A Comment on "Price Dispersion, Inflation, and Welfare" by A. Head and A. Kumar[1]

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The empirical relationship between the rate of inflation and cross seller variation in prices has a long history. However, it is primarily empirical work without a theory. This paper provides an insightful theoretical foundation for the relationship with empirical content.

The primary purpose of this comment is to illustrate the principal results of the Head and Kumar paper using a special but natural assumption about the distribution of price quotes to buyers. Where ever possible, the authors' notation is utilized. Any new variables are introduced in the text.

The model studied is a synthesis of the monetary exchange model proposed by Shi (1995, 1999), on the one hand, and the price setting model of Burdett and Judd (1983), on the other. In the Burdett-Judd model, sellers set prices to maximize expected profit, these prices are described by a price offer c.d.f. $F(p)$, and each buyer observes a random sample of offers from this distribution each period. The number of offers, k, is itself a random variable. Let q_k represent the probability that k offers will be received by any buyer. In the paper, the authors assume that at least one price offer is always received. It will be useful to extend the analysis here slightly by assuming that receiving no offer is a possibility, i.e., $q_0 \geq 0$.

In any stationary monetary equilibrium (SME) with price dispersion as defined in the paper, the authors' demonstrate that buyers pursue a reservation strategy. The optimal strategy is to purchase at the lowest price offered if and only if that price is less than the reservation price. In the model, households are identical, the reservation price is the same for all buyers, and no rational seller will set a higher price. As in the paper, the reservation price is

$$\overline{p} = \frac{u'(C)}{\Omega} \tag{1}$$

where $u'(C)$ is the marginal utility of consumption and Ω is the shadow price

[1] I thank Shouyong Shi for pointing out and correcting errors in the original draft of this comment. Those remaining are still mine.

of money. As Shi (2004) has shown,

$$\Omega = \frac{u'(C)C}{\frac{\gamma}{\beta} - q_0} \tag{2}$$

replaces equation (3.8) when there is a possibility that no offer is received by a buyer.

By applying the well known arguments of Burdett and Judd, the authors demonstrate that the offer prices posted by sellers are described by a continuous distribution $F(p)$ with a convex and compact support. Furthermore, the upper support is equal to the reservation price in any SME with price dispersion. Because a purchase is required for consumption and is made at the lowest price observed, aggregate household consumption is

$$C = \int_{\underline{p}}^{\overline{p}} \frac{(1-q_0)dJ}{p}.$$

where

$$J(p) = \sum_{k=1}^{\infty} \left[1 - (1-F(p))^k\right] \frac{q_k}{1-q_0}$$

represents the distribution of purchase prices, the lowest of the k prices observed in any period given that at least one is observed.

The assumption that the random number of prices observed by each buyer in the household is distributed Poisson is a special case of considerable interest. Namely,

$$q_k = \Pr\{z = k\} = \frac{e^{-\eta}\eta^k}{k!}, k = 1, 2, ... \tag{3}$$

where z is the number of price quotes received given there is at least one and η represents the expected number of prices observed. There are two reasons for studying this case. First, the infinite sums and integrals that enter the equilibrium conditions can be solved in closed form. Second, the specification is standard in the labor market search literature because it simplifies the analysis when the arrival rate η is regarded as endogenous. For these reasons, I use the assumption extensively in Mortensen (2003).

The principal analytic simplification obtained in this special case is a consequence of the fact that the distribution of transaction prices takes the closed form

$$J(p) = \sum_{k=1}^{\infty}\left[1-(1-F(p))^k\right]q_k = 1-\frac{e^{-\eta}}{1-e^{-\eta}}\sum_{k=1}^{\infty}\frac{[\eta(1-F(p))]^k}{k!} \tag{4}$$

$$= 1-\frac{e^{-\eta}}{1-e^{-\eta}}\left(\sum_{k=0}^{\infty}\frac{[\eta(1-F(p))]^k}{k!}-1\right) = \frac{1-e^{-\eta F(p)}}{1-e^{-\eta}}$$

Note that the last equality is implied by the definition of the exponential function, namely that $e^x = \sum_{i=0}^{\infty}\frac{x^i}{i!}$, and a bit of algebra.

Because expected sales when a price p is posted equals

$$\sum_{k=1}^{\infty}\frac{q_k}{1-q_0}k[1-F(p)]^{k-1} = \sum_{k=1}^{\infty}k[1-F(p)]^{k-1}\frac{e^{-\eta}\eta^k}{(1-e^{-\eta})k!} =$$

$$\frac{\eta e^{-\eta F(p)}}{1-e^{-\eta}}\sum_{k=1}^{\infty}\frac{e^{-\eta(1-F(p))}[\eta(1-F(p))]^{k-1}}{(k-1)!} = \frac{\eta e^{-\eta F(p)}}{1-e^{-\eta}},$$

the equal profit condition, equation (3.10) in the paper, can be written as

$$\left(\Omega-\frac{\phi}{p}\right)\eta e^{-\eta F(p)} = \left(1-\frac{\phi}{u'(C)}\right)\Omega\eta e^{-\eta}.$$

By using equation (2) to eliminate Ω, one obtains the following closed form solution for the equilibrium distribution of prices when one exists:

$$\left(\frac{p-p^*}{p}\right)e^{\eta[1-F(p)]} = \frac{\overline{p}-p^*}{\overline{p}}$$

or, equivalently,

$$1-F(p) = \frac{1}{\eta}\left[\ln\left(\frac{\overline{p}-p^*}{\overline{p}}\right)-\ln\left(\frac{p-p^*}{p}\right)\right] \tag{5}$$

where equations (1) and (2) require

$$\overline{p} = \frac{u'(C)}{\Omega} = \frac{\gamma}{\beta C}. \tag{6}$$

and where

$$p^* \equiv \frac{\phi}{\Omega} = \frac{\gamma\phi}{\beta C u'(C)} \tag{7}$$

is the "marginal cost" of any good.

Obviously, equation (5) defines a cumulative distribution function only when its support is contained in the interval $(p^*, \overline{p})$. Equivalently, a SME exists with price dispersion only if $\overline{p} > p^* \Leftrightarrow u'(C) > \phi$. From equations (4) and (5) and a change in variable of integration from p to $z = F(p)$, it follows that any equilibrium value for C solves

$$\begin{aligned} C &= \int_{\underline{p}}^{\overline{p}} \frac{(1-q_0)dJ}{p} = \int_{\underline{p}}^{\overline{p}} \frac{\eta F'(p)e^{-\eta F(p)}dp}{p} \\ &= \frac{\eta}{p^*}\int_0^1 \left[1-\left(1-\frac{p^*}{\overline{p}}\right)e^{-\eta(1-z)}\right]e^{-\eta z}dz \\ &= \frac{1}{p^*}\int_0^1 \left[\eta e^{-\eta z} - \eta e^{-\eta}\left(1-\frac{p^*}{\overline{p}}\right)\right]dz \\ &= \frac{1}{p^*}\left[1-e^{-\eta}-\eta e^{-\eta}\left(1-\frac{p^*}{\overline{p}}\right)\right]. \end{aligned} \tag{8}$$

After using equations (2), (6) and (7) to eliminate Ω, $\overline{p}$ and p^*, one finds that

$$\frac{u'(C)}{\phi} = \frac{\frac{\gamma}{\beta} - e^{-\eta} - \eta e^{-\eta}}{1 - e^{-\eta} - \eta e^{-\eta}}. \tag{9}$$

when prices are dispersed. Hence, an SME exists with price dispersion if and only if

$$u'(C) > \phi \Longleftrightarrow \gamma > \beta. \tag{10}$$

The fact that there is no price dispersion when condition (10) is violated is the author's Proposition 2. As Shi (2004) has shown, there is no monetary equilibrium when $\gamma < \beta$ in the model even when extended to allow for the possibility of receiving no price quote. Hence, every SME is characterized by price dispersion except when the Friedman rule ($\gamma = \beta$) holds.

Because condition iii of Proposition 3, namely $q_1 \in (0, 1)$, always holds is the Poisson case, condition (10) is sufficient as well as necessary for existence for a unique equilibrium when the utility function $u(C)$ is concave and satisfies $\lim_{C\to 0} u'(C) = \infty$ and $\lim_{C\to\infty} u'(C) = 0$. Hence, the assumption of a CRRA utility function used in Proposition 4 is not needed for uniqueness in the Poisson case. Also, versions of Propositions 5-7 are easily derived as corollaries of equation (9).

Finally, note that the equation also implies that price dispersion vanishes and prices converge to the p^*, marginal cost, as the expected number of price quotes, η, tends to infinity. This limit corresponds to Bertrand competition in which all buyers received two or more price quotes with certainty in the Burdett-Judd formulation of price setting.

One can introduce endogenous search simply by letting buyers choose the expected number of price quotes received per period, which the sole parameter of any Poisson arrival process. Since buyers know the distribution of price quotes, $F(p)$, but take it as given, the expected utility maximizing choice satisfies

$$\eta = \arg\max(u(C)|_F - \mu\eta) \tag{11}$$

where μ represents the cost per price observation, as in the paper. It follows that the FOC is

$$\begin{aligned}
\mu &= u'(C)\frac{\partial C}{\partial \eta}|_F = u'(C)\int_{\underline{p}}^{\overline{p}}\left(\frac{[1-\eta F(p)]e^{-\eta F(p)}}{p}\right)F'(p)dp. \qquad (12)\\
&= \frac{u'(C)}{p^*}\int_{\underline{p}}^{\overline{p}}\left(1-\left(1-\frac{p^*}{\overline{p}}\right)e^{-\eta[1-F(p)]}\right)[1-\eta F(p)]e^{-\eta F(p)}F'(p)dp\\
&= \frac{u'(C)}{p^*}\int_0^1\left(e^{-\eta z}-\left(1-\frac{p^*}{\overline{p}}\right)e^{-\eta}\right)[1-\eta z]dz\\
&= \frac{u'(C)}{p^*}\left[\frac{1}{2}\left(1-\frac{p^*}{\overline{p}}\right)\eta+\frac{p^*}{\overline{p}}\right] = \frac{\beta}{\gamma}Cu'(C)e^{-\eta}\left(\frac{1}{2}\left(\frac{u'(C)}{\phi}-1\right)\eta+\frac{u'(C)}{\phi}\right)\\
&= \frac{\beta}{\gamma}Cu'(C)e^{-\eta}\left(\frac{1}{2}\left(\frac{\frac{\gamma}{\beta}-e^{-\eta}-\eta e^{-\eta}}{1-e^{-\eta}-\eta e^{-\eta}}-1\right)\eta+\frac{\frac{\gamma}{\beta}-e^{-\eta}-\eta e^{-\eta}}{1-e^{-\eta}-\eta e^{-\eta}}\right)
\end{aligned}$$

The second equality is implied by (8), the third by (5), and the fourth by the change in variable $z = F(p)$. The rest follow by integration and algebra after applying equations (6), (7), and (9) appropriately to eliminate $\overline{p}$ and p^*.

For this generalization of the model, an SME is associated with any non-negative pair (C, η) that satisfy (9) and (12). As the term $e^{-\eta} + \eta e^{-\eta}$ is decreasing in η, the relationship implicitly defined by (9) is positively sloped in the $C \times \eta$ plane when $\gamma \geq \beta$ holds given $u(C)$ is strictly concave. For the same reasons, the right side of (12) is also decreasing in η. Hence, it defines a non-increasing relationship between C and η if $Cu'(C)$ is not increasing in C as usually assumed. Hence, at most one equilibrium exists under these conditions. Finally, since an increase in γ shifts the curve defined by (9) down but shifts the curve defined by (12) up in the positive quadrant of the

$C \times \eta$ plane, an increase in the rate of monetary expansion increases search effort and can increase consumption and welfare, as the authors demonstrate by example.

I thank the authors for an extremely interesting analysis. It should have an important role to play in future research on the relationship between inflation and price dispersion. I hope the simplicity of the results in the Poisson arrival case will be helpful in accomplishing that task.

References:

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