t$,

$$\frac{\partial U_{t-\tau}}{\partial u_t} = \sum_{s=0}^{\tau-1} \alpha^{t-s} G'(U_{t-s}) \frac{\partial U_{t-s}}{\partial u_t}.$$ 

So, for $\tau = 1$, $\frac{\partial U_{t-1}}{\partial u_t} = \alpha G'(U_t)$. More generally, for $2 \leq \tau \leq t$,

$$\frac{\partial U_{t-\tau}}{\partial u_t} = \alpha \sum_{s=0}^{(\tau-1)-1} \alpha^{(\tau-1)-s} G'(U_{t-s}) \frac{\partial U_{t-s}}{\partial u_t} + \alpha G'(U_{t-(\tau-1)}) \frac{\partial U_{t-(\tau-1)}}{\partial u_t}$$

$$= \frac{\partial U_{t-(\tau-1)}}{\partial u_t} \alpha (1 + G'(U_{t-(\tau-1)})).$$
So,
\[ \frac{\partial U_{t-s}}{\partial u_t} = \alpha^r G'(U_i) \prod_{s=1}^{\tau-1} (1 + G'(U_{t-s})). \]
Let \( \prod_{s=1}^{\tau-1} (1 + G'(U_{t-s})) = 1 \) if \( \tau = 1 \). Then,
\[ \frac{\partial U_0}{\partial u_t} = \alpha^r G'(U_i) + G'(U_i) \sum_{\tau=1}^{t-1} \alpha^r G'(U_{t-s}) \prod_{s=1}^{\tau-1} (1 + G'(U_{t-s})) \]
\[ = \alpha^r G'(U_i) \left[ 1 + \sum_{\tau=1}^{t-1} G'(U_{t-s}) \prod_{s=1}^{\tau-1} (1 + G'(U_{t-s})) \right]. \]

B.6 Proof of Corollary 6

For every \( c \in C \), consider the sequence \( \{U_s\}_{s=0} \) in the proof of Proposition 4. Using representation (5) and Axiom 7(ii) as in Lemma 12, we have that \( c \geq u' \) implies \( U_s \geq U'_s \) for all \( s \geq 0 \). It is immediate that, if \( G' \) is increasing (decreasing), then \( d(t,c) \geq (\leq) d(t,c') \) for all \( t > 0 \). On the other hand, suppose \( G' \) is not increasing, i.e., there is \( U > U' \) in \( \mathcal{U} \) such that \( G'(U) < G'(U') \)—the other case is similar. By definition and Lemma 19, \( U = U(c) \) and \( U' = U(c') \) for some constant streams \( c \) and \( c' \). By Lemma 12, \( c \geq u' \). However, for all \( t > 0 \), \( d(t,c) < d(t,c') \).

B.7 Proof of Lemma 17

Recall that for any \( \nu' > \nu \) in \( \mathcal{U} \)
\[ G(\nu') - G(\nu) < \frac{1-\alpha}{\alpha}(\nu' - \nu). \]
We will show that, for any \( \varepsilon > 0 \) small enough, there exists a constant \( K < \frac{1-\alpha}{\alpha} \) such that
\[ G(\nu') - G(\nu) \leq \max\{K(\nu' - \nu), \varepsilon\} \quad (28) \]
for all \( \nu' > \nu \) in \( \mathcal{U} \).

Case (i): Suppose first that \( \mathcal{U} \) is bounded and let \( \overline{\mathcal{U}} = \text{cl}(\mathcal{U}) \). If necessary, extend \( G \) to \( \overline{\mathcal{U}} \) by continuity. Since \( \overline{\mathcal{U}} \) is compact and \( G \) is continuous, it is also uniformly continuous. Hence, for any \( \varepsilon > 0 \), there exists \( \eta(\varepsilon) > 0 \) such that \( |\nu - \nu'| < \eta(\varepsilon) \) implies \( |G(\nu) - G(\nu')| < \varepsilon \). Let \( \Delta(\varepsilon) = \{\nu, \nu' \in \overline{\mathcal{U}}^2 \mid |\nu - \nu'| + \eta(\varepsilon) \} \). The function \( F(\nu, \nu') = \frac{G(\nu) - G(\nu')}{\nu - \nu'} \) is continuous and strictly less\(^\dagger\) than \( \frac{1-\alpha}{\alpha} \) on the compact set \( \Delta(\varepsilon) \) and thus has a strictly positive upper bound

\(^\dagger\)This is true by assumption if \( \nu \) and \( \nu' \) belong to \( \mathcal{U} \), and it is easy to show that it is still true if either \( \nu \) or \( \nu' \) belongs to \( \overline{\mathcal{U}} \setminus \mathcal{U} \). For example, if \( \nu' \) is the infimum of \( \mathcal{U} \), one can take any point \( \tilde{\nu} \in (\nu', \nu) \). By assumption \( G(\nu) - G(\tilde{\nu}) < (1-\alpha)/\alpha(\nu - \tilde{\nu}) \) and, by continuity of \( G \), \( G(\tilde{\nu}) - G(\nu') \leq (1-\alpha)/\alpha(\tilde{\nu} - \nu') \). Combining these inequalities yields the result, as is easily seen. (One way of showing this is to use the
\( K < \frac{1-\alpha}{\alpha} \). By construction, (28) holds for any \((\nu, \nu') \in \Delta(\varepsilon)\) and any \((\nu, \nu') \in \overline{U}^2 \setminus \Delta(\varepsilon)\).

**Case (ii):** Suppose that \( U \) is unbounded both above and below—the intermediate cases follow by combining the two cases shown here. Let \( \underline{G} = \inf_{\nu \in U} G(\nu) \) and \( \overline{G} = \sup_{\nu \in U} G(\nu) \), which are finite and distinct because \( G \) is bounded and strictly increasing. Fix any \( \varepsilon < \overline{G} - \underline{G} \). Let \( \nu(\varepsilon) = G^{-1}(\underline{G} + \varepsilon) \) and \( \nu'(\varepsilon) = G^{-1}(\overline{G} - \varepsilon) \). If either \( \nu \leq \nu(\varepsilon) \) and \( \nu' \leq \nu(\varepsilon) \), or \( \nu \geq \nu'(\varepsilon) \) and \( \nu' \geq \nu'(\varepsilon) \), then (28) holds by construction. Now take any \( \overline{\nu}, \underline{\nu} \in U \) with \( \overline{\nu} > \nu'(\varepsilon) + 2(\frac{1}{1-\alpha} + 1) \) and \( \underline{\nu} < \nu(\varepsilon) - 2(\frac{1}{1-\alpha} + 1) \). On the compact set \([\underline{\nu}, \overline{\nu}]\), the continuous function \( G \) is uniformly continuous, so there exists \( \eta > 0 \) and \( \eta(\varepsilon) = \min\{\eta, \frac{1}{2}(\overline{\nu} - \nu'(\varepsilon)), \frac{1}{2}(\nu(\varepsilon) - \underline{\nu})\} \) such that \( |\nu - \nu'| < \eta(\varepsilon) \) implies \( |G(\nu) - G(\nu')| < \varepsilon \). Let \( \Delta'(\varepsilon) = \{(\nu, \nu') \in [\underline{\nu}, \overline{\nu}]^2 \mid \nu \geq \nu' + \eta(\varepsilon)\} \). By the same argument as before, the function \( F(\nu, \nu') = \frac{G(\nu) - G(\nu')}{\nu - \nu'} \) has a strictly positive upper bound \( K_1 < \frac{1-\alpha}{\alpha} \) on the set \( \Delta'(\varepsilon) \).

Define \( \nu_m = \frac{1}{2}(\nu + \nu'(\varepsilon)) \) and \( \nu_m = \frac{1}{2}(\nu + \nu(\varepsilon)) \). The only difficulty is to show the claim when \( \nu' < \nu'(\varepsilon) \leq \nu - \nu' \) or \( \nu' < \nu(\varepsilon) \). We focus on the first case. If \( \nu' < \nu(\varepsilon) \), by construction \( \nu_m - \nu' \geq \eta(\varepsilon) \) and hence

\[
\frac{G(\nu_m) - G(\nu')}{\nu_m - \nu'} < K_1. \tag{29}
\]

Now note that

\[
\nu - \nu_m > \nu - \nu_m = \frac{1}{2}(\nu - \nu(\varepsilon)) > \alpha\varepsilon \left(\frac{1}{1-\alpha}\right) + 1.
\]

Hence, there exists a strictly positive \( K_2 < \frac{1-\alpha}{\alpha} \) such that, for all \( \nu > \nu(\varepsilon) \), we have \( \nu - \nu_m > \varepsilon/K_2 \). Since \( \nu > \nu(\varepsilon) \) and \( \nu_m > \nu(\varepsilon) \), it follows that

\[
\frac{G(\nu) - G(\nu_m)}{\nu - \nu_m} \leq \frac{\varepsilon}{\nu - \nu_m} < K_2. \tag{30}
\]

For any strictly positive \( a, b, c, d, (a + c)/(b + d) \leq \max\{a/b, c/d\} \). Combining this inequality to (29) and (30), we conclude that

\[
\frac{G(\nu) - G(\nu')}{\nu - \nu'} \leq \max\{K_1, K_2\}.
\]

By a similar argument, for all \( \nu' < \nu \leq \nu(\varepsilon) < \nu, \)

\[
\frac{G(\nu) - G(\nu')}{\nu - \nu'} \leq \max\{K_1, K_3\}
\]

for some strictly positive \( K_3 < \frac{1-\alpha}{\alpha} \). Letting \( K = \max\{K_1, K_2, K_3\} \) then proves the claim of the lemma.

\[\text{fact that } a/b < c/d \Rightarrow (a + b)/(c + d) < c/d \text{ for } a, b, c, d \text{ strictly positive — see the argument at the end of this proof.}\]