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LEXICOGRAPHIC PROBABILITIES AND EQUILIBRIUM REFINEMENTS

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This paper develops a decision-theoretic approach to normal-form refinements of Nash equilibrium and, in particular, provides characterizations of (normal-form) perfect equilibrium (Selten (1975)) and proper equilibrium (Myerson (1978)). The approach relies on a theory of single-person decision making that is a non-Archimedean version of subjective expected utility theory. According to this theory, each player in a game possesses, in addition to a strategy space and a utility function on outcomes, a vector of probability distributions, called a *lexicographic probability system* (LPS), on the strategies chosen by the other players. These probability distributions can be interpreted as the player's first-order and higher order theories as to how the game will be played, and are used lexicographically in determining an optimal strategy. We define an equilibrium concept, called *lexicographic Nash equilibrium*, that extends the notion of Nash equilibrium in that it dictates not only a strategy for each player but also an LPS on the strategies chosen by the other players. Perfect and proper equilibria are described as lexicographic Nash equilibria by placing various restrictions on the LPS's possessed by the players.

KEYWORDS: Nash equilibrium, equilibrium refinements, perfect equilibrium, proper equilibrium, lexicographic probabilities, non-Archimedean preferences.

1. INTRODUCTION

THIS PAPER DEMONSTRATES how a non-Archimedean version of subjective expected utility theory can be used to provide decision-theoretic foundations for normal-form refinements of Nash equilibrium. In so doing, this paper complements recent work by Aumann and others that provides Bayesian foundations for correlated equilibrium and for Nash equilibrium itself.

There are, by now, several well known intuitive arguments as to why certain Nash equilibria in a game are unreasonable prescriptions for how the game will be played. The large number of refinements of Nash equilibrium in the literature represents different attempts to try to capture these intuitive ideas in a formal solution concept. In this paper we provide decision-theoretic characterizations of two such refinements for normal-form games, namely, perfect equilibrium (Selten (1975)) and proper equilibrium (Myerson (1978)).

An explicitly decision-theoretic approach to game theory requires specifying for each player, not only a strategy space and a utility function on outcomes (as is done in the orthodox description of a game), but also beliefs over the strategies chosen by the other players. This viewpoint rests on the supposition that players are Bayesian decision makers who conform to the axioms of subjective expected utility theory (Savage (1954)), so that a player's utility function *and beliefs* are derived simultaneously from preferences over strategies.

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The inclusion of beliefs in the description of a game is central to the analysis of normal-form games developed by Aumann (1987), Bernheim (1984, 1986), Brandenburger and Dekel (1987), Pearce (1984), and Tan and Werlang (1988). In all these papers players are explicitly modelled as subjective expected utility maximizers and hence possess both utility functions on outcomes and probability distributions on strategies. The importance of beliefs has also been stressed by Kreps and Wilson (1982) in the context of refinements of Nash equilibrium based on the extensive form. Kreps and Wilson's solution concept of sequential equilibrium dictates a (behavioral) strategy for each player and, in addition, for each information set of that player a probability distribution over the nodes in the information set. Moreover, Kreps and Wilson propose a lexicographic procedure for forming beliefs at information sets (op. cit., pp. 873–874) that is similar to the assumption made in this paper about how players form beliefs. The papers by Myerson (1986) and Okada (1987, 1988) also employ lexicographic techniques in the context of equilibrium refinements.[2]

This paper is concerned with normal-form games. (The reader is referred to Kohlberg and Mertens (1986, pp. 1010–1012) for a defense of the normal-form approach to refinements of Nash equilibrium.) However, our treatment of a normal-form game is distinct from that in Aumann (1987) and the related papers mentioned above in that we begin with a different (non-Archimedean) single-person decision theory. Rather than supposing each player to be a subjective expected utility maximizer, we apply the theory of decision making developed in Blume, Brandenburger, and Dekel (1990). According to this latter theory, a player possesses, not a single probability distribution, but rather a vector of probability distributions that is used lexicographically in selecting an optimal strategy. Such a vector of probability distributions is called a *lexicographic probability system* (LPS). The first component of the LPS can be thought of as representing the player's primary theory as to how the game will be played, the second component the player's secondary theory, and so on. In this paper we define an equilibrium concept, called *lexicographic Nash equilibrium*, that extends the notion of Nash equilibrium in that it dictates not only a strategy for each player but also an LPS on the strategies chosen by the other players. We go on to show how perfect and proper equilibria can be characterized as lexicographic Nash equilibria by placing various restrictions on the LPS's possessed by the players.

The organization of the rest of the paper is as follows. Section 2 begins by reviewing the non-Archimedean version of subjective expected utility theory developed in Blume, Brandenburger, and Dekel (1990). Next, two results are presented that play a key role in applying non-Archimedean choice theory to the task of characterizing refinements of Nash equilibrium. Section 3 starts by defining a lexicographic Nash equilibrium and then provides characterizations of perfect and proper equilibrium. We treat two-person and general (N-person)

[2] Fishburn (1972) and Skala (1974, 1975) investigate existence of equilibrium in two-person zero-sum games with non-Archimedean utilities.

games separately since the latter require additional assumptions on the players' LPS's. Concluding remarks are in Section 4.

2. NON-ARCHIMEDEAN SUBJECTIVE EXPECTED UTILITY THEORY

This section begins with a review of the non-Archimedean version of subjective expected utility theory developed in Blume, Brandenburger, and Dekel (1990).[3] The framework employed follows the treatment in Fishburn (1982) of Anscombe and Aumann (1963).

The decision maker faces a finite set of states Ω and a set of pure consequences C. Let $\mathscr{P}$ denote the set of simple (i.e., finite support) probability distributions on consequences. The decision maker has preferences over acts, which are maps from the state space Ω into $\mathscr{P}$. Thus the set of acts is the product space $\mathscr{P}^{\Omega}$. The ωth coordinate of act x is denoted x_{ω}, and determines the objective lottery on consequences that obtain if act x is chosen and state ω occurs. Conventional subjective expected utility theory imposes axioms on the decision maker's weak preference relation on $\mathscr{P}^{\Omega}$ that are necessary and sufficient for the following familiar representation (Anscombe and Aumann (1963)). There is an (affine) utility function $u:\mathscr{P}\rightarrow\mathbb{R}$ and a probability measure p on Ω such that an act x is (weakly) preferred to another act y if and only if

$$(2.1)\qquad \sum_{\omega\in\Omega} p(\omega)u(x_{\omega}) \geqslant \sum_{\omega\in\Omega} p(\omega)u(y_{\omega}).$$

In Blume, Brandenburger, and Dekel (1990) a modified axiom system involving a weakened Archimedean axiom is proposed that is necessary and sufficient for the following representation. There is an (affine) utility function u: $\mathscr{P}\rightarrow\mathbb{R}$ and a vector $\rho=(p_1,\ldots,p_K)$, for some integer K, of probability measures on Ω such that an act x is (weakly) preferred to another act y if and only if

$$(2.2)\qquad \left[\sum_{\omega\in\Omega} p_k(\omega)u(x_{\omega})\right]_{k=1}^{K} \geqslant_L \left[\sum_{\omega\in\Omega} p_k(\omega)u(y_{\omega})\right]_{k=1}^{K}.$$ [4]

Such a vector $\rho=(p_1,\ldots,p_K)$ of probability measures will be referred to as a *lexicographic probability system* (LPS). The first component p_1 of the LPS ρ can be thought of as representing the decision maker's primary "theory" about the true state of the world, the second component p_2 the decision maker's secondary "theory," and so on.

A "numerical" representation of preferences which is equivalent to the representation (2.2) is possible where the "numbers" are elements in a non-Archimedean ordered field $\mathbb{F}$ that is a proper extension of the real number field $\mathbb{R}$. This representation is exactly the same as (2.1) except that while u is an $\mathbb{R}$-valued utility function, p is now an $\mathbb{F}$-valued probability distribution on Ω.

[3] The reader should refer to this paper for a fuller discussion of the material in this section.

[4] The symbol $\geqslant_L$ represents the lexicographic ordering. That is, for $a,b\in\mathbb{R}^K$, $a\geqslant_L b$ if and only if whenever $b_k>a_k$, there exists an $l<k$ such that $a_l>b_l$.

As it stands, the class of preferences having the representation (2.2) contains conventional Archimedean preferences (the special case $K=1$). The force behind the modified theory is that, unlike conventional subjective expected utility theory, it can be made to guarantee admissibility (Luce and Raiffa (1957, Chapter 13)) and the existence of well defined conditional probabilities for *any* event, without giving up some notion of probability 0 events.

DEFINITION 1: An LPS $\rho=(p_1,\ldots,p_K)$ on Ω has *full support* if for each $\omega\in\Omega$, $p_k(\omega)>0$ for some $k=1,\ldots,K$.

Blume, Brandenburger, and Dekel (1990, Section 4) show that if ρ has full support, then admissibility is satisfied and well defined conditional probabilities exist for any event. Even so, a state ω can be "infinitely unlikely" in the sense that $p_1(\omega)=0$ (of course, it must then be that $p_k(\omega)>0$ for some $k>1$). Blume, Brandenburger, and Dekel (1990, Section 5) go on to discuss a partial order on states, $\omega>_\rho\omega'$, to be read as "ω is infinitely more likely than ω' according to the LPS ρ."

DEFINITION 2: Given an LPS $\rho=(p_1,\ldots,p_K)$ on Ω and states ω, $\omega'\in\Omega$, write $\omega>_\rho\omega'$ if

$$\min\{k: p_k(\omega)>0\}<\min\{k: p_k(\omega')>0\}.$$

The weak version of this order is defined in the usual fashion by $\omega\geqslant_\rho\omega'$ if not $\omega'>_\rho\omega$. This complete and transitive weak order will turn out to be useful in characterizing proper equilibrium (see Sections 3.2 and 3.3 below).

We close this section by stating two results that are key to understanding the relationship between the non-Archimedean choice theory described in this section and the test sequences, or "trembles," used in defining refinements of Nash equilibrium. Given an LPS $\rho=(p_1,\ldots,p_K)$ on Ω and a vector $r=(r_1,\ldots,r_{K-1})\in(0,1)^{K-1}$, write $r\,\square\,\rho$ for the probability measure on Ω defined by the nested convex combination

$$(1-r_1)p_1+r_1\big[(1-r_2)p_2+r_2\big[(1-r_3)p_3 \\ +r_3\big[\cdots+r_{K-2}\big[(1-r_{K-1})p_{K-1}+r_{K-1}p_K\big]\cdots\big]\big]\big].$$

PROPOSITION 1: *Given any finite subset X of acts in $\mathscr{P}^{\Omega}$, there is a sequence $r(n)\in(0,1)^{K-1}$ with $r(n)\to 0$ such that for all $x,y\in X$, x is strictly preferred to y if and only if*

$$\sum_{\omega\in\Omega}(r(n)\,\square\,\rho)(\omega)u(x_\omega)>\sum_{\omega\in\Omega}(r(n)\,\square\,\rho)(\omega)u(y_\omega)$$

for all n.[5]

PROPOSITION 2: *Let $p(n)$ be a sequence of probability measures on Ω. Then there is an LPS $\rho=(p_1,\ldots,p_K)$ on Ω such that a subsequence $p(m)$ of $p(n)$ can be written as $p(m)=r(m)\,\square\,\rho$ for a sequence $r(m)\in(0,1)^{K-1}$ with $r(m)\to 0$.*

[5] See the Appendix for the proof of Proposition 1 and all subsequent results in the paper.

Propositions 1 and 2 make precise the sense in which LPS's and (convergent) sequences of probability measures—which correspond to the "trembles" of the refinements literature—are interchangeable. They play a central role in proving the characterizations of refinements of Nash equilibrium to be presented in the next section. Finiteness is important in both propositions: Proposition 1 applies to a finite subset X of acts in $\mathscr{P}^{\Omega}$ while Proposition 2 relies on the state space Ω being finite. In what follows, these finiteness assumptions will be satisfied since the subject matter is finite games.

3. LEXICOGRAPHIC NASH EQUILIBRIUM

3.1. *Nash Equilibrium*

A finite N-person game in normal form is a $2N$-tuple $\Gamma = \langle A^1, \ldots, A^N; u^1, \ldots, u^N \rangle$ where for each $i = 1, \ldots, N$, A^i is player i's finite set of pure strategies (henceforth actions) and u^i: $\times_{j=1}^N A^j \to \mathbb{R}$ is i's von Neumann-Morgenstern utility function. For any finite set Y, let $\Delta(Y)$ denote the set of all probability measures on Y. The set of mixed strategies of player i is then $\Delta(A^i)$, with a typical element denoted by σ^i. An N-tuple of mixed strategies is denoted by $\sigma = (\sigma^1, \ldots, \sigma^N) \in \times_{i=1}^N \Delta(A^i)$.

For the reasons discussed in the Introduction, we now augment this orthodox description of a normal-form game by specifying not only utility functions over outcomes but also, for each player, beliefs over the uncertainty that player faces. It is assumed that each player is a decision maker of the kind described in Section 2, and hence that the beliefs of each player take the form of an LPS on an appropriate state space. In order to write down this assumption formally, it will be helpful to have some more notation: given sets $Y^1, \ldots, Y^N$, let Y^{-i} denote the set $Y^1 \times \cdots \times Y^{i-1} \times Y^{i+1} \times \cdots \times Y^N$, and $y^{-i} = (y^1, \ldots, y^{i-1}, y^{i+1}, \ldots, y^N)$ a typical element of Y^{-i}. The beliefs of player i are described by an LPS $\rho^i = (p_1^i, \ldots, p_{K^i}^i)$, for some integer K^i, on A^{-i}. A collection of LPS's, one for each player, will be denoted by $\mu = (\rho^1, \ldots, \rho^N)$ and called a *belief system*. Given an LPS ρ^i for player i, and $j \neq i$, let $\rho^{ij} = (p_1^{ij}, \ldots, p_{K^i}^{ij})$ denote the marginal on A^j of ρ^i. That is, for each $k = 1, \ldots, K^i, p_k^{ij}$ is the marginal on A^j of p_k^i.

DEFINITION 3: A pair (μ, σ) is a *lexicographic Nash equilibrium* if for all $i = 1, \ldots, N$ and $j \neq i$:

(i) $p_1^{ij}(a^j) > 0$ implies

$$\left[\sum_{a^{-j} \in A^{-j}} p_k^j(a^{-j}) u^j(a^j, a^{-j}) \right]_{k=1}^{K^j} \geq_L \left[\sum_{a^{-j} \in A^{-j}} p_k^j(a^{-j}) u^j(b^j, a^{-j}) \right]_{k=1}^{K^j}$$

for all $b^j \in A^j$;

(ii) $p_1^i(a^{-i}) = \times_{j \neq i} \sigma^j(a^j)$ for all $a^{-i} \in A^{-i}$.

A lexicographic Nash equilibrium is an extension of the notion of Nash equilibrium in that it requires specifying for each player, in addition to a (mixed) strategy σ^i, an LPS ρ^i on the actions of the other players. Condition (i) of Definition 3 is a "knowledge of rationality" requirement: if player i assigns positive probability under p_1^i to an action a^j of some other player j, then a^j must be an optimal action for j given j's beliefs ρ^j. Notice that Condition (i) places no restriction on i's higher order beliefs $p_2^i, \ldots, p_{K^i}^i$. Condition (ii) is a consistency condition linking beliefs to strategies. It says that the first order probability that player i assigns to each $(N-1)$-tuple of actions a^{-i} must coincide with the weight given to a^{-i} by the mixed strategies σ^{-i}.

It is possible to drop any mention of the strategies σ and work exclusively in terms of the belief system μ. Conceptually, this approach offers some advantages. For example, it provides an interpretation of Nash equilibrium solely in terms of the players' knowledge and beliefs that does away with the troublesome notion of players randomizing their actions (cf. Aumann (1987) and the related papers mentioned in the Introduction). Nevertheless, to facilitate comparisons with the conventional definitions of Nash equilibrium and its refinements, we chose in this paper to work with strategies as well as beliefs.

As it stands, Definition 3 of a lexicographic Nash equilibrium is equivalent, in terms of equilibrium strategies, to Nash equilibrium. This is stated formally in Proposition 3 below. The next two sections demonstrate how, by placing restrictions on the players' higher order beliefs, the notion of a lexicographic Nash equilibrium can be used to characterize refinements of Nash equilibrium.

PROPOSITION 3: *The strategy profile σ is a Nash equilibrium if and only if there is some belief system μ for which (μ, σ) is a lexicographic Nash equilibrium.*

Before proceeding, it is worth examining the similarities and differences between the solution concepts of lexicographic Nash equilibrium (Definition 3) and sequential equilibrium (Kreps and Wilson (1982)). While the first is defined on the normal form and the second on the extensive form, both require the specification of beliefs as well as strategies. In developing the notion of a sequential equilibrium, Kreps and Wilson devote considerable attention to placing various "consistency" conditions on beliefs, including a restriction called "lexicographic consistency" (Kreps and Wilson (1982, pp. 873–874), Kreps and Ramey (1987)). Indeed, in the absence of restrictions on beliefs, in particular if beliefs at each information set need not even be concentrated on (i.e., assign probability 1 to) that information set, an analogous result to Proposition 3 would obtain: sequential equilibrium would be equivalent, in terms of equilibrium strategies, to Nash equilibrium. In what follows we likewise consider restrictions on the belief system μ. Thus we provide a model for analyzing normal-form refinements that is analogous to the extensive-form model of strategies and beliefs underlying sequential equilibrium.[6] A distinction between the approach taken by Kreps and Wilson and that in this paper, however, is that

[6] Moreover, sequential equilibrium has proved useful in developing and understanding further refinements, e.g., McLennan (1985), Cho and Kreps (1987), Cho (1987).

in a sequential equilibrium players take into account higher order beliefs as to how the game is being played only if their lower order beliefs are contradicted, whereas in a lexicographic Nash equilibrium players *always* take into account (lexicographically, of course) their higher order beliefs. This explains why, once the players' LPS's are required to have full support, choices in a lexicographic Nash equilibrium are admissible (see Sections 3.2 and 3.3 below) whereas choices in a sequential equilibrium need not be.

3.2. *Refinements in Two-person Games*

This section provides characterizations of normal-form perfect equilibrium and proper equilibrium in two-person games as lexicographic Nash equilibria with restrictions on the players' higher order beliefs. The N-person case, which is postponed until Section 3.3, raises two further questions. Do two players share the same beliefs about a third player's choice of action? Are a player's beliefs over the actions of the other players (stochastically) independent? The simplicity of the two-person case derives from the fact that neither of these questions arises.

Recall that in two-person games a Nash equilibrium is perfect if and only if each player's (mixed) strategy is admissible. This gives an immediate characterization of perfect equilibrium in terms of lexicographic Nash equilibrium. (In the following, to say that a belief system $\mu = (\rho^1, \dots, \rho^N)$ has full support means that each ρ^i, $i = 1, \dots, N$, has full support.)

PROPOSITION 4: *The strategy profile σ is a perfect equilibrium if and only if there is some belief system μ with full support for which (μ, σ) is a lexicographic Nash equilibrium.*

Proper equilibrium can be described as a lexicographic Nash equilibrium by ensuring that each player's beliefs respect the preference relations of the other players.

DEFINITION 4: The belief system $\mu = (\rho^1, \dots, \rho^N)$ *respects preferences* if for all $i = 1, \dots, N$ and $j \neq i$, each $a^j \in A^j$ satisfies

$$\left[\sum_{a^{-j} \in A^{-j}} p_k^j(a^{-j}) u^j(a^j, a^{-j})\right]_{k=1}^{K^j} \geqslant_L \left[\sum_{a^{-j} \in A^{-j}} p_k^j(a^{-j}) u^j(b^j, a^{-j})\right]_{k=1}^{K^j}$$

for all $b^j \in A^j$ with $a^j \geqslant_{\rho^{ij}} b^j$.

Definition 4 says that player i's LPS ρ^i should respect player j's preference relation—as determined by j's utility function u^j and LPS ρ^j— in the following sense. Suppose player i believes that an action b^j of player j is not infinitely more likely than some other action a^j. Then b^j must not be strictly better than a^j for player j. Definition 4 is a strengthening of Condition (i) of Definition 3 in that the latter applies only to those actions a^j of j for which $p_1^{ij}(a^j) > 0$.

Player 2

Player 1	L	C	R
T	1, 1	1, 1	1, 0
M	1, 1	2, 2	2, 2
B	0, 1	2, 2	3, 3

Γ_1

FIGURE 1

PROPOSITION 5: *The strategy profile σ is a proper equilibrium if and only if there is some belief system μ that has full support and respects preferences for which (μ, σ) is a lexicographic Nash equilibrium.*

We illustrate Propositions 4 and 5 by means of the game Γ_1 depicted in Figure 1. This game has three Nash equilibria: ((1, 0, 0), (1, 0, 0)), ((0, 1, 0), (0, 1, 0)), and ((0, 0, 1), (0, 0, 1)). Only the last two are perfect and only the last is proper. Given $\sigma = (\sigma^1, \sigma^2) = ((0,1,0),(0,1,0))$, define a belief system $\mu = (\rho^1, \rho^2)$ by

$$p_1^1 = (0,1,0) = \sigma^2, \quad p_1^2 = (0,1,0) = \sigma^1,$$

$$p_2^1 = (1,0,0), \quad p_2^2 = (1,0,0),$$

$$p_3^1 = (0,0,1), \quad p_3^2 = (0,0,1).$$

Condition (i) of Definition 3 is satisfied: $p_1^2(M) > 0$ and, while M is indifferent to B under p_1^1, M is strictly preferred to B under p_2^1. Similarly, $p_1^1(C) > 0$ and C is optimal under ρ^2. Hence (μ, σ) is a lexicographic Nash equilibrium. Moreover, μ has full support which confirms that σ is a perfect equilibrium. Notice, however, that μ does *not* respect preferences (Definition 4): player 1 strictly prefers B to T, yet player 2 believes T is infinitely more likely than B. In fact, since $p_1^1 = \sigma^2$ by itself implies that player 1 strictly prefers B to T, if μ is to respect preferences, it must be the case that player 2 believes B to be infinitely more likely than T. But then player 2 strictly prefers R to C. This demonstrates that σ is not a proper equilibrium. Finally, given $\tilde{\sigma} = (\tilde{\sigma}^1, \tilde{\sigma}^2) = ((0,0,1),(0,0,1))$, define a belief system $\tilde{\mu} = (\tilde{\rho}^1, \tilde{\rho}^2)$ by

$$\tilde{p}_1^1 = (0,0,1) = \tilde{\sigma}^2, \quad \tilde{p}_1^2 = (0,0,1) = \tilde{\sigma}^1,$$

$$\tilde{p}_2^1 = (0,1,0), \quad \tilde{p}_2^2 = (0,1,0),$$

$$\tilde{p}_3^1 = (1,0,0), \quad \tilde{p}_3^2 = (1,0,0).$$

Player 2

Player 1	L	C	R
T	1, −1	−1, 1	1, −3
B	−1, 1	1, −1	0, 2

Γ_2

FIGURE 2

The pair $(\tilde{\mu}, \tilde{\sigma})$ is a lexicographic Nash equilibrium where $\tilde{\mu}$ has full support and respects preferences. This confirms that $\tilde{\sigma}$ is a proper equilibrium.

Observe that in each of the LPS's constructed in the preceding example, the component probability distributions had disjoint supports. Blume, Brandenburger, and Dekel (1990, Section 5) provide an axiomatic derivation of LPS's $\rho = (p_1, \ldots, p_K)$ in which the p_k's have disjoint supports. However, despite the appeal of this class of LPS's, it is sometimes necessary to consider LPS's in which the p_k's do have overlapping supports. To see this, refer to the game Γ_2 depicted in Figure 2. This game has a unique Nash—hence perfect and proper—equilibrium: $((\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2}, 0))$. Thus, in any lexicographic Nash equilibrium of Γ_2, player 1's first-order belief is given by $p_1^1 = (\frac{1}{2}, \frac{1}{2}, 0)$. If player 1's LPS is to have full support, while at the same time p_1^1 and p_2^1 are to have disjoint supports, then it must be the case that $p_2^1 = (0, 0, 1)$. But then player 1 strictly prefers T to B, which upsets the equilibrium. A second order belief for player 1 that preserves indifference between T and B is $p_2^1 = (0, \frac{1}{3}, \frac{2}{3})$, but notice that now p_1^1 and p_2^1 have overlapping supports. If, however, we consider a game with the same reduced normal form as Γ_2, that is, if duplication of pure strategies is permitted, then disjoint supports can be guaranteed (Blume, Brandenburger, and Dekel (1990, Section 5)).[7]

We close this section by observing that the characterization of proper equilibrium offered in Proposition 5 is helpful in understanding the distinction between proper equilibrium and the solution concept of stability due to Kohlberg and Mertens (1986). Recall that stable sets of equilibria satisfy the following property ("Forward Induction"): given a stable set S of Nash equilibria, if a strategy a^i that is an inferior response in all equilibria in S is deleted, then S contains a stable set of the reduced game (Kohlberg and Mertens (1986, Proposition 6)). The strength of this property derives from the fact that it does not rely on a specific order of deletion. Proper equilibrium satisfies an analogous, albeit weaker, property that *is* order dependent. The following result is an immediate consequence of Proposition 5.

[7] Let player 2's new strategy space be $\{L, C, \tilde{C}, R\}$ where the column $\tilde{C}$ duplicates the column C, and let $\tilde{p}_1^1 = (\frac{1}{2}, \frac{1}{2}, 0, 0)$, $\tilde{p}_2^1 = (0, 0, \frac{1}{3}, \frac{2}{3})$. Then $\tilde{p}_1^1$ and $\tilde{p}_2^1$ have disjoint supports and correspond to the original equilibrium.

COROLLARY 1: *Given a lexicographic Nash equilibrium* (μ, σ) *where the belief system* μ *has full support and respects preferences, for some player i delete all* $a^i \in A^i$ *that are least preferred by i. Then the restriction of* (μ, σ) *to the reduced game is again a lexicographic Nash equilibrium where the (restricted) belief system* μ *has full support and respects preferences.*

3.3. *Refinements in N-person Games*

This section extends the characterizations of perfect and proper equilibrium in two-person games to general (N-person) games. In order to carry out this extension, we need to ensure that the players' beliefs satisfy the following two conditions: (1) any two players share the same beliefs about a third player's choice of action; and (2) each player's beliefs over the actions of the other players are stochastically independent.

Condition (1) can be satisfied by assuming a non-Archimedean analog to the Common Prior Assumption in Aumann (1987).

DEFINITION 5: The belief system $\mu = (\rho^1, \ldots, \rho^N)$ satisfies the *Common Prior Assumption* (CPA) if there is an LPS ρ on $\times_{i=1}^N A^i$ such that for all $i = 1, \ldots, N$, ρ^i is the marginal on A^{-i} of ρ.

The meaning of Condition (2) is more ambiguous. In Blume, Brandenburger, and Dekel (1990, Section 7) it is shown that three possible definitions of stochastic independence, while equivalent in standard probability theory, are distinct in the lexicographic setting. We briefly review each of the three definitions within the framework of Section 2. Suppose that the set of states Ω is a product space $\Omega^1 \times \cdots \times \Omega^N$. An LPS $\rho = (p_1, \ldots, p_K)$ on Ω satisfies S-independence if p_1 is a product measure (see Blume, Brandenburger, and Dekel (1990, Definition 7.2)). Axiom 6 in Blume, Brandenburger, and Dekel (1990) provides a stronger, preference-based, notion of stochastic independence. The axiom states that for any $i = 1, \ldots, N$ and $\omega^i, \tilde{\omega}^i \in \Omega^i$, the decision maker's conditional preference relation given $\{\omega^i\} \times \Omega^{-i}$ is the same as that given $\{\tilde{\omega}^i\} \times \Omega^{-i}$. Being preference-based, this definition of stochastic independence is perhaps the most compelling from a decision-theoretic perspective. Finally, given an LPS $\rho = (p_1, \ldots, p_K)$ on Ω, the strongest version of independence (Blume, Brandenburger, and Dekel (1990, Definition 7.1)) requires there to be an equivalent $\mathbb{F}$-valued probability measure that is a product measure. In this case we will say that ρ satisfies *strong independence*. It is straightforward to show that ρ satisfies strong independence if and only if there is a sequence $r(n) \in (0,1)^{K-1}$ with $r(n) \to 0$ such that $r(n) \square \rho$ is a product measure for all n. Notice that in the special case $K = 1$, strong independence coincides with S-independence, but in general the former is more restrictive.[8]

[8] Since $\lim_n r(n) \square \rho = p_1$, strong independence implies S-independence.

In a lexicographic Nash equilibrium (μ, σ) of a game Γ, each player i's LPS $\rho^i = (p_1^i, \ldots, p_{K^i}^i)$ automatically satisfies S-independence since Condition (ii) of Definition 3 requires the first order belief p_1^i to be a product measure. But neither of the stronger notions of independence is necessarily satisfied. Without imposing either of these stronger versions of independence, or the CPA, the following N-person analog to Proposition 4 obtains. This proposition also provides a characterization of c-perfect equilibrium (Fudenberg, Kreps, and Levine (1988, p. 377)).

PROPOSITION 6: *The strategy profile σ is a Nash equilibrium in admissible strategies if and only if there is some belief system μ with full support for which (μ,σ) is a lexicographic Nash equilibrium.*

In order to characterize perfect and proper equilibrium in N-person games, both the CPA and strong independence must be imposed. To apply strong independence to a game Γ, we suppose that the belief system $\mu = (\rho^1, \ldots, \rho^N)$ satisfies the CPA. That is, there is an LPS ρ on $\times_{i=1}^N A^i$ such that each ρ^i is the marginal on A^{-i} of ρ. In this case we say that μ satisfies strong independence if ρ satisfies strong independence.

PROPOSITION 7: *The strategy profile σ is a perfect equilibrium if and only if there is some belief system μ that has full support and satisfies the CPA and strong independence for which (μ, σ) is a lexicographic Nash equilibrium.*

PROPOSITION 8: *The strategy profile σ is a proper equilibrium if and only if there is some belief system μ that has full support, respects preferences, and satisfies the CPA and strong independence for which (μ, σ) is a lexicographic Nash equilibrium.*

Proposition 8 immediately provides the following N-person analog to Corollary 1.

COROLLARY 2: *Given a lexicographic Nash equilibrium (μ, σ) where the belief system μ has full support, respects preferences, and satisfies the CPA and strong independence, for some player i delete all $a^i \in A^i$ that are least preferred by i. Then the restriction of (μ, σ) to the reduced game is again a lexicographic Nash equilibrium where the (restricted) belief system μ has full support, respects preferences, and satisfies the CPA and strong independence.*

4. CONCLUDING REMARKS

This paper has shown how non-Archimedean choice theory provides a decision-theoretic characterization of equilibrium refinements. The non-Archimedean approach is useful both to deepen our understanding of existing refinements and to develop alternatives.

For example, we have seen (in Section 3.3) that strong independence is the appropriate notion of stochastic independence for characterizing perfect and proper equilibrium. However, strong independence is obviously a stringent requirement and perhaps not totally compelling from a decision-theoretic perspective. Hence, it would be of interest to characterize lexicographic Nash equilibria in which weaker notions of stochastic independence are satisfied. Figure 7.1 in Blume, Brandenburger, and Dekel (1990) shows that Axiom 6 is strictly weaker than strong independence in single-person decision problems. The same example can be used to construct a game in which there is a lexicographic Nash equilibrium (μ, σ), where the belief system μ has full support, respects preferences, and satisfies the CPA and stochastic independence in the guise of Axiom 6, but where σ is *not* a proper equilibrium. Hence Axiom 6 is also strictly weaker than strong independence in the context of equilibrium. A characterization of the effect of substituting Axiom 6 for strong independence would be useful. Related issues are discussed in Binmore (1987, 1988), Dekel and Fudenberg (1990), Fudenberg, Kreps, and Levine (1988), and Kreps and Ramey (1987). Along the same lines of weakening stochastic independence assumptions, another possible extension would be to use the non-Archimedean choice theory described in Section 2 of this paper to characterize refinements of *correlated* equilibrium, such as acceptable correlated equilibrium and predominant correlated equilibrium (Myerson (1986)).

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APPENDIX

Proof of Proposition 1: *Necessity*. Assume first that $X = \{x, y\}$. Suppose that x is strictly preferred to y. That is, there is a $k \leqslant K$ such that

$$\sum_{\omega \in \Omega} p_l(\omega)[u(x_\omega) - u(y_\omega)] = 0 \qquad \text{for all } l < k,$$

$$\sum_{\omega \in \Omega} p_k(\omega)[u(x_\omega) - u(y_\omega)] > 0.$$

If $k = K$, then set $(\frac{1}{2}, \ldots, \frac{1}{2}) = r(1) > r(2) > \ldots$. Otherwise set $\frac{1}{2} = r_l(1) > r_l(2) > \ldots$ for all $l \neq k$. Let

$$W = \min\{u(x_\omega) - u(y_\omega) : \omega \in \Omega\},$$

$$B = \sum_{\omega \in \Omega} p_k(\omega)[u(x_\omega) - u(y_\omega)].$$

Note that $B > 0$, and $\Sigma_{\omega\in\Omega} q(\omega)[u(x_\omega) - u(y_\omega)] \geq W$ for all $q \in \Delta(\Omega)$. Choose $r_k(1) \in (0,1)$ to solve $(1 - r_k(1))B + r_k(1)W > 0$ and set $r_k(1) > r_k(2) > \dots$. Then

$$\sum_{\omega\in\Omega} [(1 - r_k(n))p_k(\omega) + r_k(n)q(\omega)][u(x_\omega) - u(y_\omega)] > 0$$

for all n and $q \in \Delta(\Omega)$. But

$$\sum_{\omega\in\Omega} (r(n)\square\rho)(\omega)[u(x_\omega) - u(y_\omega)] =$$

$$\sum_{\omega\in\Omega} r_1(n)\cdots r_{k-1}(n)[(1 - r_k(n))p_k(\omega) + r_k(n)(r'(n)\square\rho')(\omega)][u(x_\omega) - u(y_\omega)]$$

where $r'(n) = (r_{k+1}(n), \dots, r_{K-1}(n))$ and $\rho' = (p_{k+1}, \dots, p_K)$, and is therefore greater than 0 for all n.

If X consists of more than two elements, first calculate $r(1)$ as above for all pairs $x, y \in X$ for which x is strictly preferred to y; then choose the minimum $r(1)$.

Sufficiency. Suppose $\Sigma_{\omega\in\Omega}(r(n)\square\rho)(\omega)[u(x_\omega) - u(y_\omega)] > 0$ for all n. Letting $r_1(n) \to 0$ gives $\Sigma_{\omega\in\Omega} p_1(\omega)[u(x_\omega) - u(y_\omega)] \geq 0$. If strict inequality holds, then x is strictly preferred to y. So suppose equality holds. Then it must be true that $\Sigma_{\omega\in\Omega}(r'(n)\square\rho')(\omega)[u(x_\omega) - u(y_\omega)] > 0$ where $r'(n) = (r_2(n), \dots, r_{K-1}(n))$ and $\rho' = (p_2, \dots, p_K)$. Letting $r_2(n) \to 0$ gives $\Sigma_{\omega\in\Omega} p_2(\omega)[u(x_\omega) - u(y_\omega)] \geq 0$. Continuing in this manner, strict inequality must arise at some stage, since otherwise $\Sigma_{\omega\in\Omega}(r(n)\square\rho)(\omega)[u(x_\omega) - u(y_\omega)] = 0$. Hence x is strictly preferred to y. *Q.E.D.*

REMARK: The proof of sufficiency in Proposition 1 shows that for *any* sequence $r(n) \in (0,1)^{K-1}$ with $r(n) \to 0$, if

$$\sum_{\omega\in\Omega} (r(n)\square\rho)(\omega)u(x_\omega) > \sum_{\omega\in\Omega} (r(n)\square\rho)(\omega)u(y_\omega)$$

for all n, then x is strictly preferred to y.

PROOF OF PROPOSITION 2: Define a function $\psi: \Delta(\Omega)\times\Delta(\Omega) \to \mathbb{R}$ by

$$\psi(p, \tilde{p}) = \sup\{r \in \mathbb{R}: p(\omega) - r\tilde{p}(\omega) \geq 0 \text{ for all } \omega \in \Omega\}.$$

The following properties of ψ will be used:

(P1) $\psi(p, \tilde{p}) = 1$ if and only if $p = \tilde{p}$.

(P2) If $\operatorname{Supp}\tilde{p} \subseteq \operatorname{Supp} p$, then $p(\omega) - \psi(p, \tilde{p})\tilde{p}(\omega) = 0$ for some $\omega \in \operatorname{Supp} p$ (Supp denotes the support of a measure).

(P3) ψ is continuous in its first argument.

The proof proceeds by a number of steps. (In what follows we do not distinguish notationally a subsequence from its parent sequence.)

Step 1: Choose a subsequence of $p(n)$ such that if for any $\omega \in \Omega$, $p(n)(\omega) = 0$ infinitely often (i.o.), then $p(n)(\omega) = 0$ for all n.

Step 2: Choose a convergent subsequence of $p(n)$, converging to p_1, say.

Step 3: If $p(n) = p_1$ i.o., the proof is complete. So suppose $p(n) \neq p_1$ for all sufficiently large n. In this case, by (P1) there is a well defined new sequence $p_2(n) \in \Delta(\Omega)$ given by

$$p_2(n) = \frac{p(n) - \psi(p(n), p_1)p_1}{1 - \psi(p(n), p_1)}. \tag{A.1}$$

Note that if $p(n)(\omega) = 0$ for all n, then $p_2(n)(\omega) = 0$ for all n. That is, $\operatorname{Supp} p_2(n) \subseteq \operatorname{Supp} p(n)$ for all n. In fact, since $\operatorname{Supp} p_1 \subseteq \operatorname{Supp} p(n)$, it follows from (P2) that for each n, $p_2(n)(\omega) = 0$ for some $\omega \in \operatorname{Supp} p(n)$. That is, for each n, $\operatorname{Supp} p_2(n) \subsetneq \operatorname{Supp} p(n)$.

Step 4: Observe that from (A.1), $p(n)$ can be written as

$$p(n) = \psi(p(n), p_1)p_1 + [1 - \psi(p(n), p_1)]p_2(n) \tag{A.2}$$

where $\psi(p(n), p_1) \in (0,1)$, and $\psi(p(n), p_1) \to 1$ using (P3). Letting $r_1(n) = 1 - \psi(p(n), p_1)$, (A.2) can be rewritten as

$$p(n) = [1 - r_1(n)]p_1 + r_1(n)p_2(n) \tag{A.3}$$

where $r_1(n) \in (0,1)$ and $r_1(n) \to 0$.

Now repeat Steps 1–4 on the sequence $p_2(n)$. This yields a subsequence of $p_2(n)$ that can be written as

$$p_2(n) = \psi(p_2(n), p_2)p_2 + [1 - \psi(p_2(n), p_2)]p_3(n) \tag{A.4}$$

where $p_2(n) \to p_2$ and, for each n, $\operatorname{Supp} p_3(n) \subsetneq \operatorname{Supp} p_2(n)$. Letting $r_2(n) = 1 - \psi(p_2(n), p_2)$, (A.3) and (A.4) can be combined to give

$$p(n) = [1 - r_1(n)]p_1 + r_1(n)[[1 - r_2(n)]p_2 + r_2(n)p_3(n)].$$

Repeating Steps 1–4 on the sequence $p_3(n)$, and continuing in this fashion, shows that a subsequence of $p(n)$ can be written as $p(n) = r(n) \square \rho$, where the length K of the LPS ρ is bounded by the cardinality of Ω since $\operatorname{Supp} p(n) \supsetneq \operatorname{Supp} p_2(n) \supsetneq \operatorname{Supp} p_3(n) \supsetneq \dots$. *Q.E.D.*

PROOF OF PROPOSITION 3: *Sufficiency*. Suppose (μ, σ) is a lexicographic Nash equilibrium. It follows from Conditions (i) and (ii) of Definition 3 that if $\sigma^j(a^j) > 0$ for a player j, then

$$\sum_{a^{-j} \in A^{-j}} \sigma^{-j}(a^{-j})u^j(a^j, a^{-j}) \geqslant \sum_{a^{-j} \in A^{-j}} \sigma^{-j}(a^{-j})u^j(b^j, a^{-j})$$

for all $b^j \in A^j$. This is precisely the condition that σ is a Nash equilibrium.

Necessity. Suppose σ is a Nash equilibrium. For each player i let ρ^i be the one-level hierarchy given by $p_1^i(a^{-i}) = \sigma^{-i}(a^{-i})$ for $a^{-i} \in A^{-i}$. Then (μ, σ) is a lexicographic Nash equilibrium. *Q.E.D.*

PROOF OF PROPOSITION 4: *Sufficiency*. Suppose (μ, σ) is a lexicographic Nash equilibrium where μ has full support. By Proposition 3, σ is a Nash equilibrium. Moreover, it follows from Proposition 1 that there is an $r^1 \in (0,1)^{K^1 - 1}$ such that each a^1 assigned positive weight by σ^1 is a best reply to $r^1 \square \rho^1 \in \Delta(A^2)$. For any finite set Y, let $\Delta^{\circ}(Y)$ denote the subset of $\Delta(Y)$ consisting of those probability measures on Y with full support. Then, in fact, $r^1 \square \rho^1 \in \Delta^{\circ}(A^2)$ since ρ^1 has full support. That is, σ^1 is a best reply to $r^1 \square \rho^1 \in \Delta^{\circ}(A^2)$, or σ^1 is admissible. A similar argument establishes that σ^2 is admissible.

Necessity. Suppose σ is a perfect equilibrium. Then there are $\hat{\sigma}^1 \in \Delta^{\circ}(A^1)$, $\hat{\sigma}^2 \in \Delta^{\circ}(A^2)$ such that σ^1 is a best reply to $\hat{\sigma}^2$ and σ^2 is a best reply to $\hat{\sigma}^1$. Give player 1 the two-level hierarchy $\rho^1 = (p_1^1, p_2^1)$ where $p_1^1 = \sigma^2$ and $p_2^1 = \hat{\sigma}^2$. Give player 2 the analogous two-level hierarchy. Then (μ, σ) is a lexicographic Nash equilibrium and μ has full support. *Q.E.D.*

PROOF OF PROPOSITION 5: *Sufficiency*. Suppose (μ, σ) is a lexicographic Nash equilibrium where μ has full support and respects preferences. For $k = 1, \dots, K^1$, let

$$\pi_k^2 = \{a^2 \in A^2 \colon p_k^1(a^2) > 0 \text{ and } p_l^1(a^2) = 0 \text{ for } l < k\}.$$

Since ρ^1 has full support, $\{\pi_1^2, \dots, \pi_{K^1}^2\}$ is a partition of A^2. Since μ respects preferences, each $a^2 \in \pi_k^2$ is optimal for player 2 among actions in $\bigcup_{l=k}^{K^1} \pi_l^2$. Hence by Proposition 1, for every $k = 1, \dots, K^1$ there is a sequence $r^{2,k}(n) \in (0,1)^{K^2 - 1}$ with $r^{2,k}(n) \to 0$ such that for each $a^2 \in \pi_k^2$, $b^2 \in \bigcup_{l=k}^{K^1} \pi_l^2$

$$\sum_{a^1 \in A^1} (r^{2,k}(n) \square \rho^2)(a^1)[u^2(a^1, a^2) - u^2(a^1, b^2)] \geqslant 0 \tag{A.5}$$

for all n. Moreover, (A.5) holds with equality if $b^2 \in \pi_k^2$. For each n, let $r^2(n) = \min\{r^{2,k}(n) \colon k = 1, \dots, K^1\}$.

We have found a set of mixed strategies $M^2 = \{r^2(n)\square\rho^2: n = 1, 2, \ldots\} \subset \Delta^\circ(A^1)$, with $r^2(n) \to 0$, and a partition $\{\pi_1^2, \ldots, \pi_{K^1}^2\}$ of A^2 such that against a given $r^2(n)\square\rho^2 \in M^2$ all actions in π_k^2 yield the same expected utility to player 2, and yield at least as great expected utility to player 2 as does any action in π_{k+1}^2. By a similar construction, we can find a set of mixed strategies $M^1 = \{r^1(n)\square\rho^1: n = 1, 2, \ldots\} \subset \Delta^\circ(A^2)$, with $r^1(n) \to 0$, and a partition $\{\pi_1^1, \ldots, \pi_{K^2}^1\}$ of A^1 with analogous properties.

Given the sequence $(r^2(n)\square\rho^2, r^1(n)\square\rho^1) \in \Delta^\circ(A^1) \times \Delta^\circ(A^2)$ with $r^2(n) \to 0, r^1(n) \to 0$, the proof of sufficiency is completed by finding a sequence of positive numbers $\varepsilon(n) \to 0$ such that

$$a^1 \in \pi_k^1 \text{ and } b^1 \in \pi_{k+1}^1 \Rightarrow \left(r^2(n)\square\rho^2\right)(b^1) < \varepsilon(n)\left(r^2(n)\square\rho^2\right)(a^1),$$

$$a^2 \in \pi_k^2 \text{ and } b^2 \in \pi_{k+1}^2 \Rightarrow \left(r^1(n)\square\rho^1\right)(b^2) < \varepsilon(n)\left(r^1(n)\square\rho^1\right)(a^2).$$

For $k = 1, \ldots, K^2$, let $m_k^2 = \min\{p_k^2(a^1): a^1 \in \operatorname{Supp} p_k^2\}$. Let $m^2 = \min\{m_k^2: k = 1, \ldots, K^2\}$. Note that for $a^1 \in \pi_k^1$

$$\begin{aligned}\left(r^2(n)\square\rho^2\right)(a^1) &\geq r_1^2(n)\cdots r_{k-1}^2(n)\left[1 - r_k^2(n)\right]p_k^2(a^1)\\ &\geq r_1^2(n)\cdots r_{k-1}^2(n)\left[1 - r_k^2(n)\right]m^2\end{aligned}$$

while for $b^1 \in \pi_{k+1}^1$

$$\left(r^2(n)\square\rho^2\right)(b^1) \leq r_1^2(n)\cdots r_k^2(n).$$

Hence $(r^2(n)\square\rho^2)(b^1) < \varepsilon(n)(r^2(n)\square\rho^2)(a^1)$ if

$$r_1^2(n)\cdots r_k^2(n) < \varepsilon(n)r_1^2(n)\cdots r_{k-1}^2(n)\left[1 - r_k^2(n)\right]m^2$$

or

$$\varepsilon(n) > \frac{r_k^2(n)}{\left[1 - r_k^2(n)\right]m^2}.$$

Let

$$\varepsilon^2(n) = \max\left\{\frac{2r_k^2(n)}{\left[1 - r_k^2(n)\right]m^2}: k = 1, \ldots, K^2 - 1\right\}.$$

The argument for $a^2 \in \pi_k^2$ and $b^2 \in \pi_{k+1}^2$ is similar. Hence defining $\varepsilon^1(n)$ in an analogous fashion, a suitable sequence $\varepsilon(n)$ is given by $\varepsilon(n) = \max\{\varepsilon^1(n), \varepsilon^2(n)\}$.

Necessity. Suppose σ is a proper equilibrium. That is, there is a sequence $\sigma(n) = (\sigma^1(n), \sigma^2(n)) \in \Delta^\circ(A^1) \times \Delta^\circ(A^2)$ of $\varepsilon(n)$-proper equilibria converging to σ. By Proposition 2 there is an LPS $\rho = (p_1, \ldots, p_K)$ on $A^1 \times A^2$ such that a subsequence $\sigma(m)$ of $\sigma(n)$ can be written as $\sigma(m)(a^1, a^2) = (r(m)\square\rho)(a^1, a^2), (a^1, a^2) \in A^1 \times A^2$, for a sequence $r(m) \in (0,1)^{K-1}$ with $r(m) \to 0$. Let player 1's LPS $\rho^1 = (p_1^1, \ldots, p_K^1)$ be the marginal on A^2 of ρ and let player 2's LPS $\rho^2 = (p_1^2, \ldots, p_K^2)$ be the marginal on A^1 of ρ. Note that ρ^1 and ρ^2 have full support since $\sigma(m) \in \Delta^\circ(A^1) \times \Delta^\circ(A^2)$, and that $p_1^1 = \sigma^2, p_1^2 = \sigma^1$. It remains to show that $\mu = (\rho^1, \rho^2)$ respects preferences. Suppose not. Then there are, say, $a^2, b^2 \in A^2$ such that

$$\left[\sum_{a^1 \in A^1} p_k^2(a^1)u^2(a^1, a^2)\right]_{k=1}^K <_L \left[\sum_{a^1 \in A^1} p_k^2(a^1)u^2(a^1, b^2)\right]_{k=1}^K$$

but $a^2 \geq_{\rho^1} b^2$. By the remark following the proof of Proposition 1

$$\sum_{a^1 \in A^1}\left(r(m)\square\rho^2\right)(a^1)u^2(a^1, a^2) < \sum_{a^1 \in A^1}\left(r(m)\square\rho^2\right)(a^1)u^2(a^1, b^2)$$

for all sufficiently large m. That is

$$\text{(A.6)} \qquad \sum_{a^1 \in A^1} \sigma^1(m)(a^1)u^2(a^1,a^2) < \sum_{a^1 \in A^1} \sigma^1(m)(a^1)u^2(a^1,b^2)$$

for all sufficiently large m. But if $p_k^1(a^2) > 0$ and $p_l^1(b^2) = 0$ for $l < k$, then

$$\sigma^2(m)(a^2) \geqslant r_1(m) \cdots r_{k-1}(m)[1 - r_k(m)]p_k^1(a^2)$$

and

$$\sigma^2(m)(b^2) \leqslant r_1(m) \ldots r_{k-1}(m).$$

If m is large, then for any $\varepsilon(m) > 0$

$$r_1(m) \cdots r_{k-1}(m)[1 - r_k(m)]p_k^1(a^2) > \varepsilon(m)r_1(m) \cdots r_{k-1}(m) \quad \text{or}$$

$$\text{(A.7)} \qquad \sigma^2(m)(a^2) > \varepsilon(m)\sigma^2(m)(b^2).$$

Equations (A.6) and (A.7) contradict the fact that $(\sigma^1(m), \sigma^2(m))$ is an $\varepsilon(m)$-proper equilibrium. *Q.E.D.*

PROOF OF COROLLARY 1: The proof follows immediately from the proof of sufficiency in Proposition 5. *Q.E.D.*

PROOF OF PROPOSITION 6: The proof follows exactly the lines of the proof of Proposition 4. *Q.E.D.*

PROOF OF PROPOSITION 7: *Sufficiency*. Suppose (μ, σ) is a lexicographic Nash equilibrium where μ has full support and satisfies the CPA and strong independence. It follows that there is an LPS $\rho = (p_1, \ldots, p_K)$ on $\times_{i=1}^N A^i$ and a sequence $r(n) \in (0,1)^{K-1}$ with $r(n) \to 0$ such that $r(n) \square \rho$ is a product measure for all n. Define a sequence $\sigma(n) = (\sigma^1(n), \ldots, \sigma^N(n)) \in \times_{i=1}^N \Delta(A^i)$ by letting, for each i, $\sigma^i(n)$ be the marginal on A^i of $r(n) \square \rho$. Note that $\lim_n \sigma(n) = \sigma$ and, for each $i, \sigma^i(n) \in \Delta^\circ(A^i)$ since μ has full support. Also, for each i, $(r(n) \square \rho^i)(a^{-i}) = \times_{j \neq i} \sigma^j(n)(a^j)$ for $a^{-i} \in A^{-i}$. Hence by Condition (i) of Definition 3 and the Remark following the proof of Proposition 1, $\sigma^i(a^i) > 0$ implies a^i is a best reply to $\sigma^{-i}(n)$ for all sufficiently large n.

Necessity. Suppose σ is a perfect equilibrium. That is, there is a sequence $\sigma(n) = (\sigma^1(n), \ldots, \sigma^N(n)) \in \times_{i=1}^N \Delta^\circ(A^i)$ with $\sigma(n) \to \sigma$ and such that for each i, σ^i is a best reply to $\sigma^{-i}(n)$ for all n. By Proposition 2 there is an LPS $\rho = (p_1, \ldots, p_K)$ on $\times_{i=1}^N A^i$ such that a subsequence $\sigma(m)$ of $\sigma(n)$ can be written as $\sigma(m)(a^1, \ldots, a^N) = (r(m) \square \rho)(a^1, \ldots, a^N), (a^1, \ldots, a^N) \in \times_{i=1}^N A^i$, for a sequence $r(m) \in (0,1)^{K-1}$ with $r(m) \to 0$. Note that $p_1(a^1, \ldots, a^N) = \sigma(a^1, \ldots, a^N)$ since $\sigma(m) \to \sigma$. Also, since $\sigma(m) \in \times_{i=1}^N \Delta^\circ(A^i)$, ρ has full support and satisfies strong independence. For each i, let ρ^i be the marginal on A^{-i} of ρ. (Hence the CPA is satisfied by construction.) It remains to show that Condition (i) of Definition 3 is satisfied. But $\sigma^i(a^i) > 0$ implies a^i is a best reply to $\sigma^{-i}(m)$ for all m, and $\sigma^{-i}(m)(a^{-i}) = (r(m) \square \rho^i)(a^{-i})$ for $a^{-i} \in A^{-i}$. It follows by an identical argument to that used in the proof of sufficiency in Proposition 1 that a^i is optimal under the LPS ρ^i. *Q.E.D.*

PROOF OF PROPOSITION 8: *Sufficiency*. Suppose (μ, σ) is a lexicographic Nash equilibrium where μ has full support, respects preferences, and satisfies the CPA and strong independence. It follows that there is an LPS $\rho = (p_1, \ldots, p_K)$ on $\times_{i=1}^N A^i$ and a sequence $r(n) \in (0,1)^{K-1}$ with $r(n) \to 0$ such that $r(n) \square \rho$ is a product measure for all n. Define a sequence $\sigma(n) = (\sigma^1(n), \ldots, \sigma^N(n)) \in \times_{i=1}^N \Delta(A^i)$ by letting, for each $i, \sigma^i(n)$ be the marginal on A^i of $r(n) \square \rho$. Note that $\lim_n \sigma(n) = \sigma$ and, for each i, $\sigma^i(n) \in \Delta^\circ(A^i)$ since μ has full support. Also, for each $i, (r(n) \square \rho^i)(a^{-i}) = \times_{j \neq i} \sigma^j(n)(a^j)$ for $a^{-i} \in A^{-i}$. Since μ respects preferences, a similar argument to that used in the proof of sufficiency in Proposition 5 shows that for each player i there is a set of mixed strategies $M^i = \{\sigma^{-i}(n) : n > n^i \text{ for some } n^i\} \subset \times_{j \neq i} \Delta^\circ(A^j)$, and a partition $\{\pi_1^i, \ldots, \pi_K^i\}$ of A^i with the following property. For $k = 1, \ldots, K-1$, all actions in π_k^i yield the same expected utility to player i against any given $\sigma^{-i}(n) \in M^i$, and yield at least as great expected utility to i as does any action in

π^i_{k+1}. The proof of sufficiency is completed by finding a sequence of positive numbers $\varepsilon(n) \to 0$ such that for each i,

$$a^i \in \pi^i_k \text{ and } b^i \in \pi^i_{k+1} \Rightarrow \sigma^i(n)(b^i) < \varepsilon(n)\sigma^i(n)(a^i).$$

Such a sequence $\varepsilon(n)$ can be constructed by a similar argument to that used in the proof of sufficiency in Proposition 5.

Necessity. Suppose σ is a proper equilibrium. That is, there is a sequence $\sigma(n) \in \times_{i=1}^N \Delta^\circ(A^i)$ of $\varepsilon(n)$-proper equilibria converging to σ. By Proposition 2 there is an LPS $\rho = (p_1, \dots, p_K)$ on $\times_{i=1}^N A^i$ such that a subsequence $\sigma(m)$ of $\sigma(n)$ can be written as $\sigma(m)(a^1, \dots, a^N) = (r(m) \Box \rho)(a^1, \dots, a^N), (a^1, \dots, a^N) \in \times_{i=1}^N A^i$, for a sequence $r(m) \in (0,1)^{K-1}$ with $r(m) \to 0$. Note that $p_1(a^1, \dots, a^N) = \sigma(a^1, \dots, a^N)$ since $\sigma(m) \to \sigma$. Also, since $\sigma(m) \in \times_{i=1}^N \Delta^\circ(A^i)$, ρ has full support and satisfies strong independence. For each i, let ρ^i be the marginal on A^{-i} of ρ. (Hence the CPA is satisfied by construction.) It remains to show that $\mu = (\rho^1, \dots, \rho^N)$ respects preferences. This follows by an identical argument to that used in the proof of necessity in Proposition 5. *Q.E.D.*

Proof of Corollary 2: The proof follows immediately from the proof of sufficiency in Proposition 8. *Q.E.D.*

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