Competitive Pricing and Efficiency in Search Equilibrium

Dale T. Mortensen
Northwestern Unversity
Randall Wright
Unversity of Pennsylvania

February 1997

Abstract

The purpose of this paper is to consolidate, extend, and relate a set of results on price determination and efficiency that apply to the two-sided search equilibrium model of pure exchange. Gains from trade are due to different trader valuations, buyers and sellers meet at random at rate determined by a linearly homogenous matching function, and prices are bilateral bargaining outcomes. The principal results follow: The price of every exchange is the same and the common price is equal to the Walrasian market clearing price in the limit as *both* the rate of time discount and direct search costs converges to zero, i.e. Gale's (1987) limit theorem holds. However, given strictly positive search costs, social efficiency obtains if and only if the price rule satisfies Hosios' (1990) condition and the rate of time discount is zero. Finally, if transaction prices are set exante by competing third party "market makers", then the set of search equilibria and the set of Pareto efficient trades are equivalent, a result which generalized Moen (1994) and Shimer (1995).

1 Introduction

The starting point for the analysis of economic exchange is the competitive theory of market clearing supply and demand interaction. The dual purposes of the approach are to identify the determinants of trade flows and to illuminate the social role of exchange. Although this standard paradigm is a very powerful tool, there are many questions on which it sheds little light. For example, what is the process by which competitive equilibrium prices are set? Why does it appear that some markets, those for labor services and housing in particular, fail to clear? How robust are the claims of the theory to modification of the perfect information and no transactions cost assumptions? These are some of the questions that recent studies in search equilibrium have attempted to answer.

The purpose of this paper is to consolidate, extend, and relate a set of results on price determination and allocative efficiency under conditions of decentralized exchange obtained in the context of two-sided search and matching models. Rubinstein and Wolinsky (1985) study a simple model of search equilibrium in which each buyer wishes to sell a single unit of an indivisible good for some amount of another perfectly divisible substitute and sellers have one unit to sell. In this market, buyer and seller pairs are randomly matched, trade whenever both parties benefit, and split the gains according to the outcome of a symmetric strategic bargaining game. In particular, trade takes place at a price that provides each party with the option value of continuing to seek an alternative trading partner plus half the remaining trade surplus. In a modified version of their model in which perpetual potential flows of buyers and seller, who generally value the good differently, choose to enter the market or not, Gale (1987) shows that all steady state exchange prices limit to a common value as "exchange friction" vanishes and that the common price is "market clearing" in the sense that it equates demand and supply as represented by the flows of entering buyers and sellers. Here, we show that the law of one price holds when the discount rate is zero even when search costs are positive and that Gale's limit theorem follows for every possible division of the trade surplus between buyer and seller and any matching technology as search costs of transaction vanish.

Of course, market clearing in the classical sense is not necessary for efficiency if time and resources are required by the transactions process. Still search equilibria are generally inefficient, as we have known since Diamond (1982) and Mortensen (1982). For any reasonable specification of the matching process, the marginal participant increases the expected time required to find a trading partner for all agents on the same side of the market and decreases the expected time for all on the other side. We also know from the analysis of Hosios (1990) that these search externalities can offset one another in search equilibrium if the matching process exhibits constant returns to scale and agents on two sides of the market are respectively identical. To maximize the aggregate gains from trade less transaction costs, the traders' shares of realized gain from trade net of the sum of the values of the option to continue searching must reflect the marginal contribution of each agent to the value of the aggregate transaction flow. The Hosios condition is satisfied in the case of a linearly homogenous matching function if and only if the buyer and seller shares respectively equal the elasticities of the matching function with respect to the stocks of buyers and sellers in the market.

Although trade surplus shares reflect relative "bargaining power" in a bilateral bargaining formulation of price determination rather than satisfy the Hosios' condition for social optimality, the efficiency loss associated with an arbitrary sharing rule is small when transaction friction is small as a consequence of Gale's limit theorem. Still, many real markets are characterized by significant delay, search, and other forms of transaction costs. Prices and exchange quantities in search equilibrium need not be close to the price-quantity pair suggested by standard market clearing conditions. Is there a competitive market story that yields the efficient pricing rule as an equilibrium outcome in this case? Moen (1994) and Shimer (1995) have independently provide affirmative answers.

Here, we derive the Moen and Shimer result in a search equilibrium framework by invoking the existence of third party "market makers" willing and able to provide the means by which buyers and seller with different trade-offs between expected waiting time and price in the Rubinstein-Wolinsky-Gale framework can improve their values of participation. In what we call "competitive search equilibrium", each trader optimally selects to participate in the submarket preferred among those available before the matching process begins, all sub-markets that can attract traders exist, and all existing sub-markets clear in the sense that the numbers of buyers and sellers participating in each adjust to equate the steady state inflow of participating buyers and sellers to the transaction flow. As all gains from trade are exploited in equilibrium, including those obtained through the implicit exchange of preferred expected waiting time for a less advantageous price for the good, the set of competitive search equilibrium and the set of Pareto optimal trades are equivalent. Furthermore, because equilibrium is unique when the discount rate is sufficiently small, the aggregate value of participation is maximal in this case. Interestingly, the equilibium price in each submarket can be interpreted as a trade surplus sharing bilateral bargain outcome that satisfies the Hosios condition.

2 Search Equilibrium

2.1 Definitions

Each buyer demands one unit of the indivisible good in exchange for a divisible substitute only if the price of the former in terms of the latter, p, is less than the buyer's valuation or demand price, x, and each seller supplies one unit only if the price exceed the its cost or supply price, y. The potential buyer flow, b, and the continuous distribution of buyer demand prices, described by the c.d.f. $F(x)$, characterize flow demand while the potential seller flow s and the continuous distribution of seller supply prices, denoted by the c.d.f. $G(y)$, determine flow supply.[1] There is a continuum of buyer types represented by the interval $[\underline{x}, \overline{x}] \subseteq \Re_+$ and a continuum of seller types represented by $[\underline{y}, \overline{y}] \subseteq \Re_+$. Of course, $\overline{x} > \underline{y}$ is necessary for trade. For future reference, let p^* and q^* represent the perfectly *competitive market clearing* price and quantity, the solutions to

$$b[1 - F(p^*)] = sG(p^*) = q^*. \tag{1}$$

[1] The assumption of continuous types distributions is a simplification that permits the use of differential methods. All the results presented here can be extended to the more realistic case in which the distributions of types are discrete given the appropriate technical fixes.

In a market with friction, buyers and sellers are matched as potential trading partners at an aggregate flow rate determined by some matching function $M(B,S)$ where B and S are the stocks of buyers and sellers respectively currently in the market.[2] The matching function is increasing, concave, and homogeneous of degree one by assumption. Pissarides' measure of *market tightness*, the buyer to seller ratio B/S, is denoted as θ. Given that the specific buyer and seller matches are determined at random, the probabilistic rate at which a seller is matched with some buyer, the duration hazard rate $M(B,S)/S = M(\theta,1)$, is represented as $m(\theta)$ while the rate at which a buyer is matched is $M(B,S)/B = m(\theta)/\theta$. The respective expected duration of time spent waiting to be matched is the inverse of the meeting rate (waiting time duration hazard) in each case. A common exponential rate of time preference r is assumed. Out of pocket transactions costs take the form of positive search or participation cost flows denoted as c_b per period for each buyer and c_s per period for each seller.

When a buyer and seller meet, one of the two announces a take it or leave it price offer and the other accepts or rejects. Who sets the price is determined at random; let β represent the exogenous probability that the seller sets the price. If the price is rejected, both continue to search as though they never met. If the price is accepted, the exchange is made and both exit the market. Under these assumptions, the responding trader accepts the price if and only the utility received exceeds the value of continued search.

Let $V_b(x)$ represent the steady state value of market participation to a buyer characterized by demand price x and let $V_s(y)$ denote the analog for a seller with supply price y. As formally defined below, each is the expected present value of an exchange less the seach costs incurred during the random time interval required to make it. Hence, the take it or leave it price p is acceptable to a buyer if and only if the utility realized by accepting it, $x - p$, is at least as large as the value of continued search, $V_b(x)$, given that the buyer's limit price is x. Similarly, a seller with cost y accepts the price p only if $p - y$ is no less than $V_s(y)$. Under the assumption of complete information, the optimal response of the agent who sets the price is to offer the other party's reservation value, i.e., $p_b(y) = y + V_s(y)$ when the buyer sets the price knowing the seller is of type y and $p_s(x) = x - V_b(x)$ when the seller sets the price knowing that the buyer is of type x. The implied expected price for a transaction between buyer and seller of type pair (x, y) is

$$\begin{aligned} p(x,y) &= \beta p_s(x) + (1-\beta) p_b(y) \\ &= \beta[x - V_b(x)] + (1-\beta)[y + V_s(y)] \\ &= y + V_s(y) + \beta\left[x - y - V_b(x) - V_s(y)\right], \end{aligned} \tag{2}$$

[2] See Pissarides (1990) for a discussion of the origins and usefulness of the construct.

As a consequence of the last expression, the expected share of joint trade surplus, equal to $x - y - V_b(x) - V_s(y)$, received by the seller, β, can also be interpreted as the "bargaining power" parameter in a generalized Nash bargain over a pie of size $x - y$ with "threat point" equal to $(V_b(x), V_s(y))$. Specifically, the seller's realized gain from trade, $p - y$, is equal to the value of continued search plus a share of the joint trade surplus.

As any buyer-seller type pair (x, y) expects an exchange price $p(x, y)$, the expected values of participation for a buyer with valuation x in a steady state solves the continuous time Bellman or asset pricing equation

$$rV_b(x) = \frac{m(\theta)}{\theta} \int (\max\{x - p(x,z), V_b(x)\} - V_b(x))\, d\Gamma(z) - c_b \tag{3}$$

where $M(B, S)/B = m(\theta)/\theta$ is the expected frequency with which a buyer meets a seller and $\Gamma(y)$ is the distriubtion of types over the steady state stock of sellers in the market. Analogously, for a seller with cost y the value of steady state value of participation solves

$$rV_s(y) = m(\theta) \int (\max\{p(z,y) - y, V_s(y)\} - V_s(y))\, d\Phi(z) - c_s \tag{4}$$

where $M(B, S)/S = m(\theta)$ is the expected time frequency with which sellers meet buyers and $\Phi(x)$ is the distribution of types over the steady state stock of buyers in the market.

Equation (2) - (4) imply that the expected search cost plus interest holding cost per meeting incurred by an individual agent is equal to the expected share of the gain from trade attributable to meeting an agent on the other side of the market in the sense that

$$(rV_b(x) + c_b)\, \frac{\theta}{m(\theta)} = (1 - \beta) \int \max\{x - z - V_b(x) - V_s(z), 0\} d\Gamma(z) \tag{5}$$

and

$$(rV_s(y) + c_s)\, \frac{1}{m(\theta)} = \beta \int \max\{z - y - V_b(z) - V_s(y), 0\} d\Phi(z). \tag{6}$$

One can easily verify from these functional relations that seller's and buyer's value functions are respectively monotone increasing in x and decreasing in y. Hence, only buyers with demand price $x \geq R_b$ enter the market while participating sellers are those with supply price $y \leq R_s$ where

$$V_b(R_b) = 0 = V_s(R_s) \tag{7}$$

The solution pair (R_b, R_s) identify the marginal participating buyer-seller type in the market.

In a steady state, the inflow of each participating type must equal the per period flow of that type's stock in the market who affect a trade. Because exchange takes place when pairs meet with a gain from trade surplus, $x - y$, no less than the sum of the values of continued search

$$
\begin{aligned}
bdF(x) &= \frac{m(\theta)}{\theta} Bd\Phi(x) \int \phi(x - z - V_b(x) - V_s(z)) d\Gamma(z) \\
\forall\, x &\geq R_b \text{ and } d\Phi(x) = 0 \text{ otherwise}
\end{aligned} \tag{8}
$$

and

$$
\begin{aligned}
sdG(y) &= m(\theta) Sd\Gamma(y) \int \phi(z - y - V_b(z) - V_s(y)) d\Phi(x) \\
\forall\, y &\leq R_s \text{ and } d\Gamma(y) = 0 \text{ otherwise}
\end{aligned} \tag{9}
$$

where $\phi(z)$ is the indicator function that takes on the value unity if its argument z is non-negative. Since

$$
\theta \equiv B/S \tag{10}
$$

where (B, S) is the steady state pair of buyer and seller stocks participating in the market at any point in time, the fact that F and Φ are both c.d.f.s defined on $[R_b, \overline{x}]$, that G and Γ are both c.d.f.s defined on $\left[\underline{y}, R_s\right]$, and $m(\theta)S = M(S, B)$ imply

$$
b\left[1 - F(R_b)\right] = sG(R_s) = M(S, B) \int \int \phi(x - y - V_b(x) - V_s(z)) d\Phi(x) d\Gamma(y) \tag{11}
$$

in steady state by virtue of (8) and (9).

A steady state *search equilibrium* is a pair of marginal participating types (R_b, R_s), a pair distribution functions characterizing type frequencies of the steady state stocks of buyer and seller $(\Phi(x), \Gamma(y))$, and a pair of steady state buyer-seller stocks (B, S) that satisfy equations (7) - (11). Because the first three equations uniquely defined the value functions and the marginal participating types for any fixed distributions of buyer and seller types, the general existence problem is one of finding a fixed point of the map from the space of distribution functions in which the pair $(\Phi(x), \Gamma(y))$ resides into itself implicitly defined by the steady state conditions. Rather than dwell on the technicalities required to establish a general existence result, we focus in the sequel on the special case that arises when the time discount rate r is small relative to the meeting rates at which participants meet.

2.2 Equilibrium Properties

In the limiting case of no time preference, $r = 0$, the right side of equation (5) must be the same for all x and analogously the right side of (6) is invariant with respect to y. Hence, continuity with respect to r and the definitions of the marginal demand and supply prices imply that

$$
\lim_{r \to 0} V_b(x) = x - R_b \tag{12}
$$

and

$$\lim_{r\to 0} V_s(y) = R_s - y \tag{13}$$

where equations (5) and (6) require that the limiting reservation values and market tightness solve

$$\begin{aligned} (1-\beta)\max\{R_b - R_s, 0\} &= \frac{c_b\theta}{m(\theta)} \\ \beta\max\{R_b - R_s, 0\} &= \frac{c_s}{m(\theta)}. \end{aligned} \tag{14}$$

Obviously, the equations of (14) imply that the marginal demand price exceeds the marginal supply price. As a consequence, the trade surplus exceeds the sum of the values of continued search for every participating pair,

$$\begin{aligned} \lim_{r\to 0}\{x - y - V_b(x) - V_s(y)\} &= R_b - R_s = \frac{c_b\theta}{m(\theta)} + \frac{c_s}{m(\theta)} > 0 \\ \forall\, x &\geq R_b \text{ and } y \leq R_s. \end{aligned} \tag{15}$$

In other words, every meeting results in trade in the limiting case. Indeed, continuity implies that the same condition holds for all sufficiently small values of the discount rate if buyers and sellers both face a strictly positive cost of participation. Finally, the steady state condition (??) reduces to

$$b[1 - F(R_b)] = sG(R_s) = M(B, S). \tag{16}$$

Hence, the unique equilibrium that obtains when $r = 0$ is fully characterized by triple (θ, R_b, R_s) which solves the two equations of (14) and the first equation of (16).

2.3 The Law of One Price: $r = 0$

Gale proves that a single price prevails and that price is the competitive equilibrium when in the limit as the ratio of the discount rate to the meeting rates, $r/m(\theta)$, vanish given $\beta = 1/2$. Actually, the law of one price holds in the limiting case of zero time preference even when there are costs of seach. Indeed, equations (2), (12) and (13) imply that the common price in all trade is the following weighted average of the marginal demand and supply prices, i.e.,

$$\lim_{r\to 0} p(x,y) = \widehat{p} = \beta R_b + (1-\beta)R_s \;\forall\, (x,y). \tag{17}$$

To fully characterize the other properties of equilibrium, first note that the equations of (14) imply that the ratio of buyers to sellers in the market is the unique solution to

$$\frac{1-\beta}{\beta} = \frac{c_b}{c_s}\theta. \tag{18}$$

Given this solution for market tightness, the difference between the marginal buyer's limit price and the marginal seller's cost is

$$R_b - R_s = \frac{c_b \theta}{m(\theta)} + \frac{c_s}{m(\theta)}. \tag{19}$$

Because the first steady state condition (16) defines a negative relationship between R_b and R_s, a unique solution for (R_b, R_s) exists.

All the interesting characteristics of a search equilibrium in the case of $r = 0$ are illustrated in Figure 1. As in the standard supply and demand diagram, the vertical axis represents price p, the horizontal is quantity q, the demand curve DD is the graph of the inverse demand function, which in our case is $F^{-1}(1 - \frac{q}{b})$, and the supply curve SS is the curve of the inverse supply function $G^{-1}(\frac{q}{s})$. The competitive market clearing price-quantity pair (p^*, q^*) is at the intersection of the two curves. The equilibrium transaction flow $\widehat{q} = M(B, S) = sG(R_s) = b[1 - F(R_d)]$ is the value of q such that the vertical difference between the marginal demand price and the marginal supply price is equal to the expected cost of search per transaction. The competitive market clearing price is bracketed by the limit price of the marginal buyer and the cost of the marginal seller, i.e. $p^* \in (R_s, R_b)$, as illustrated in Figure 1. The search equilibrium price $\widehat{p} = \beta R_b + (1-\beta) R_s$ is a simple average of the values placed on the good by the marginal buyer and seller type in the market. Consequently, the pair $(\widehat{p}, \widehat{q})$ converges to the competitive market clearing pair (p^*, q^*) defined by equation (1) as the participation costs c_b and c_s vanish holding their ratio constant by virtue of (18) for any measure of relative market power, reflected in the value of β.[3]

2.4 Price Dispersion: $r > 0$

The case of a small positive time discount rate provides an illustration of how prices depend on the buyer and seller types involved in an exchange in general. By virtue of (15), it follows that every meeting results in an exchange for all sufficiently small values of r. Hence, the value functions that solve equations (5) and (6) are of the linear form

$$V_b(x) = \frac{(1-\beta)m(\theta)}{(1-\beta)m(\theta) + r\theta} (x - R_b) \text{ and } V_s(y) = \frac{\beta m(\theta)}{\beta m(\theta) + r} (R_s - y). \tag{20}$$

By substitution into (2), it follows that the type contingent exchange price,

$$p(x, y) = \beta \left(\frac{r\theta x + (1-\beta)m(\theta)R_b}{r\theta + (1-\beta)m(\theta)} \right) + (1-\beta) \left(\frac{ry + \beta m(\theta) R_s}{\beta m(\theta) + r} \right), \tag{21}$$

increases with both the buyer's willingness to pay for the good, x, and the seller's cost of providing it, y. Still these deviations from the law of one price are small when the rate of time discount is small relative to the rates at which buyers and sellers meet one another, i.e., when $r/m(\theta)$ and $r\theta/m(\theta)$ are small.

[3] Furthermore, the steady state ratio of the stock of buyers to seller in the market waiting to trade tends to 0 as the market structure tends to monopsony ($\beta \to 1$) and to ∞ as the market structure tends to monopsony ($\beta \to 0$) by virtue of (6), a result that suggests the relative numbers of traders on the two sides of the market is determined by "bargaining power" rather than vice versa.

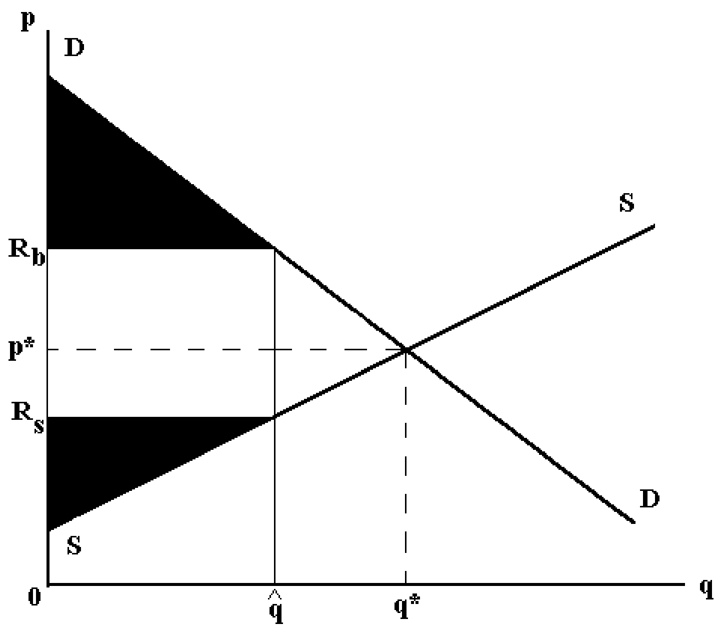

Figure 1: Search Equilibrium: $r = 0$

By substitution from (20) back into (5) and (6), one obtains

$$(1-\beta)\int\left[R_b - y - \frac{\beta m(\theta)}{\beta m(\theta)+r}(R_s - y)\right]\frac{d\Gamma(y)}{S} = \frac{c_b\theta}{m(\theta)}$$

and

$$\beta\int\left[x - R_s - \frac{(1-\beta)m(\theta)}{(1-\beta)m(\theta)+r\theta}(x - R_b)\right]\frac{d\Phi(x)}{B} = \frac{c_s}{m(\theta)}.$$

Hence, the steady state condition (8) and (9) and a rearrangement of terms yield

$$(1-\beta)(R_b - R_s) + \frac{(1-\beta)r}{\beta m(\theta)}\left(R_b - \int_{\underline{y}}^{R_s}\frac{ydG(y)}{G(R_s)} - \frac{c_b\theta}{(1-\beta)m(\theta)}\right) = \frac{c_b\theta}{m(\theta)} \quad (22)$$

and

$$\beta(R_b - R_s) + \frac{\beta r\theta}{(1-\beta)m(\theta)}\left(\int_{R_b}^{\overline{x}}\frac{xdF(x)}{1-F(R_b)} - R_s - \frac{c_s}{\beta m(\theta)}\right) = \frac{c_s}{m(\theta)}. \quad (23)$$

These two equations and the market clearing condition (16) determine the equilibrium triple (θ, R_b, R_s). A unique solution to these equations exists satisfying the following participations $R_b \leq \overline{x}$ and $R_s \geq \underline{y}$ if the discount rate r and costs of participation c_b and c_s are small enough.

However, if the discount rate is too large, the meetings of certain buyers and sellers fail to generate trades because the sum of the values of continued search exceeds the potential gains from trade. As an implication of (20), the necessary and sufficient conditions for trade $x - y - V_b(x) - V_s(y) \geq 0$ holds for all participants, i.e., for all $x \geq R_b$ and $y \leq R_s$, if and only if the limit price of the marginal buyer exceeds the cost the margin seller, i.e., $R_b \geq R_s$. Hence, the critical discount rate is that for which the two are equal to one another. As the two are both equal to the competitive price when equal to one another by virtue of the market clearing condition (16), which can be regarded as a parameter p^* dependent only the flow demand and supply conditions, equations (22) and (23) imply that the critical discount rate and its associated equilibrium market tightness pair is the solution $(\widehat{r}, \widehat{\theta})$ to

$$\widehat{r} = \frac{c_b\widehat{\theta}\left(\frac{\beta}{1-\beta}\right)}{p^* - \int_{\underline{y}}^{p^*}\frac{ydG(y)}{G(p^*)} - \frac{c_b\widehat{\theta}}{(1-\beta)m(\widehat{\theta})}} = \frac{c_s\left(\frac{1-\beta}{\beta}\right)}{\widehat{\theta}\left(\int_{p^*}^{\overline{x}}\frac{xdF(x)}{1-F(R_b)} - p^* - \frac{c_s}{\beta m(\widehat{\theta})}\right)}. \quad (24)$$

Of course $p^* - \int_{\underline{y}}^{p^*}\frac{ydG(y)}{G(p^*)} > 0$ and $\int_{p^*}^{\overline{x}}\frac{xdF(x)}{1-F(p^*)} - p^* > 0$ given the assumption of both buyer and seller diversity. These facts, $m(\theta)$ increasing and strictly concave, and $\lim_{\theta\to 0}\{\frac{\theta}{m(\theta)}\} = 0$ imply the second term is strictly increasing in $\widehat{\theta}$ and equals zero at $\widehat{\theta} = 0$ while the last term is strictly decreasing and converges to infinity as $\widehat{\theta} \to 0$. Consequently, the solution for $\widehat{\theta}$ to the second equality is unique and positive and the associated solution for the critical discount rate $\widehat{r}$ which is strictly positive if and only if both buyers and sellers face a cost of search. In sum, all meetings result in trade if and only if the traders are sufficiently patient in the sense that their common discount rate is no larger than the strictly positive critical value $\widehat{r}$ defined in equation (24).[4]

2.5 Social Efficiency

Although a search equilibrium is "approximately" market clearing when friction is small for any allocation of "market power" as reflected in the shares of trading surplus obtained by seller and buyer by virtue of Gale's Limit Theorem, Diamond (1982) and Mortensen (1982) long ago pointed out that search equilibria are generally inefficient. The reason is that trader welfare is inversely related to the time required to make an exchange except in the limiting case of zero friction and these expected waiting times depend non-trivially on the participation and search behavior of other agents. Although inefficiency characterizes search equilibrium for an arbitrary allocation of the gains from trade between buyer and seller, Hosios (1990) shows that a social optimum is achieved if the trade surplus shares equal the elasticities of the matching function with respect to buyers and seller provided that the matching function is homogeneous of degree one and that all buyers and sellers are respectively identical. In this case, the two search externalities, the congestion diseconomy imposed by every other agent on the same side of the market and the "thick market" external economy attributable to the participation of any agent on the other side, just off set one another.

In the case of a zero discount rate and a linear homogenous matching function, Hosios' result holds even in the case of heterogenous trader types. In this case, one can think of the social planner as choosing the best among the set of search equilibria parameterized by β.[5] Given the assumption of linear preferences, the appropriate social welfare criterion is the steady state gains from trade realized by buyers and sellers per period less the search cost flow per period. In other words, the optimal seller share solves

$$\max_{\beta \in (0,1)} \left\{ b \int_{R_b}^{\overline{x}} (x - \widehat{p}) dF(x) + s \int_{\underline{y}}^{R_s} (\widehat{p} - y) dG(y) - c_b B - c_s S \right\} \tag{25}$$

subject to conditions that the marginal participant types and steady state numbers of buyers and sellers satisfy the equilibrium conditions, (18) and (19) and steady condition (16) where the equilibrium exchange price $\widehat{p}$ is that specified in (17).

[4] Because $p^* \in [\underline{x}, \overline{x}] \cap [\underline{y}, \overline{y}]$, $\int_{p^*}^{\overline{x}} \frac{x dF(x)}{1-F(p^*)} - p^* \to 0$ as the support of the distribution of buyer types collapses while $p^* - \int_{\underline{y}}^{p^*} \frac{y dG(y)}{G(p^*)} \to 0$ as seller dispersion vanishes. Hence, $\widehat{r} \to \infty$ in either case. In other words, if either all buyers are identical or all sellers are identical, then all meetings result in trade for any positive time rate of discount and, consequently, the competive search equilibrium in either case solves equations (22), (23) and (16).

[5] This formulation of the planner's problem would be too restrictive in the case of a matching function that is not homogenous of degree one simply because the optimal solution is not supported by any search equilibrium in this more general case. However, in the linearly homogenous case it is an insightful way to look at the problem.

Note that the integral in (??) is consumer plus producer surplus over all value of q out to the equilibrium trade flow $\widehat{q} = M(B, S)$. As the total search cost flow satisfies

$$c_b B + c_s S = \left[\frac{c_b \theta}{m(\theta)} + \frac{c_s}{m(\theta)}\right] M(B, S) = [R_b - R_s]\widehat{q}$$

by virtue of the homogeneity of the matching function and the equilibrium condition (19), the optimal choice of β is that which maximizes aggregate surplus net of transactions cost, the shaded area in Figure 1. But, maximizing the shaded area is equivalent to minimizing the search cost incurred by both parties per trade, equal in equilibrium to the difference between the demand and supply prices of the marginal buyer and seller. As the necessary and sufficient condition is

$$\frac{m'(\theta)}{m(\theta) - \theta m'(\theta)} = \frac{c_b}{c_s}, \tag{26}$$

a comparison with (18) verifies that the optimal choice of the seller's share parameter is uniquely determined by the *Hosios condition*

$$\beta = 1 - \frac{\theta m'(\theta)}{m(\theta)} = \frac{S M_s(B, S)}{M(B, S)}. \tag{27}$$

2.6 Three Results

Along the way, we have established the following results: Assume: (i) $m(\theta)$ is increasing, strictly concave, and satisfies $\lim_{\theta \to 0}\{\frac{\theta}{m(\theta)}\} = 0$. (ii) The distributions of buyer types $F : [\underline{x}, \overline{x}] \to [0, 1]$ and the distribution seller types $G : [\underline{y}, \overline{y}] \to [0, 1]$ are both continuous.

Proposition 1 *(Existence and Uniqueness): Under the assumptions, a unique steady state search equilibrium exists for all $r > 0$ but sufficiently small with the property that every participating buyer-seller pair trades when they meet.*

Proposition 2 *(Gale's Limit Theorem): Consider any sequence of search equilibria defined by a sequence of strictly positive triples $(r, c_{b,} c_s)$ limiting to $(0, 0, 0) = 0$. The expected price of an exchange between any participating pair converges to the competitive equilibrium price. Namely, given continuity*

$$\lim_{(r, c_b, c_s) \to 0} p(x, y) = \lim_{(c_b, c_s) \to 0} \{\beta R_b + (1 - \beta) R_s\} = p^* \; \forall \, x \geq R_b \text{ and } y \leq R_s.$$

Proof: The first equality is a corollary of (21) and the second follows from the fact that equations (22) and (23) require that the marginal demand and supply prices converge to a common value as participation costs vanish and the fact that the market clearing condition (16) requires that their common limit is the competitive equilibrium price as defined in equation (1).

Proposition 3 *(Hosios) In the case of $r = 0$, search equilibrium maximizes aggregate trade surplus less search costs if and only if sellers and buyers shares of match surplus respectively equal the elasticities of the matching function with respect to the steady state stocks of participating sellers and buyers.*

3 Competitive Search Equilibrium

Of course, there is no reason that expost bilateral bargaining should yield prices that satisfy the socially efficient trade surplus sharing rule. However, Moen (1994) and Shimer (1995) independently demonstrate that Hosios' condition does obtain in extended formulations of the search equilibrium problem in which price offers and waiting times are known prior to the matching process. Here we interpret their equilibrium constructs as a complete markets Walrasian equilibrium, one in which the expected duration of time until a trade is made is implicitly priced, and extend them to the case of heterogeneity on both sides of the market. Because search externalities are internalized in this competitive formulation of search equilibrium, participation and sorting of buyers and sellers into a collection of submarkets that offer different expected waiting times is Pareto efficient and, conversely, and any Pareto efficient solution to the participation and sorting problem can be interpreted as a competitive search equilibrium. In the case of a zero rate of discount, the competitive search equilibrium is the search equilibrium with surplus shares that satisfy the Hosios condition. More generally, the equilibrium pricing across submarkets can be formally interpreted as the particular bilateral bargaining outcome in each submarket for which the Hosios condition holds. Finally, there is only one competitive search equilibrium for all positive sufficiently small rates of discount and that equilibrium necessarily maximizes the aggregate values of buyer and seller participation.

3.1 Definitions

Imagine that each trader can participate in one of a collection of submarkets Θ where in each one buyers and sellers find one another at rates determined by the numbers of buyers and sellers who participate. Since the meeting rates, $m(\theta)$ for sellers and $m(\theta)/\theta$ for buyers, are uniquely determined by the ratio of buyers to sellers θ in the submarket visited, the *set of submarket* Θ is simply the set of distinct values of the market-tightness parameter offered by the collection of market. Differences in trading prices across the submarkets, given by the endogenous *price function* $p : \Theta \to \Re_+$, implicitly price differences in expected waiting times. Each participating trader selects among the trade-off available by choosing the most preferred submarket taking this price function and the set of submarkets as given. An equilibrium price function in *market clearing* in the sense that the steady state numbers of participating buyers and sellers in each submarket adjust to equate the flow of buyers and sellers who enter each submarket with the common transactions flow. Finally, the equilibrium set of market Θ satisfies two conditions: (i) Each submarket is *open*, i.e., each is the preferred choice of some buyer and some seller type. (ii) Markets are *complete*, i.e., no other submarket exists that would be preferred by a buyer and seller pair to those that are open.

In a competitive search equilibrium, buyers and sellers who participate in each submarket are willing to trade when they meet as a consequence of the completeness condition. Therefore, the type-contingent stationary values of buyer and seller participation in a particular submarket characterized by a particular price-tightness pair (p, θ) are

$$U_b(x, p, \theta) = \frac{m(\theta)\,(x - p) - \theta c_b}{r\theta + m(\theta)} \tag{28}$$

and

$$U_s(y, p, \theta) = \frac{m(\theta)\,(p - y) - c_s}{r + m(\theta)} \tag{29}$$

by virtue of the Bellman equations (3) and (4). Given any price function $p(\theta)$, the values of voluntary participation for each type is equal to the value of participating in the preferred sub-market, i.e.,

$$\begin{aligned} V_b(x) &= \max_{\theta \in \Theta}\{U_b(x; p(\theta), \theta)\} \\ V_s(y) &= \max_{\theta \in \Theta}\{U_s(y, p(\theta), \theta)\}. \end{aligned} \tag{30}$$

Again, the value of buyer participation increases with the demand price and the value of seller participation decrease with supply price. Hence, all buyers $x \geq R_b$ and all seller types $y \leq R_s$ participate where the marginal types again solve

$$V_b(R_b) = V_s(R_s) = 0. \tag{31}$$

Let $\gamma_b(x, \theta)$ and $\gamma_c(y, \theta)$ represent an *assignment* of buyer and seller equal respectively to the fraction of the entering flow of each type assigned to the submarket indexed by the steady state buyer to seller ratio θ. A feasible steady state or *market clearing* assignment equates the flows of participating buyers and sellers into each market, i.e.,

$$\begin{aligned} b\int \gamma_b(x, \theta)dF(x) &= s\int \gamma_s(y, \theta)dG(y)\ \forall\, \theta \in \Theta \\ \text{where } \int_\Theta \gamma_b(x, \theta)d\theta &= \int_\Theta \gamma_b(x, \theta)d\theta = 1\ \forall\, (x, y). \end{aligned} \tag{32}$$

An assignment is a *competitive* if and only if it is market clearing and respects the submarket self-selection and individually rational participation conditions (30) and (31), i.e.,

$$\begin{aligned} \gamma_b(x, \theta) &= 0 \text{ if either } x < R_b \text{ or } \theta \notin \arg\max\{U_b(x; p(\theta), \theta)\}\ \forall\, x \geq R_{b,} \\ \gamma_b(x, \theta) &= 0 \text{ if either } y > R_s \text{ or } \theta \notin \arg\max\{U_b(x; p(\theta), \theta)\}\ \forall\, y \leq R_s. \end{aligned} \tag{33}$$

In the usual Walrasian formulation of exchange equilibrium, the commodity space is taken as given. Here we need to extend the concept by allowing the set of commodities, Θ, in our case, to be endogenous. There are two natural conditions to impose. First, each market in the set is open in the sense that it attracts some type of both participating buyers and sellers.

$$\begin{aligned}\text{For every } p &\in \Theta \text{ there exists a } (x,y) \in [R_b, \overline{x}] \times [\underline{y}, R_s] \\ \text{such that } \theta &= \arg\max U_b(x, p(\theta), \theta) = \arg\max U_s(y, p(\theta), \theta).\end{aligned} \tag{34}$$

Second, an equilibrium set of submarkets is complete in the sense that there is no other strictly preferred by some pair of buyer and seller types.

$$\begin{aligned}\text{There exists no } \theta &\notin \Theta,\ p \in [\underline{y}, \overline{x}] \text{ and } (x,y) \in [\underline{x}, \overline{x}] \times [\underline{y}, \overline{y}] \\ \text{such that } U_b(x,p,\theta) &> \max\{V_b(x), 0\} \text{ and } U_s(y,p,\theta) > \max\{V_s(y), 0\}.\end{aligned} \tag{35}$$

One way to interpret these conditions is to invoke the existence of notional third party “market makers” with a profit motive who could set up submarkets. Then, if there were positive but sufficiently small costs of setting up and operating a submarket, any that failed to attract both buyers and sellers would not exist while those that could attract both buyers and sellers from existing markets would be created.[6]

Finally, a *competitive search equilibrium* is a pair of marginal participant types (R_s, R_b), a market assignment $(\gamma_b(x,\theta), \gamma_s(y,\theta))$, a set of submarkets Θ, and a price function $p : \Theta \rightarrow \Re_+$ that satisfy (31) - (35).

3.2 The Equilibrium Price Function

Because there are no externalities under market completeness, the sets of feasible steady state Pareto efficient assignmnents of buyers and sellers to submarkets and the set of steady state equilibrium competitive assignments are equivalent. Specifically, the marginal rates of substitution between price and market tightness for any buyer-seller pair (x, y) who are assigned to the same submarket must be equal, i.e.,

$$\begin{aligned}\frac{dp}{d\theta}|_{U_s} &= -\frac{m'(\theta)\,[r(p-y)+c_s]}{m(\theta)\,[r+m(\theta)]} = p'(\theta) \\ &= -\frac{[m(\theta)-\theta m'(\theta)]\,[r(x-p)+c_b]}{m(\theta)\,[r\theta+m(\theta)]} = \frac{dp}{d\theta}|_{U_b}\end{aligned} \tag{36}$$

because otherwise completeness (35) would be violated.

This tangency condition collapses to the transaction cost minimization condition (26) for all (x, y) in the case of a zero rate of time discount. Because the marginal

[6] This interpretation is borrowed from Greenwald and Stiglitz (1988).

rates of substitution are the same across buyer and seller types respectively when $r = 0$, all buyers and sellers participate in a common market in this case and the price, that which induces optimal buyer and seller participation, is given by (17) where the share parameter β satisfied the Hosios condition (27).When the discount rate is strictly positive, the trangentcy condition can be written as the pricing rule (2) with the seller's share parameter β equals the elasticity of the matching function with respect to the number of participating buyer each submarket, i.e.,

$$\begin{aligned} p &= \left(1 - \frac{\theta m'(\theta)}{m(\theta)}\right)[x - V_b(x)] + \left(\frac{\theta m'(\theta)}{m(\theta)}\right)[y + V_s(y)] \\ \forall\, \theta &\in \Theta \text{ and } (x,y) \text{ such that } (\gamma_b(x,\theta), \gamma(y,\theta)) > 0. \end{aligned} \tag{37}$$

In other words, the price in every submarket can be viewed as the bilateral bargaining outcome for which the Hosios condition holds. Note that an equilibrium price function $p(\theta)$ is a fixed point of the map defined by (37) and a competitive assignment of types to submarkets. Below, we show that the set of fixed points contains a single element for all sufficiently small discount rates r.

In the general case, transactions costs include the forgone interest on the potential gain from trade as well as the cost of search. Indeed, the total cost per period spent searching equals the sum $r(p-y)+c_s$ for a seller of type y and the sum $r(x-p)+c_b$ for a buyer of type x. The marginal rate of substitution between price and market tightness for a seller, represented by the left side of (36), decreases with supply price y for this reason and for a buyer, the right side of (36), increases with the demand price x. As a consequence, sellers with higher supply prices strictly self select into markets with buyers who place a higher value on the good. The prices in markets chosen by buyers and sellers with higher demand and supply prices are higher while market tightness is lower because buyers are only willing to accept higher prices for a shorter expected times to trade while sellers will accept longer expected waiting times in exchange for higher prices.

To prove these assertion, consider Figure 2. In the figure, the curve EE represents the a particular price function $p(\theta)$, which is downward sloping because the indifference curves of all buyer and seller types are negatively sloped. The buyer's and seller's indifference curves through any particular point (p', θ'), those for the type pair (x', y'), are labeled $I'_bI'_b$ and $I'_sI'_s$ respectively in Figure 2. Now consider a different point (p'', x'') preferred by the buyer-seller type pair (x'', y''). At the point where the indifference curves of the two seller types cross, A in the figure, the indifference curve for type y'', represented by the $I''_sI''_s$, is necessarily less steeply sloped than the indifference curve for type y'. This fact and the left side of (36) imply that $y'' > y'$. Analogously, the fact that the indifference curve for buyer type x'', $I''_bI''_b$ in Figure 2, is more steeply sloped than that of buyer type x' at their point of intersection B requires $x'' > x'$ by virtue of the right side of (36).

The steady state equality of the flow of buyers and sellers into every submarket,

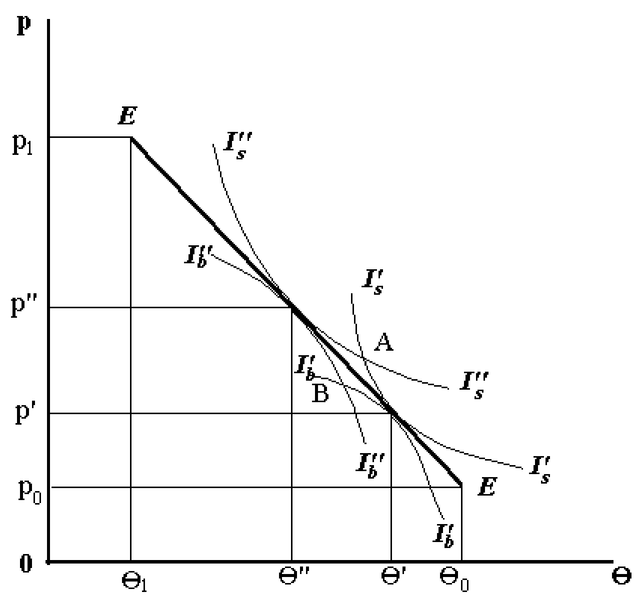

Figure 2: Socially Efficient Market Segmentation

implied by the equality of both to the common rate at which buyers and sellers transact in every submarket, requires the following specific positive matching of buyer and seller types across the submarkets:

$$\begin{aligned} b[F(x) - F(R_b)] &= sG(y) \; \forall \, y \in [\underline{y}, R_s] \\ & \quad \text{where} \\ b[1 - F(R_b)] &= sG(R_s) \end{aligned} \tag{38}$$

As the tangency condition (36), rewritten here as

$$\frac{m'(\theta)\left[r(p-y)+c_s\right]}{r+m(\theta)} = \frac{\left[m(\theta)-\theta m'(\theta)\right]\left[r(x-p)+c_b\right]}{r\theta+m(\theta)}, \tag{39}$$

defines a decreasing relationship between x and y for every pair (p, θ), these two equations uniquely identify the assignment of both the buyer and the seller types who would choose to participate in the particular market characterized by that price and market tightness pair. In other words, there is one and only one buyer type assigned to every market, i.e., a unique pair (x, y) exists for every θ such that $\gamma_b(x, \theta) = \gamma_x(y, \theta) = 1$ and that pair is determined by the functions

$$\begin{aligned} x &= f(p(\theta), \theta, R_b) \\ y &= g(p(\theta), \theta, R_b) \end{aligned} \tag{40}$$

implicitly defined by (38) and (39). Note in passing that $f(\cdot)$ is increasing and $g(\cdot)$ is decreasing in R_b.

Substituting for y in the first equation of (36) yields the ordinary first order differential equation

$$p'(\theta) = -\frac{m'(\theta)\left[r(p - g(p, \theta, R_b)) + c_s\right]}{m(\theta)\left[r + m(\theta)\right]} \tag{41}$$

Let

$$\begin{aligned} p &= p(\theta) = p_0 + h(\theta - \theta_0, R_b), \\ h(0, R_b) &= 0, \; \frac{\partial h}{\partial \theta} < 0, \; \text{and} \; \frac{\partial h}{\partial R_b} > 0, \end{aligned} \tag{42}$$

represent its general solution where the properties follow by virtue of (41), the fact that $g(\cdot)$ is decreasing in R_b, and the boundary conditions that follow. Because

$$\begin{aligned} R_b &= p_0 + \frac{\theta_0 c_b}{m(\theta_0)} \\ R_s &= p_1 - \frac{c_s}{m(\theta_1)} \end{aligned} \tag{43}$$

are implied by the participation conditions of (30), equations (38) and (39) require

$$\begin{aligned} r\left(p_0-\underline{y}\right)+c_s &= \frac{[r+m(\theta_0)]\,[m(\theta_0)-\theta_0 m'(\theta_0)]\,[r(R_b-p_0)+c_b]}{m'(\theta_0)\,[r\theta_0+m(\theta_0)]} \\ &= \frac{[r+m(\theta_0)]\,[m(\theta_0)-\theta_0 m'(\theta_0)]\,c_b}{m'(\theta_0)m(\theta_0)}, \end{aligned} \tag{44}$$

$$\begin{aligned} r\left(\overline{x}-p_1\right)+c_b &= \frac{m'(\theta_1)\,[r\theta_1+m(\theta_1)]\,[r(p_1-R_s)+c_s]}{[r+m(\theta_1)]\,[m(\theta)-\theta m'(\theta)]} \\ &= \frac{m'(\theta_1)\,[r\theta_1+m(\theta_1)]\,c_s}{[m(\theta_1)-\theta m'(\theta_1)]\,m(\theta_1)}, \end{aligned} \tag{45}$$

and

$$b\left[1-F\left(p_0+\frac{\theta_0 c_b}{m(\theta_0)}\right)\right]=sG\left(p_1-\frac{c_s}{m(\theta_1)}\right). \tag{46}$$

These three equations together with the requirement that

$$p_1=p_0+h\left(\theta_1-\theta_0,p_0+\frac{\theta_0 c_b}{m(\theta_0)}\right) \tag{47}$$

determine the endpoints (p_0,θ_0) and (p_1,θ_1) of the set of submarkets that satisfy the steady state conditions together with participation and self-selection conditions.

3.3 Three More Results

In sum, we have established the following:

Proposition 4 *The sets of steady state competitive search equilibrium and steady state Pareto efficient assignments of buyer and seller types to submarkets are equivalent.*

Proposition 5 *In every submarket, the seller's and buyer's shares of realized match surplus respectively equal the elasticities of the matching function with respect to the stocks of participating buyers and sellers, i.e., the Hosios condition holds in every submarket.*

Proposition 6 *A unique competitive search equilibrium exists for all non-negative r sufficiently small.*

Proof. In the case of $r=0$, the efficient price function degenerates to the point

$$\begin{aligned} p_1 &= p_0=p=\left(1-\frac{\theta m'(\theta)}{m(\theta)}\right)R_b+\left(\frac{\theta m'(\theta)}{m(\theta)}\right)R_s \\ \theta_1 &= \theta_0=\theta \end{aligned}$$

where (R_b, R_s, θ) is the unique solution to

$$\begin{aligned}\frac{m'(\theta)}{m(\theta)-\theta m'(\theta)} &= \frac{c_b}{c_s} \\ R_b - R_s &= \frac{c_b\theta}{m(\theta)} + \frac{c_s}{m(\theta)} \\ b[1-F(R_b)] &= sG(R_s).\end{aligned}$$

Given the continuity of the equation system (44) - (47) with respect to r, we need only demonstrate that equations (44) - (47) have at most one solution when r is sufficiently small. The strict concavity of $m(\theta)$ implies that the solution to (44), call it $p_0(\theta_0)$, and the solution to (45), $p_1(\theta_1)$, are both strictly increasing for all $r > 0$ sufficiently small. Hence,

$$b\left[1 - F\left(p_0(\theta_0) + \frac{\theta_0 c_b}{m(\theta_0)}\right)\right] = sG\left(p_1(\theta_1) - \frac{c_s}{m(\theta_1)}\right)$$

implicitly defines a decreasing functional relation between θ_0 and θ_1 while the relation defined by

$$p_1(\theta_1) - p_0(\theta_0) = h\left(\theta_1 - \theta_0, p_0 + \frac{\theta_0 c_b}{m(\theta_0)}\right)$$

is strictly decreasing given the properties of $h(\cdot)$. The unique solution defines the two endpoints $(p_0(\theta_0), \theta_0)$ and $(p_1(\theta_1), \theta_1)$ which solve the larger system.

Because Pareto optimality is necessary, it follows that any unique search equilibrium also maximized the aggregate value of participation of the entering flow as defined by

$$b\int_{R_b}^{\overline{x}} V_b(x)dF(x) + \int_{\underline{y}}^{R_s} V_s(y)dG(y).$$

4 Conclusions and Future Research

It is our contention that we have learned quite a lot about price determination and allocative efficiency through this exercise of collating and extending existing results in the literature on search equilibrium. First, the framework provides a simple but convincing story about how the price-quantity pair in a decentralized market is determined and why it is approximated by the market clearing pair when transactions costs are small. Second, when the conditions for a good approximation do not hold, then the search equilibrium solution for price and quantity are not those associated with the intersection of the usual supply and demand curves and there are stocks of buyers and sellers waiting to exchange. In short, *these market do not clear* either in appearance or in fact. Although this fact does not provide a *prima facia* case for market failure given that transactions costs exist, search equilibria do not generally minimize transaction costs when prices are determined by ex post bargaining between the traders after they meet as assumed. To obtain efficiency a complete set of markets is needed that offer traders the opportunity to shorten the time required to find a trading partner at the appropriate implicit price. The question that remains is who provides these markets?

The lessons learned in the paper are all predicated on the assumption that transaction technologies as embodied in the matching function exhibit constant returns to scale. The literature provides examples in which this condition need not hold and demonstrations that social efficiency cannot be attained in search equilibrium when it does not. A question left open for future study is whether the "complete market" model of search equilibrium developed in this paper might resolve this difficulty as well?

[1] Diamond, P.A. (1981), "Mobility Costs, Frictional Unemployment, and Efficiency." *Journal of Political Economy* **89**: 798-812.

[2] Gale, D. (1987), "Limit Theorems for Markets with Sequential Bargaining." *Journal of Economic Theory* **43**: 20-54.

[3] Greenwald, B., and J.E. Stiglitz (1988), "Pareto Inefficiency of Market Economies: Search and Efficiency Wage Models." *American Economic Review, AEA Papers and Proceeding* **78**(2): 351-356.

[4] Hosios, A. J. (1990), "On the Efficiency of Matching and Related Models of Search and Unemployment." *Review of Economic Studies* **57**: 279-298.

[5] Moen, E.R. (1994), "A Search Model with Wage Announcement", London School of Economics working paper.

[6] Mortensen, D.T. (1982), "The Matching Process as a Noncooperative/Bargaining Game" in J.J. McCall (ed.) *The Economics of Information and Uncertainty*, 233-254, University of Chicago Press.

[7] Pissarides, C.A. (1990), *Equilibrium Unemployment Theory*, Basil Blackwell.

[8] Rubinstein, A., and A. Wolinsky (1985), "Equilibrium in a Market with Sequential Bargaining", *Econometrica* **53**, 1133-1150.

[9] Shimer, Robert (1995), "Contacts in a Frictional Labor Market," MIT working paper.