Equilibrium Unemployment Dynamics[1]

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Abstract

The global dynamics of Pissarides' (1990) equilibrium model of aggregate unemployment are studied in the case of increasing returns to scale in production and constant returns to scale in the matching process. An equilibrium is a dynamic path for the aggregate number of matches generated by best response search and recruiting investment decisions under rational expectations. Necessary and sufficient conditions for multiple equilibria, including limit cycles, are derived and illustrative examples are computed. The application of saddle-loop bifurcation theory is a novel feature of the analysis. As one equilibrium Pareto dominates all the others, a macroeconomic coordination problem exists.

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"*At any moment in time, there is some level of unemployment which has the property that it is consistent with equilibrium in the structure of real wage rates.*"... Milton Friedman [1968].

"*...nevertheless, every state of expectation has its definite corresponding level of long-period employment.*"... John Maynard Keynes [1936]

1 Introduction

The conflicting views of macroeconomic dynamic equilibrium reflected in these introductory quotes are still central in academic and policy debates. That a market economy is a self stabilizing system that seeks out a unique 'natural' rate of employment after every shock has become the dominant view since Friedman's famous AEA address in 1968. Still, this position is seriously questioned by those who sympathize with Keynes' earlier insight that real economic outcomes can reflect the self fulfilling prophecies of investors in a capitalist economy.

The academic debate generated by these two views has major implications for practical policy discussion over when and whether monetary and fiscal policy can have any

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important long run effect on the unemployment rate without detrimental inflationary or employment consequences. For example, the first view provides the intellectual justification for the current (Greenspan) approach to monetary policy in the U.S. summed up in the maxim: ‘Control inflation and unemployment will take care of itself.’ However, many in Europe view the high levels of unemployment persisting there as a bad equilibrium in an economic environment containing others that might be achieved were economic policy appropriately designed and implemented.

This paper represents a contribution to the formal argument that multiple Pareto ranked dynamic macro economic equilibria generically exist, each supported by different but rational expectations about the future.[2] The fact that the equilibrium set for the model studied also includes periodic solutions under reasonable conditions is of particular interest. As in the existing literature on non-unique steady states and endogenous cycles within the framework of the neoclassical growth model, the source of the multiplicity here is increasing returns in production.[3] The innovation of this paper is the demonstration that the incorporation of a job-worker matching process into a version of the neoclassical model in which labor is the only input will also generate multiple Pareto ranked equilibria and equilibrium limit cycles. Indeed, lags in the formation of match capital is enough to enable equilibrium macro dynamic outcomes that depend on investor expectations.

The model studied is a variant of Pissarides’ (1990) theory of “equilibrium unemployment.” A single good is produced and consumed by risk neutral worker-employer pairs. The dynamic process by which employers and workers form such matches is the focus of the analysis. Search and recruiting activities serve as inputs in a matching technology that produces a flow of new productive pairs as output. In equilibrium, unemployed workers and employers with vacancies make matching investment decisions so as to maximize expected discounted future consumption flows. These decisions depend on match surplus, the expected present value of the future quasi-rent flow attributable to an existing match. Its division is determined by a generalized Nash wage bargain.

The mathematical model is developed in Section 2. The case under study is one in which a production externality exists responsible for aggregate increasing returns in the sense that average output per match increases with the level of employment. The

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[2] The paper is another on macro economic coordination games. See Cooper (1999) for a complete treatment of the topic and literature.

[3] The general problem in the context of equilibrium growth models is well stated and studied in Farmer (1993). Endogenous cycles can also be present in a search model when the production technology is standard but the matching technology is subject to increasing returns. Diamond and Fudenberg (1989) establish the existence of limit cycles in a related model under these conditions. Drazen’s (1988) model and analysis is mostly closely related to those of this paper. Finally, Mortensen (1989) presents a statement of the model studied here.

matching technology, the relation between matching output and inputs, is concave and exhibit constant returns. In general, participants in the matching process do not capture their marginal benefit of their participation under the assumption that wages are set via bilateral bargaining. However, it is well known that matching efficiency obtains if the worker's (employer's) share of match capital equals the elasticity of the matching function with respect to aggregate search (recruiting) effort when agents are risk neutral.

The model's equilibria, defined in Section 3, are bounded solutions to a planar system of differential equations involving the number of matched employer-worker pairs and the value of the typical match. In the case of a zero pure rate of time discount and efficient wage bargaining, the system's global dynamics are shown to be Hamiltonian in Section 4, i.e., the solution paths are level curves of a known function of the number of matches and their value. As a corollary, multiple steady state solutions generally exist composed of alternating saddle points and centers under the assumption that productivity per match increases with aggregate employment. Although non-periodic equilibria converge to the saddle points, a family of closed orbits, representing oscillating periodic equilibria, surrounds each center. Many equilibria, sometimes a continuum, generically exist when there is more than one steady state solution. The alternative equilibria correspond to different self fulfilling expectations about the future course of the economy.

Because the model is structurally unstable given no discounting and efficient wage bargaining, the conclusions drawn don't extend even approximately to the small positive discount rate and inefficient bargaining cases without qualification. For example, that no periodic equilibria exist given an efficient wage bargain and a positive discount rate is the principal result of Section 5. Still, a perturbation argument, presented in Section 6, implies that there is a limit cycle for each element in an open set of positive discount rates when the workers' share of match surplus is less than the elasticity of the matching function with respect to worker search effort provided that these deviations from efficiency and no discounting are sufficiently small. The limit cycle can be either stable or unstable and is unstable if and only if the economy is sufficiently productive. Finally, any equilibrium unemployment cycle is Pareto dominated by a alternative non-periodic solution, a fact that justifies collective economic policy designed to coordinate on the better equilibrium.

Computed examples are reported to illustrate and extend the analytic results. The special case studied, introduced in Section 7, is characterized by constant elasticity production, matching, and cost of search functions and a constant workers' share of match surplus. In this case, there are three steady state if the economy is sufficiently productive and if the elasticity of the matching rate per unemployed worker with respect to the value of a match and the elasticity of match productivity with respect to the aggregate number of matches are positive and large enough. Parameter values based on explicit and implicit estimates reported in the literature suggest that this necessary condition for multiple equilibria is empirically reasonable. Finally, actual cycles in unemployment are identified for these same parameter values. These computed solutions to the constant elasticity version of the model are reported and illustrated in Section 8. The final section, Section 9, includes a brief description of the empirical properties of the cycle.

2 A Matching Economy

Agents are of two types, workers and employers. A single consumption good is produced by each matched worker-employer pair at a rate that depends on aggregate employment. Let $f(n)$ denote the output of the typical match expressed as a function of the current employment, n. The instantaneous utility function for each agent is linear in consumption and future consumption is discounted at a constant rate r. Hence, individual agents act as expected present value maximizers.[4]

Match formation is the output of a transactions process with aggregate search effort and recruiting activity serving as inputs. Every worker is either matched with a job, employed, or seeking such a match, unemployed. Letting the unit interval represent the total available worker force, the fraction seeking employment is $1-n$. Aggregate search effort is the product $s(1-n)$ where s denotes the search effort of the representative unemployed worker. The cost per searching worker expressed in terms of the produced good is denoted as $c(s)$. Employers both participate in existing job matches and seek new employees. The recruiting cost flow per vacancy is denoted as a and the aggregate recruiting effort is reflected in the number of job vacancies, represented as v. The rate at which new matches form is a function of aggregate search and recruiting effort; it is denoted as $m(v, s(1-n))$. Finally, the law of motion for employment (its time derivative) is

$$\dot{n} = m(v, s(1-n)) - \delta n \tag{1}$$

where δ represents an exogenous job destruction rate.

Increasing returns in production is assumed in the sense that match productivity, $f(n)$, is an increasing function of the aggregate number of matches. Hall (1988) suggests the possibility of increasing returns in production at the aggregate level in the U.S. and Caballero and Lyons (1989) provide evidence that external economies across sectors can account for it. The assumption that the job/worker matching function, $m(v, s(1-n)$, is concave and homogeneous of degree one is the second principal specification restriction. Empirical justification can be found in Blanchard and Diamond (1989) for the U.S. labor market and in Pissarides (1986) for the U.K. Finally, both the job destruction rate, δ, and the cost of filling a vacancy, a , are strictly positive by assumption throughout the paper and the cost of search function, $c(s)$, is strictly increasing and convex with the property that $c(0) = c'(0) = 0$.

Under the assumption that existing vacancies and unemployed workers meet at random, the probability that a particular vacancy will be matched per instant of length dt is equal to the aggregate meeting flow divided by the total number of vacancies, i.e., mdt/v. Analogously, the probability that an unemployed worker is matched is equal to the meeting rate divided by aggregate search effort $mdt/s(1-n)$. Given the linear homogeneity of the matching function, these meeting rates are functions of what

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[4] The model presented here is a version of that developed in Pissarides (1990) stated and studied in Mortensen (1989). The reader is referred to these works for motivation and detail.

Pissarides calls *market tightness*, the ratio of vacancies to search effort,

$$\theta = \frac{v}{s(1-n)}. \tag{2}$$

Formally, an individual worker is matched at the Poisson frequency $s\lambda(\theta)$ where s is the worker's chosen search intensity and

$$\lambda(\theta) \equiv \frac{m(v, s(1-n))}{s(1-n)} = m(\theta, 1) \tag{3}$$

depends on the recruiting and search decisions of the other agents. Analogously, vacancies are matched at the Poison rate

$$\frac{m(v, s(1-n))}{v} = \frac{\lambda(\theta)}{\theta}. \tag{4}$$

Note in passing that $\lambda(\theta)$ is increasing and concave while $\theta/\lambda(\theta)$ is increasing in θ given the assumption that $m(v, s(1-n))$ is increasing, concave and homogenous of degree one in its arguments.

Given these expressions for the matching rates, the expected present value of the future consumption stream attributable to holding a vacancy to an employer, V, and the present value of the future consumption that a searching unemployed worker can expect, U, solve the following asset pricing equations

$$rV = \frac{\lambda(\theta)}{\theta}[J - V] - a + \dot{V} \tag{5}$$

$$rU = s\lambda(\theta)[W - U] - c(s) + \dot{U} \tag{6}$$

where r is the given rate of time discount common to all agents and where J and W represent the expected present values of future consumption to employer and worker respectively once a match forms. In each case, the return on the asset value, the left side, equals current income plus expected capital gain where the latter is composed of two parts, the expected gain from search activity plus pure appreciation.

Workers and employers decide the extent of their individual search and recruiting investment in response to current market conditions and expectations about future private returns. As vacancies can be created without limit in the aggregate, free entry drives their value V to zero. Equivalently, the cost of holding a vacancy is equal to the expected return, i.e.,

$$V = 0 \Longleftrightarrow a = \frac{\lambda(\theta)}{\theta} J \tag{7}$$

when individual employers regard aggregate market tightness as given. Each unemployed worker choose a search intensity s to maximize the value of unemployed search given market tightness. As a result, the marginal cost of search effort equals the expected return,

$$c'(s) = \lambda(\theta)[W - U]. \tag{8}$$

The value of an occupied job to the employer, J, and of employment to the worker, W, solve the asset pricing equations

$$rJ = f(n) - w - \delta[J - V] + \dot{J} \tag{9}$$

$$rW = w - \delta[W - U] + \dot{W} \tag{10}$$

where w denotes the wage paid by the employer to the worker which can vary with market conditions as specified below. Match surplus p is defined as the value of a match to its parties less the value of the option of remaining separate, i.e.,

$$p \equiv J + W - V - U. \tag{11}$$

When worker and employer meet, a wage is negotiated which is viewed as the outcome of a bilateral bargaining problem. The outcome determines how match surplus will be shared once a match forms. Let the values of continued search and recruiting represent the "threat point" (U, V) in a generalized Nash solution to the problem of the total value of a match $J + V$. The wage outcome satisfies

$$w = \arg\max\{\beta \ln(W - U) + (1 - \beta)\ln(J - U)\}$$

subject to (11) where β denotes the worker's "bargaining power". As the first order condition requires

$$(1 - \beta)[W - U] = \beta[J - V], \tag{12}$$

β is the worker's share of match surplus, i.e. $W - U = \beta p$ from (11).

Although equity considerations in a bilateral bargaining situation might suggest sharing equally, there is no general reason for even assuming constancy of the shares. For example, the wage determination rules implied by the alternative competitive wage setting models suggested by Moen (1997) and Shimer (1995) can be interpreted in our framework as a bilateral bargaining outcome with shares that are endogenous functions of market tightness. Indeed, because the marginal contribution of recruiting and search effort are respectively equal to the private return in their formulation, the match value shares solve Hosios' (1990) necessary condition for social efficiency:

$$p\lambda'(\theta) = (1 - \beta)\frac{p\lambda(\theta)}{\theta} \Longleftrightarrow \beta = [1 - \frac{\theta\lambda'(\theta)}{\lambda(\theta)}]. \tag{13}$$

In this case, the two search externalities, the congestion effect that agents on the same side of the market impose by reducing the matching rate of their fellows, and the thick market benefit that agents on the other side bestow by increasing the matching rate of their potential match partners, just cancel. Equivalently, the total cost of the recruiting and search effort, $av + c(s)(1 - n)$, required to achieve any given matching rate $\overline{m} = m(v, s(1 - n))$ is minimized if and only if (13) holds as Mortensen and Wright (1997) show.

In order to include the important case of efficient bargaining as characterized by the Hosios condition, we allow the worker's share to depend on market tightness. However, when the Hosios condition does not hold exactly, we restrict the analysis to the case in which worker bargaining power increases as vacancies rise relative to worker search effort, i.e.,

$$\beta'(\theta) \geq 0. \tag{14}$$

Given the standard interpretation of β in the generalized Nash bargaining model, this restriction rules out counter cyclical worker 'bargaining power'.

Equation (11) and (12) and the equilibrium conditions (7) and (8) implicitly define market tightness and search intensity as functions of match surplus. Formally,

$$a = \frac{\lambda(\theta(p))}{\theta(p)}(1 - \beta(\theta(p)))p \tag{15}$$

and

$$c'(s(p)) = \lambda(\theta(p))\beta(\theta(p))p \tag{16}$$

where the solutions $\theta(p)$ and $s(p)$ are the equilibrium functions that determine vacancies and search intensity.

Because $\lambda(\theta)$ is an increasing concave function and $c(s)$ is an increasing convex function by assumption, these two equations unique determine search intensity s and market tightness θ as increasing functions of match surplus p given either (13) or (14). Furthermore, $\lambda(0) = c'(0) = 0$ imply $s(0) = s'(0) = \theta(0) = 0$. Consequently, net expected income derived from search activity

$$g(p) \equiv \max_{s \geq 0}\{s\lambda(\theta(p))\beta(\theta(p)))p - c(s)\} \tag{17}$$

and the matching rate per unemployed worker

$$h(p) \equiv s(p)\lambda(\theta(p)) \tag{18}$$

have these same properties.

Proposition 1 *If either the Hosios condition, (13), holds or the worker's share of match surplus is pro-cyclical in the sense of (14), then the matching rate per unemployed worker $h(p)$ and net search income $g(p)$ are increasing, continuous and differentiable functions for all $p > 0$. Furthermore, $g(0) = h(0) = h'(0) = 0$ and*

$$g'(p) = h(p)\left[1 + \left(\frac{p\theta'(p)}{\theta(p)}\right)\left(\frac{\theta\lambda'(\theta)}{\lambda(\theta)} - [1 - \beta(\theta)]\right)\right]. \tag{19}$$

Proof. All but the last assertion have been established. Differentiate equations (17) and (15) with respect to p applying the envelope theorem appropriately in the first case. The results are

$$g'(p) = s[\beta\lambda + p(\beta\lambda)'\theta']$$

and

$$a\theta' = (1 - \beta)\lambda + p\lambda'\theta' - p(\beta\lambda)'\theta'$$

where $(\beta\lambda)'$ represents the derivative of the product with respect to the common argument θ of both the functions $\beta(\theta)$ and $\lambda(\theta)$. Use the second result to eliminate the derivative of the product in the first. The result is

$$g' = s\lambda \left[1 + \frac{p\lambda'\theta' - a\theta'}{\lambda} \right].$$

After substituting from (15) for a and from (18) for $s\lambda$, rearrange the terms to obtain (19). ■

Since $m = s\lambda(1-n) = h(1-n)$, the law of motion for employment, equation (1), can be written

$$\dot{n} = h(p)(1-n) - \delta n. \tag{20}$$

By differentiating the identity (11) with respect to time and then combining results from above appropriately, one finds that the surplus value of a match, p, satisfies the following differential equation:

$$\dot{p} = (r+\delta)p + g(p) - f(n). \tag{21}$$

Furthermore, the wage that supports the division of surplus characterized by equation (12)

$$w = \beta(\theta(p))f(n) + (1 - \beta(\theta(p)))g(p). \tag{22}$$

In other words, match surplus is the expected present value of the difference between match output and net search income earned were the worker unemployed and the wage is a weighted average of match output per employed and forgone net search income per unemployed worker.

To derive equations (21) and (22), first combine equations (5), (9), and (7) to obtain

$$\begin{aligned}
(r+\delta)(J-V) - (\dot{J} - \dot{V}) &= p - w - rV + \dot{V} \\
&= p - w - \left[\left(\frac{\theta}{\lambda(\theta)} \right) [J - V] - a \right] \\
&= p - w
\end{aligned}$$

Analogously, a combinations of equations (6), (10), (8), (11), (12), and (17) together yield

$$\begin{aligned}
(r+\delta)(W-U) - (\dot{W} - \dot{U}) &= w - rU + \dot{U} \\
&= w - [s\lambda(\theta)[W-U] - c(s)] \\
&= w - \max_{s \geq 0} \{ s\lambda(\theta)\beta(\theta(p))p - c(s) \}.
\end{aligned}$$

Given the match surplus identity (11), equation (21) is the sum of these two equations. Second, multiply both sides of the first equation by $\beta(\theta)$ and both sides of the second by $1 - \beta(\theta)$. The fact that the rights sides of the results must be equal to another from equation (12) together with equation (21) imply (22).

3 Equilibrium

Given the assumption of rational expectations, agents understand the laws of motion for both employment and the value of a job-worker match characterized by equation (20) and (21) and use them to forecast future values of both variables. Although each agent's forecast depends on the expectations of the others, all must make the same forecast. Hence, agents have perfect foresight in the sense that a dynamic market equilibrium is a particular solution to this system of differential equations. The following definition rules out future aggregate dynamics that either are inconsistent with voluntary market participation in the sense that the surplus value of a match become negative or are infeasible in the infinite future.

Definition: *An equilibrium is any particular solution* $(\overrightarrow{p}, \overrightarrow{n})$ *to (20) and (21) defined by initial employment,* $n(0) = n_0$, *that is consistent with feasibility,* $p(t) \geq 0$ *and* $0 \leq n(t) \leq 1$ *for all* $t \geq 0$, *and transversality,* $\lim_{t\to\infty} \{e^{-rt}p(t)\} = 0$.

Although initial employment is historically predetermined, the initial value of a match, $p(0) = p_0$, is a jump variable that depends on future expectations of the agents but is continuous in t there after. The restriction that match value satisfies the transversality condition specified in the definition is simply a recognition of limits of economic feasibility and is typically a necessary condition for individual economic rationality. The non-negative match value requirement rules out solutions that generate future match values dominated by autarchy.

Proposition 2 *A particular solution* $(\overrightarrow{p}, \overrightarrow{n})$ *to the planar system composed on (20) and (21) is an equilibrium if and only if its trajectory originates and remains in the rectangle* $B = [0, \overline{p}] \times [0, 1]$ *where* $\overline{p}$ *is the unique solution to*

$$(r + \delta)\overline{p} + g(\overline{p}) = f(1). \tag{23}$$

Proof. Sufficiency follows directly from the definition of equilibrium. Necessity is an implication of the transversality condition and the feasibility conditions. Obviously, any trajectory that implies a negative match value or an employment level larger than unity or less than zero at some finite date is ruled out as infeasible. Any other trajectory which leaves the rectangle violates transversality. This assertion is implied by

$$\dot{p} = (r + \delta)p + g(p) - f(n) > (r + \delta)p - f(1) \; \forall p \geq \overline{p} = p(s)$$

where $s > 0$ is the finite date at which the trajectory crosses the boundary $p = \overline{p}$ of the rectangle B. The inequality follows from definition of $\overline{p}$ and the fact that $g(p)$ is increasing . Consequently,

$$p(t) \geq \overline{p}e^{(r+\delta)(t-s)} \; \forall t \geq s$$

which implies that

$$\lim_{t\to\infty} p(t)e^{-rt} \geq \lim_{t\to\infty} \overline{p}e^{-rs+\delta(t-s)} = \infty.$$

Of course, this argument also applies to any trajectory for which $p(0) \geq \overline{p}$. ■

The dynamics in any neighborhood of a the various possible steady states are illustrated in the phase diagram, Figure 1. The singular curves of the planar system, the two functional relationships between p and n implied by $\dot{p} = 0$ and $\dot{n} = 0$, are both upward sloping by Proposition 1 given the assumption of increasing returns in production, $f'(n) > 0$. Because

$$\dot{n} = 0 \Longleftrightarrow n = \frac{h(p)}{\delta + h(p)}, \tag{24}$$

values of n on the curve are bounded above by 1. Because

$$\dot{p} = 0 \Longleftrightarrow (r + \delta)p + g(p) = f(n), \tag{25}$$

the values of p on this curve are bounded above by $\overline{p}$ for all $n \in (0, 1)$ and equation (23).

Furthermore, because $h(0) = g(0) = f(0) = 0$, the origin is a point of intersection of the two curves, i.e., a steady state pair. Because $h'(0) = g'(0) = 0$, $f'(0) > 0$ implies that the $\dot{p} = 0$ singular curve has a finite positive slope at the origin while the singular curve defined by $\dot{n} = 0$ has an infinite slope, it may also be the only equilibrium. However, if there are any positive equilibrium then there must be at least two. Finally, the arrows in Figure 1 indicate the directions of motion implied by the following global facts:

$$\begin{aligned} \frac{\partial \dot{n}}{\partial n} &= -h(p) - \delta < 0 \\ \frac{\partial \dot{p}}{\partial p} &= r + \delta + g'(p) > 0. \end{aligned} \tag{26}$$

Although Figure 1 depicts the phase portrait only for the case of three steady states, it is representative of the more general cases in the following sense: A no-trade steady state exists at the origin 0 in all cases under the assumptions made. In general, there are an odd number of steady states. Counting out from and including the origin, odd numbered steady states are saddle points, like H, separated by the even numbered steady states, like M. As indicated in the figure by the direction arrows, each intermediate state is either a source, center, or sink depending on the sign of the sum of the two eigen values at the point. This sum is

$$\begin{aligned} \frac{\partial \dot{p}}{\partial p} + \frac{\partial \dot{n}}{\partial n} &= r + \delta + g'(p) - h(p) - \delta \\ &= r + h(p)\left(\frac{p\theta'(p)}{\theta(p)}\right)\left(\frac{\theta\lambda'(\theta)}{\lambda(\theta)} - [1 - \beta(\theta)]\right). \end{aligned} \tag{27}$$

by virtue of equations (20), (21), and (19). Hence, when the discount rate is zero and the wage bargain is efficient in the sense that the sharing rule satisfied the Hosios condition, equation (13), the eigen values at M are purely imaginary. Although very special, the global dynamics of the model can be fully characterized in this case.

4 Hamiltonian Dynamics

Local information about eigen values provides an incomplete description of the model's dynamics. However, the global properties of the solutions to a system of differential equations can be characterized in the special case in which the solution trajectories are level curves of a known real valued function. Such a function is called a *Hamiltonian*.[5] In our case,

$$H(p,n) = \int_0^n f(x)dx + g(p)(1-n) - \delta pn \tag{28}$$

is a Hamiltonian function given no time discounting and the efficient match surplus sharing rule.

Note that the value of the function is equal to the market output of the employed plus the net value of the search effort of the unemployed less the value of the flow of matches destroyed, "depreciation". In other words, the function represents the net aggregate income of the economy modeled at every possible employment and match value combination on an equilibrium trajectory. As a corollary of the next result, net aggregate income is a constant along any equilibrium path that the economy might take and equilibria are ranked according to aggregate net income in this special case.

Proposition 3 *In the case of no pure time discounting, $r = 0$, and a match surplus sharing rule that satisfies the Hosios condition, equation (13), every solution trajectory to the differential equation system (20) and (21) is a level curve of the Hamiltonian function $H(p,n)$.*

Proof. Under the hypothesis $H_p(p,n) = g'(p)(1-n) - \delta n = \dot{n}$ by Proposition 1 and $H_n(p,n) = f(n) - g(p) - \delta n = -\dot{p}$. Since $\dot{H} = \dot{n}\dot{p} - \dot{p}\dot{n} = 0$ everywhere as a consequence, any particular solution $(\overrightarrow{p}, \overrightarrow{n})$ satisfies $H(p(t), n(t)) = H(p_0, n_0)$ for all $t > 0$. ■

Periodic solutions are connected trajectories in the phase portrait known as *closed orbits*. If such a path does not include a steady state, it is a cycle of finite period. As equation (27) implies that the sum of the eigen values at any point are identically zero everywhere in the case under study, M is necessarily a center surrounded by a family of cycles.

Indeed, the Hamiltonian takes on a local minimum value at M. To prove this assertion, simply note that $H_n(p,n) = -\dot{p} > (<)0$ for all values of n to the right (left) of M in Figure 1 holding p at its steady state value and that $H_p(p,n) = \dot{n} > (<)\ 0$ for all p above (below) M holding the value of n constant at its steady state value. Hence, there are only two different possible topological cases given three steady states, those illustrated as Figures 3 and 4 and the boundary case that separates them, illustrated as Figure 2.

The distinctions between the phase portraits associated with the three cases illustrated in Figures 2, 3, and 4 reflect differences in the relative sizes of the value of the

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[5] The terminology introduced in the sequel is taken from Guckenheimer and Holmes (1986).

Hamiltonian $H(p,n)$ at H and at the origin 0. In the boundary case, the two values are both equal to $H(0,0)=0$ by definition. As a consequence, the unstable solution trajectory from 0 converges to H from the left while the unstable trajectory diverging to the left from H converges to 0 as illustrated in Figure 2. The object formed by these two trajectories is a *heteroclinic orbit*, a closed loop connecting two or more steady states of a system.

Because the Hamiltonian function takes on a local minimum at M, the geometry of the phase portrait and the fact that each solution trajectory traverses a level curve of $H(p,n)$ imply that the Hamiltonian is larger at 0 and along the stable manifold converging to 0 than along the *homoclinic orbit* connecting H to itself in Figure 3 and is larger in value at H and along its stable manifold than at 0 on the homoclinic orbit connecting 0 to itself in Figure 4. As a consequence, an equilibrium solution that converges to the origin for all initial values of employment exists only in Figure 3 while an equilibrium solution which converges to H for every initial value of employment exists only in Figure 4. In general, these are the only two cases that arise when there are three steady sates. In the sequel, we refer to the first as case I and the second as case II.

As we known, the set of equilibria for any initial value of employment $n_0 \in (0,1)$ are those that originate in the vertical interval $[0,\overline{p}]$ and remain thereafter in the rectangle $B=[0,\overline{p}]\times[0,1]$. These include the stable manifolds associated with the two saddles and the family of closed orbits around M. That a continuum of equilibria exist for many initial condition is an obvious implication of this observation. Indeed, in the boundary case illustrated in Figure 2, all trajectories that initiate with employment no greater than the steady state value at H and match values between the trajectories converging to the origin 0, on the one hand, and the positive saddle point H, on the other, are equilibria. In the generic cases illustrated in Figures 3 and 4, all trajectories with initial match value between the stable manifolds converging to the steady states are equilibrium given initial employment in some open set which includes its value at M.

Multiple equilibria arise because the expectation that a given solution path will be realized by all agents is self fulfilling for more than one trajectory. Specifically, "bullish" expectations are self fulfilling along the solution trajectory leading to H, "bearish" expectations generate a future history leading to 0, and the expectation of future oscillations with a fixed period and amplitude are self fulfilling on the equilibria represented by any one of the closed orbits surrounding M. Of course, the indeterminacy of equilibrium poses a coordination problem for the aggregate economy.

When there is more than one equilibrium for a given initial level of employment, the equilibria are ranked by net aggregate income as reflected by the value of the Hamiltonian function. In case I, all equilibria except that represented by the trajectory converging to the origin yield a negative net income since the value of the Hamiltonian on that trajectory are zero and the value of the Hamiltonian on any other equilibrium path is between zero and the minimum value of Hamiltonian attained at the steady state point M. In case II, the only equilibria that yield a positive net income is represented by the stable manifold of H. Note that H is attainable from any initial level of employment only in case II. In case I, a sufficiently large initial level of employment is required for the existence of such an equilibrium path.

5 No Limit Cycles

The transparency of the model's dynamics vanishes with positive discounting. However, Bendixson's criterion, a well known sufficient condition for the non-existence of closed orbits, can be applied to obtain global results. (See Guckenheimer and Holmes (1986, p. 44) for a formal statement and proof.) In our case, the condition requires that the right side of (27) not be identically zero and not change sign. Hence, if they exist at all, limit cycles require a workers' share of match value strictly less than that consistent with efficiency given positive discounting.

Proposition 4 *If $r > 0$ and $\beta(\theta) \geq 1 - \theta\lambda'(\theta)/\lambda(\theta)$ for all θ, then no equilibrium is periodic.*

The global dynamics of the system given three steady states for case I and case II under the conditions of Proposition 4 are illustrated in Figures 5 and Figure 6 respectively. As there are no closed orbits and M is unstable (The right side of (27) is positive under the hypothesis.), M is necessarily the source of the stable manifold branch that converges to H in Figure 5 and to 0 in Figure 6. The stable manifold branch converging to the origin in Figure 5 and converging to H in Figure 6 also represent equilibrium trajectories. Hence a continuum of equilibria continue to exist for some initial employment levels. However, note that there is no equilibrium path converging to H in Figure 5 for sufficiently small initial values of employment. In other words, all the equilibrium paths tend toward the 'no-trade' steady state for sufficiently small initial values of employment in that case.

6 Limit Cycles

Solutions to a differential equation system are said to be *structurally stable* if the topological properties of the phase portrait are invariant to small perturbations of the system's parameters. Bifurcations occurs at points in the parameter space associated with structurally instability. Homoclinic orbits such those illustrated in Figure 3 and 4 are structurally unstable. *Saddle-loop bifurcation* is said to occur as a consequence of variation in a parameter, the discount rate when the Hosios conditions hold, through the bifurcation point, $r = 0$ in the case at hand. On either side of the bifurcation point, the homoclinic orbit splits into two separate paths, the stable and unstable manifolds associated with the saddle point on the original homoclinic orbit are distinct, as illustrated in Figures 5 and 6.

In this section, perturbation techniques (See Guckenheimer and Holmes 1986, Chapter 4.) are applied to establish that a saddle loop bifurcation occurs as well at positive discount rates given a sufficiently small positive difference between the employers' share of match value and the elasticity of the matching function with respect to vacancies. Typically saddle-loop bifurcation signals the existence of a family of limit cycles associated with values of the parameters near the bifurcation point. (See Guckenheimer and Holmes 1986, Sections 6.1.) Furthermore, under regularity conditions satisfied by our model, the limit cycle identified is stable (unstable) if the sum of the eigen values evaluated at the saddle connected by the homoclinic orbit is negative (positive) at the bifurcation point in parameter space. This test implies that the identified limit cycle is stable in case I and is unstable in case II when a saddle loop bifurcation occurs for a positive discount rate.

The generic existence of a limit cycle is established and a formula for finding discount rates that yield such cycles is derived using *Melnikov's perturbation method*. For the purpose of presenting results, define the differential vector system

$$\dot{x} = F(x) + \varepsilon G(x) \text{ where } x = \begin{pmatrix} p \\ n \end{pmatrix}, \tag{29}$$

$$F(x) = \begin{pmatrix} F_1(x) \\ F_2(x) \end{pmatrix} = \begin{pmatrix} \delta p + \int_0^p h(q)dq - f(n) \\ h(p)[1-n] - \delta n \end{pmatrix}, \tag{30}$$

$$G(x) = \begin{pmatrix} G_1(x) \\ G_2(x) \end{pmatrix} = \begin{pmatrix} rp + \left[g(p) - \int_0^p h(q)dq\right] \\ 0 \end{pmatrix}, \tag{31}$$

where ε is a small positive number. Note that $F(x)$ is the vector field associated with the Hamiltonian function $H(x)$, defined in (28), and that ε is a (small) perturbation of that field representing the distortion attributable to positive discounting and deviation from the Hosios condition. Of course, by design the system (29) approximates the equilibrium differential equation system defined by equations (20) and (21) when the discount rate and the deviation of the workers' share from the search elasticity of the matching function are both small. Finally, let the saddle point of interest, denoted as S, be H when the parameters are such that case I obtains and 0 when case II holds. Finally, define

$$\Gamma(S) = \left\{x \in \Re^2 | H(x) \leq H(S)\right\} \tag{32}$$

as the region in the phase portrait surrounded by the graph of the homoclinic orbit containing S.

For any small perturbation of the system away from the Hamiltonian case, indexed by the value of ε, the saddle point of interest, H in case I and 0 in case II, continues to exist but is displaced slightly (See Guckenheimer and Holmes 1986, Lemma 4.5.1 and 4.52.). The following result pertains to these perturbed saddle points:

Proposition 5 *If $\beta(\theta) < 1 - \theta\lambda'(\theta)/\lambda(\theta)$ for all θ, then a homoclinic orbit connecting the saddle S to itself exists for $\varepsilon > 0$ sufficiently small when $r = \widehat{r}$ where*

$$\widehat{r} = \frac{\int_{\Gamma(S)} \left[h(p)\left(\frac{p\theta'(p)}{\theta(p)}\right)\left(1 - \frac{\theta(p)\lambda'(\theta(p))}{\lambda(\theta(p))} - \beta(\theta(p))\right)\right] dpdn}{\int_{\Gamma(S)} dpdn}$$

for $S \in \{0, H\}$. Furthermore, for all $0 < r < \widehat{r}$, the solutions to (29) include a stable limit cycle when $S = H$ and an unstable limit cycle when $S = 0$.

Proof. As the proposition is a direct implication of Theorems 4.5.3 and Theorems 6.1.1 in Guckenheimer and Holmes (1986) taken together, the task is to verify the conditions of both.

By virtue of equation (4.5.15) in the same source, the conclusion of Theorem 4.5.3 is the first assertion of this proposition because the *Melnikov function* is

$$M = \int_{\Gamma(S)} \left[\frac{\partial G_1(x)}{\partial p} + \frac{\partial G_2(x)}{\partial n} \right] dpdn$$

$$= \int_{\Gamma(S)} \left[r + h(p) \left(\frac{p\theta'(p)}{\theta(p)} \right) \left(\frac{\theta(p)\lambda'(\theta(p))}{\lambda(\theta(p))} - [1 - \beta(\theta(p))] \right) \right] dpdn$$

where the second equality follows from equations (31) and (19). In particular, the graphs of the stable and unstable manifolds associated with the saddle S form a homoclinic loop connecting S with itself if the parameters are such that M is zero by virtue of the referenced theorem.

Given the homoclinic orbit which exists when $r = \widehat{r}$, Theorem 6.1.1 asserts the existence of a limit cycle for values of r near and on one side of $\widehat{r}$. Furthermore, the limit cycle is stable (unstable) if and only if the sum of the eigen values evaluated at the saddle point contained in the homoclinic orbit is negative (positive). Because $F(x)$ is the vector field of the $H(p, n)$ so that $dF_1(x)/dx + dF_2(x)dx = 0$, the sum of the eigen values at any point is given by

$$trace\{\frac{d\dot{x}}{dx}\} = \varepsilon \left[\frac{\partial G_1(x)}{\partial p} + \frac{\partial G_2(x)}{\partial n} \right]$$

$$= \varepsilon \left[r + h(p) \left(\frac{p\theta'(p)}{\theta(p)} \right) \left(\frac{\theta(p)\lambda'(\theta(p))}{\lambda(\theta(p))} - [1 - \beta(\theta(p))] \right) \right].$$

As $h(p) > 0$ at H, the sign is negative when $\beta(\theta) < 1 - \theta\lambda'(\theta)\lambda(\theta)$ at H given that $r < \widehat{r}$ by the definition of $\widehat{r}$ above. As $h(0) = 0$, the sign at 0 is always positive when r is positive. ■

7 The Coordination Problem

In this section, we show that the equilibrium path converging to the saddle point with the highest match value and employment level, call it H, Pareto dominates all others associated with the same initial employment if the worker's share of match surplus increases with market tightness. In other words, a coordination problem is generic in the modeled economy. This assertion is a consequence of the fact that the value sequence associated with the trajectory converging to H dominates that of any other equilibrium path.

Let $(\overrightarrow{p}^0, \overrightarrow{n}^0)$, where $n^0(0) = n_0$ is initial employment, denote the equilibrium trajectory converging to H and let $(\overrightarrow{p}, \overrightarrow{n})$, $n(0) = n_0$, represent any other equilibrium path associated with the same initial employment level.

Proposition 6 *If the workers' share of match surplus is pro-cyclical in the sense of condition (14), then the value sequence $\overrightarrow{p}^0$ dominates $\overrightarrow{p}$ in the sense that $p^0(t) > p(t)$ for all $t \geq 0$.*

Proof. As $p^0(t)$ is the maximum value of a match over all equilibria values associated with employment equal to $n = n^0(t)$ and the trajectory converging to H is upward sloping, it follows that $p^0(t) > p(t)$ if $n(t) \leq n^0(t)$ for all $t \geq 0$. In other words, it is sufficient to demonstrate that the latter inequality holds. Suppose otherwise, namely $n(t) > n^0(t)$ for some $t > 0$. In this case, the fact that $n^0(0) = n(0) = n_0$ implies that some date s exists such that $0 \leq s < t$, $n^0(s) = n(s)$, and

$$\dot{n}(s) = h(p(s))(1 - n(s)) - \delta n(s) > \dot{n}^0(s) = h(p^0(s))(1 - n^0(s)) - \delta n^0(s).$$

In words, n must overtake n^0 at some date in the interval $[0, t)$. But, if so, then $h'(p) > 0$ implies $p^0(s) < p(s)$, an inequality which contradicts the fact that $p^0(s)$ is the maximum equilibrium value of a match at employment level $n = n^0(s)$. ■

At any point in time, there are two employer types, matched and unmatched, and two worker types, matched and unmatched. As the value of a vacancy V is zero across all equilibria, an employer holding a vacancy is indifferent among them. However, an matched employer ranks trajectories according the their value J which in all equilibria satisfies the free entry condition

$$J = [1 - \beta(\theta(p))]p = \frac{a\theta(p)}{\lambda(\theta(p))}.$$

As the term on the far right side is increasing in market tightness and market tightness $\theta(p)$ is an increasing function of p by the second equality, the value of a filled job to an employer is increasing in p across equilibria.

Equations (6), (12), and (17) imply that the worker value of unemployment is the forward solution to the differential equation $\dot{U} = rU - g(p)$. As the value of unemployment at time zero is

$$U = \int_0^\infty g(p(t))e^{-rt}dt,$$

the proposition above guarantees that any worker unemployed at the initial date prefer the equilibrium offering the largest initial match value. Finally, because the value of employment is $W = \beta(\theta(p))p + U$ and because $\beta(\theta)$ and $\theta(p)$ are both increasing functions, any initially employed worker also prefers the match value maximizing equilibrium.

Pareto ranked equilibria pose a coordination problem. The implementation of a policy designed to induce the economy to select the best equilibrium is obviously suggested. What that policy might be is question left for future analysis.

8 The Constant Elasticity Case

The derivation of empirically verifiable conditions under which multiple steady states actually exist, conditions that are necessary for a cycle, generally requires a more concrete specification of the model. In this section, a parametric version of the model is studied, that obtained under the assumptions of a constant elasticity production function, a constant elasticity cost of search function, a constant matching function and a constant workers' share of match surplus. Formally, let

$$f(n) = an^{\alpha}, \tag{33}$$

$$m(v, s(1-n) = bv^{\eta}\left[s(1-n)\right]^{1-\eta}, \tag{34}$$

and

$$c(s) = cs^{\gamma}. \tag{35}$$

In this case, the worker's job finding rate is

$$h(p) = kp^{\kappa} \tag{36}$$

where

$$k = \left(\frac{1-\beta}{ab}\right)^{\frac{\eta\gamma}{(1-\eta)(\gamma-1)}}\left(\frac{b\beta}{c\gamma}\right)^{\frac{1}{\gamma-1}} \tag{37}$$

and

$$\kappa = \frac{1-\eta+\gamma\eta}{(1-\eta)(\gamma-1)}. \tag{38}$$

Because equation (15) implies $p\theta'/\theta(p) = 1/(1-\eta)$ where $\eta = \theta\lambda'(\theta)/\lambda(\theta)$ in this log linear case, Proposition 1 implies

$$g(p) = \frac{\beta}{1-\eta}\int_0^p h(q)dq = k\left(\frac{\beta}{1-\eta}\right)\left(\frac{p^{1+\kappa}}{1+\kappa}\right) \tag{39}$$

the equilibrium differential equation system (20) and (21) can be written as follows:

$$\dot{n} = kp^{\kappa}(1-n) - \delta n$$

$$\dot{p} = (r+\delta)p + k\left(\frac{\beta}{1-\eta}\right)\left(\frac{p^{1+\kappa}}{1+\kappa}\right) - an^{\alpha}.$$

Consequently, the singular curves are

$$\dot{n} = 0 \Longleftrightarrow n = \frac{kp^{\kappa}}{\delta + kp^{\kappa}} \tag{40}$$

and

$$\dot{p} = 0 \Longleftrightarrow n = \left(\frac{(r+\delta)(1-\eta)(1+\kappa)p + \beta kp^{1+\kappa}}{a(1+\kappa)(1-\eta)}\right)^{\frac{1}{\alpha}}. \tag{41}$$

Although the origin is always a steady state and another exists if either the productivity parameter a is large enough, whether there are three or more steady state solutions depends on the values of the other parameters. For any $\alpha \leq 1$, the value p of along the $\dot{p} = 0$ singular curve is a strictly concave function of n by virtue of (41). That the value of p along the $\dot{n} = 0$ singular curve is a convex function of n when $\kappa \leq 1$ and has a logistic shape (first concave and then convex) when $\kappa > 1$ is an implication of (40). The following result provides a sufficient condition for exactly three non-negative steady state solutions.

Proposition 7 *In the constant elasticity case, three non-negative steady state solutions exist for all $\kappa \geq 1/\alpha > 1$ and a greater than some finite critical level.*

Proof. Given the other parameters, the two singular curves cross in the positive quadrant at least once if and only if a is sufficiently large. When this condition is satisfied, the fact that 0 is a steady state, that the largest steady state H is a saddle point, i.e., $\dot{n} = 0$ intersects $\dot{p} = 0$ from below in Figure 1, and that there are at most three steady states implies that there are exactly three when the origin 0 in also a saddle. A differentiation of equations (40) and (41) imply

$$\frac{dp}{dn}|_{\dot{n}=0} = \frac{\delta\kappa k p^{\kappa-1}}{(\delta + kp^{\kappa})^2}$$

$$\frac{dp}{dn}|_{\dot{p}=0} = \left(\frac{(r+\delta)(1-\eta) + \beta/kp^{\kappa}}{a\alpha(1-\eta)}\right) \times \left(\frac{(r+\delta)(1+\kappa)(1-\eta)p + \beta kp^{1+\kappa}}{a(1+\kappa)(1-\eta)}\right)^{\frac{1}{\alpha}-1}.$$

The origin is a saddle in Figure 1 if and only if the value of the first of these two derivatives near 0 is less than the second. As

$$\frac{\frac{dp}{dn}|_{\dot{p}=0}}{\frac{dp}{dn}|_{\dot{n}=0}} = \left(\frac{(\delta + kp^{\kappa})^2\left[(r+\delta)(1-\eta) + \beta kp^{\kappa}\right]p^{1-\kappa}}{a\alpha\delta\kappa k(1-\eta)}\right) \times \left(p^{\frac{1}{\alpha}-1}\right)\left(\frac{(r+\delta)(1+\kappa)(1-\eta) + \beta kp^{\kappa}}{a(1+\kappa)(1-\eta)}\right)^{\frac{1}{\alpha}-1},$$

$$\lim_{p \to 0}\left\{\frac{\frac{dp}{dn}|_{\dot{p}=0}}{\frac{dp}{dn}|_{\dot{n}=0}}\right\} = \lim\left\{\frac{\delta^2(r+\delta)^2(1+\kappa)p^{\frac{1}{\alpha}-\kappa}}{a\alpha\delta\kappa k}\right\} > 0$$

under the hypothesis. ■

Caballero and Lyons (1989) estimate of the aggregate returns to scale is about 1.3 after taking account of external effects which suggests a value for α of about 0.3. Hence the sufficient condition for three steady states is satisfied for all $\kappa \geq 3.33$. Blanchard and Diamond (1989) obtain a point estimate for β equal to 0.4.This estimate combined with values of γ implicit in the estimated magnitudes of the transition rates to employment elasticities with respect to predicted wage rates reported in Burdett et al. (1984) are consistent with values of κ between 4 to 12. In sum, the sufficient condition for three steady state solutions is empirically plausible for values of α as low as 0.1.

For the parameterized model, the wage bargain is efficient when the worker's share of match surplus, β, equals the elasticity of matching function with respect to aggregate search effort, the parameter $1-\eta$. When in addition there is no discounting, the solution trajectories are the level curves of the Hamiltonian

$$H(p,n) = \frac{an^{1+\alpha}}{1+\alpha} + \frac{kp^{1+\kappa}}{1+\kappa}(1-n) - \delta pn$$

by virtue of equations (28), (33) and (36). Finally, since the partial derivatives of the Hamiltonian function are zero at steady states, the difference between its value at H and 0 is increasing in the productivity of both the matching technology and production technology as represented by the parameters a and k respectively. Hence, Hamiltonian dynamics case II is associated with relatively high values of these two parameters and case I with relatively low values.

9 Computed Examples with Cycles

Proposition 5 can guarantee the existence of a limit cycle only for small negative deviations of workers' share from the elasticity of the matching function with respect to search effort. In this section, limit cycles are computed and displayed for plausible values of the discount rate and the other parameters of a log linear version of the model. These examples extend the analytic results by showing that cycles exist even when the differences between workers' share and the search elasticity of the matching function are quite large.

The values of the elasticity of the transition to employment and the elasticity of the production function used in the computations are $\kappa = 10$ and $\alpha = 0.3$ respectively. Letting the unit time period equal one quarter, reasonable values for the discount and job separation rates are $r = 0.01$ and $\delta = 0.15$. For the purposes of the demonstration, k is set equal to unity. To generate an examples of both case I and case II, the production function scale parameter, a, is set at 0.12 and at 0.13 respectively. Finally, the value of the ratio $\beta/(1-\eta)$ is varied to obtain sub-cases of interest. The values of p and n at the stationary points M and H and the sums of the real parts of the eigen values at each stationary point, represented as $\sum \mathrm{Re}(M)$ and $\sum \mathrm{Re}(H)$, for different values of the worker share to search effort elasticity ratio are reported in Table 1.

Table 1: Stationary Points and Eigen Value Sums

a	$\frac{\beta}{1-\eta}$	$(p,n)=M$	$\sum \mathrm{Re}(M)$	$(p,n)=H$	$\sum \mathrm{Re}(H)$
0.12	0.724	(0.7176,0.2259)	0	(0.8611,0.6916)	-0.0519
0.12	0.760	(0.7184,0.2680)	0.0012	(0.8572,0.6816)	-0.0414
0.13	0.452	(0.6701,0.1543)	0	(0.9377,0.8401)	-0.2779
0.13	0.440	(0.6700,0.1542)	-0.0002	(0.9396,0.8428)	-0.2902

Hopf bifurcation is a local form of structural instability that occurs at any point in the parameter space for which the eigen values are purely imaginary at a specific stationary point. Since the eigen values at M form a complex pair, Hopf bifurcation occurs at M

for a ratio of the workers' share to the search elasticity with respect to search effort that equates the sum of the two real parts to zero. As indicated in Table 1, this condition holds when either the share to match elasticity ratio equals 0.724 given $a = 0.12$ or equals 0.452 given $a = 0.13$..

According to the Hopf bifurcation theorem (See Guckenheimer and Holmes 1986, pp.150-153.), closed orbits exist revolving around M in a one sided neighborhood of the bifurcation value of the ratio. However, any limit cycle that exist in this case can be either stable or unstable and the criterion for determining which is difficult to evaluate. For this reason the test associated with saddle-loop bifurcation can be useful. Although it is admittedly more difficult to verify the condition for a saddle loop bifurcation, the associated closed orbit is stable in case I and unstable in case II by Proposition 5. Because case I is associated with small values of the production function scale parameter, these results suggest that one should find a stable limit cycle when a is relatively small and a unstable cycle when a is larger for appropriate values of the ratio of the workers' share to the matching elasticity with respect to search input.

In Figure 7, plots of two computed trajectories associated with the parameters specified in the second row of Table 1. At these parameter values, $a = 0.12$ and $\beta/(1-\eta) = 0.76,$ the stationary point M is a source, i.e., $\sum \mathrm{Re}(M) > 0$ as reported in Table 1. For the same parameter values, the negative eigen value at the saddle point H is larger is absolute value than the positive root, equivalently the $\sum \mathrm{Re}(H) < 0$. These conditions and Proposition 5 suggest that a stable limit cycle surrounds $M.$ As the inner trajectory represented in Figure 7 is diverging away from M while the outer one is converging toward $M,$ the figure confirms the conjecture. Namely, a stable limit cycle necessarily exists to which both trajectories are converging.

Similarly, the plots of two computed trajectories associated with the parameter values given in the fourth row of Table 1 are illustrated in Figure 8. At the parameter values assumed, $a = 0.13$ and $\beta/(1-\eta) = 0.44,$ M is a sink and the positive eigen value at the origin exceeds the negative. These conditions and Proposition 5 suggest the existence of an unstable limit cycle surrounding $M,$ a conjecture verified by the fact that the inner trajectory is converging toward M while the outer trajectory is diverging. Given this pattern, an unstable closed orbit surrounding M lies between the two trajectories portrayed..

10 The Economics of the Cycle

I conclude this admittedly technical study with a more heuristic discussion of the time series properties of the employment cycle identified in the paper. Obviously, the source of the cumulative process driving fluctuations is the positive external economy that generates increasing returns to production. However, the existence of cyclical movements in employment also requires a propagation mechanism, supplied in this case by the delays implicit in the matching process and the incentive structure that determines investment in matching inputs.

An important reason for interest in this simple model is the fact that the cyclic movements in unemployment and vacancies implied by this propagation mechanism are consistent with important stylized facts. In addition, the predicted timing of the turning points in important aggregate measures of economic activity are also consistent with those observed.

The expectation of rising employment in the future stimulates investment in match formation now as a consequence of increasing returns. Given sufficiently optimistic (pessimistic) expectations, this cumulative process can result in smooth monotone convergence to the high (low) employment steady state. However, if agents are a little less (more) optimistic, the only perfect foresight employment time series turns down (up) before the high (low) employment steady state is reached. In the downswing, the cumulative process works in reverse; the expectation of falling employment in the near future induces less investment in matching now which ultimately confirms the expectation. Eventually the process reverses itself again to complete the cycle, even though search intensity and vacancies are low, simply because there are so many unemployed.

This cycle can be characterized in two different but theoretically equivalent ways. The first is an output/asset market description based on the fact that the stock price of an operating firm in the model is proportional to match capital. As aggregate gross income, represented by $\int_0^n f(x)dx + g(p)(1-n),$ is increasing in both p and n, gross income is maximum (minimum) at some point on the closed orbit representing a cycle between the points at which p and n respectively attain their maximum (minimum) values. Hence, as in the real world, the model implies the value of firm-match, its stock price value $p,$ lead income and income leads employment at both the top and the bottom of the cycle.

The second characterization is a description of the cycle in labor market terms. As vacancies increase with p, the model provides an explanation for counter-clockwise movement in vacancies and unemployment about the Beveridge Curve, the average downward sloping relationship observed between vacancies and unemployment. During the upswing, the ratio of vacancies to unemployment increases and search effort rises relative to its average so that the realized vacancy-unemployment pair move in a counter-clockwise direction above the Beveridge curve. Eventually, this process reverses as vacancies and search effort fall in response to falling expectations about future productivity. In the downturn, the labor market moves in the direction of higher unemployment relative to vacancies along a path below the Beveridge curve.

Although convergence to a endogenous deterministic limit cycle is a generic theoretical possibility in the model, I do not argue that observed business fluctuations are necessarily generated by such dynamic behavior. The reason I find the model of interest arises from the fact that there are other equilibria associated with intermediate employment levels that share the cycle's qualitative time series properties even when their paths eventually converge to the saddle point H. This fact suggests that these same properties would also be evident in a real business cycle version of the model in which the primary source of disturbance in economic activity is an exogenous technology shock process. In sum, increasing returns in production and the lags implicit in the matching process propagate productivity shocks in the manner consistent with the stylized business cycle facts summarized above.

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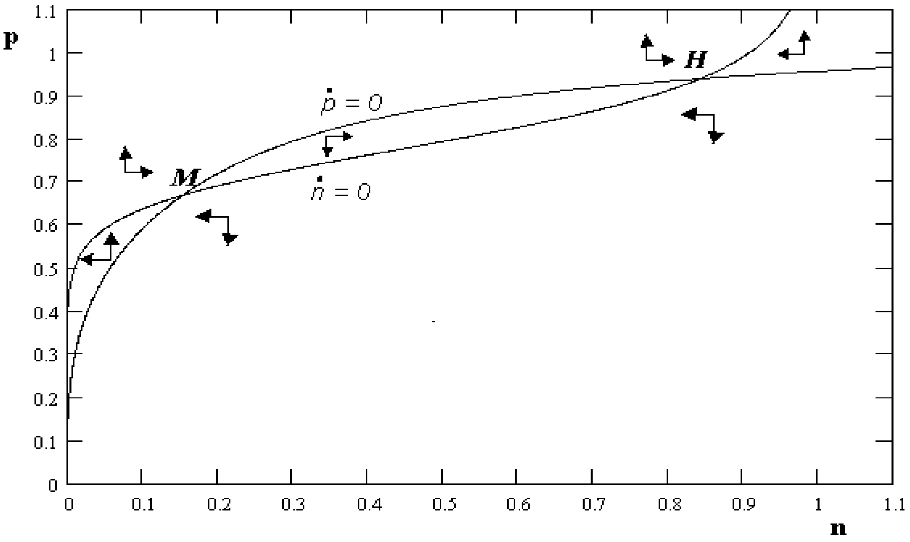

Figure 1: The Phase Portrait

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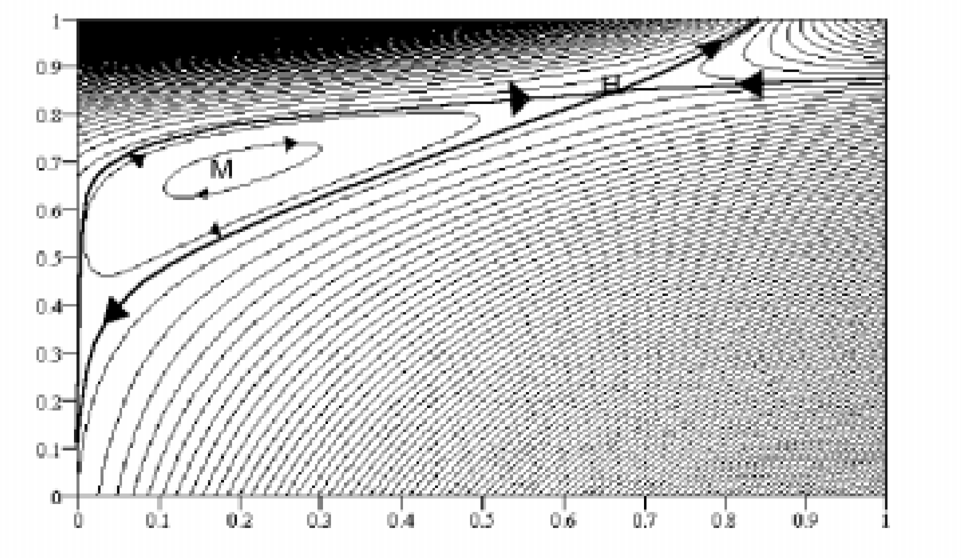

Figure 2: Hamiltonian Dynamics: Boundary Case

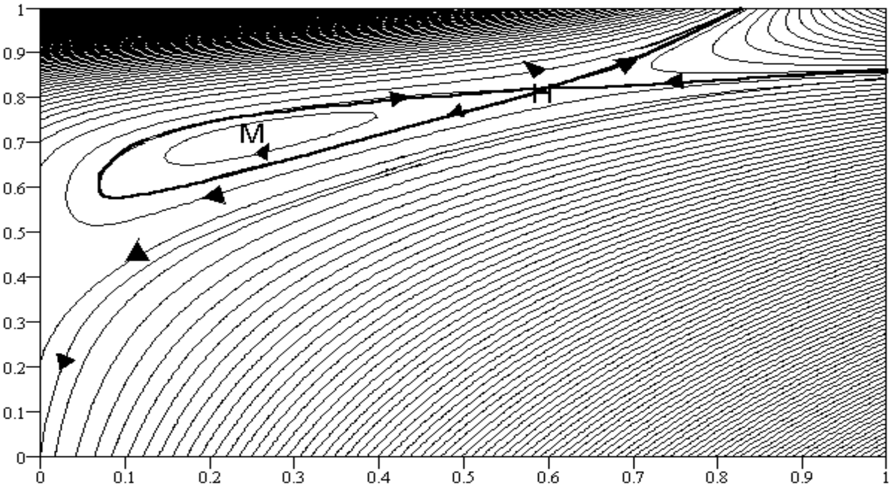

Figure 3: Hamiltonian Dynamic: Case I

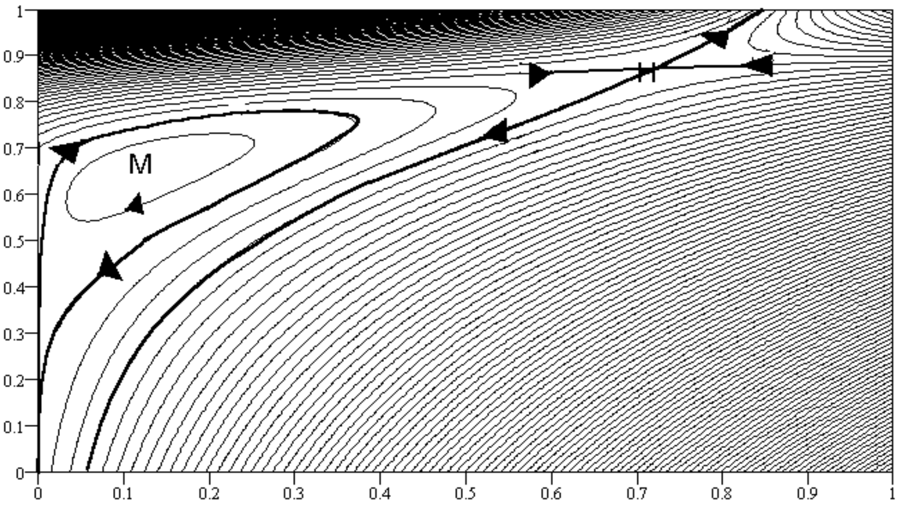

Figure 4: Hamiltonian Dynamic: Case II

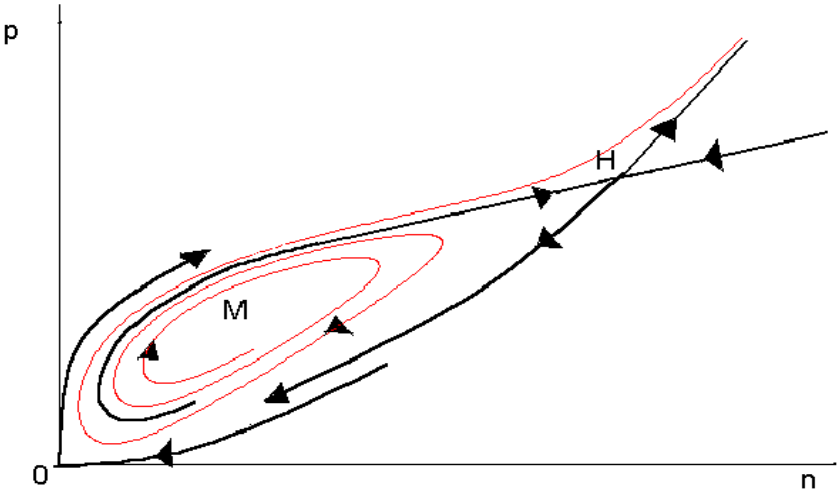

Figure 5: No Cycle: Case I

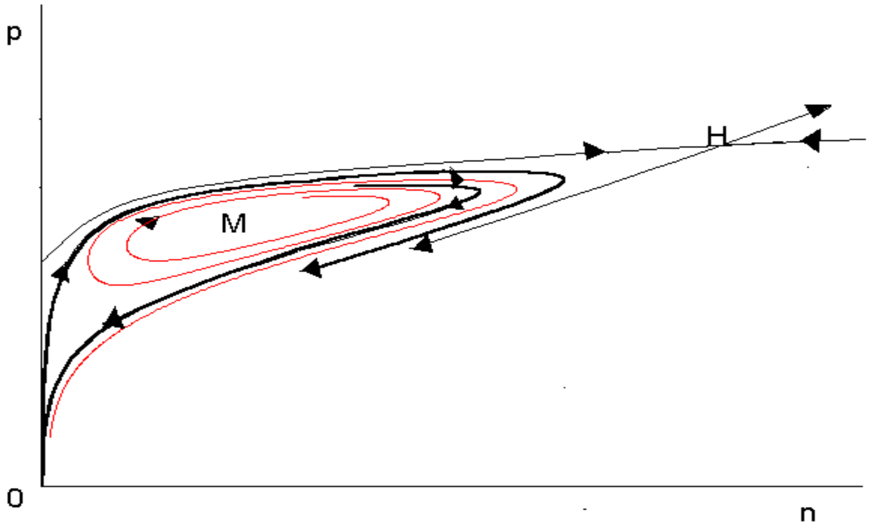

Figure 6: No Cycle: Case II

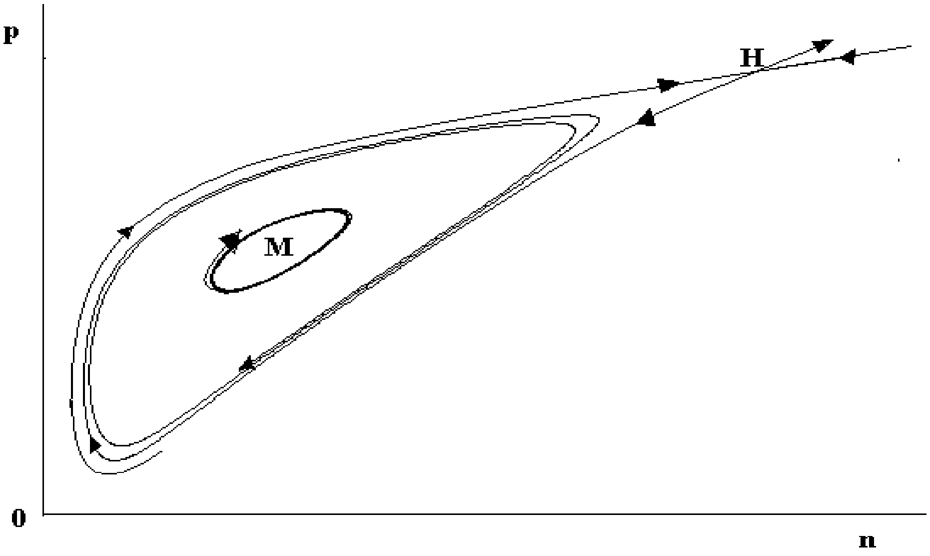

Figure 7: Stable Limit Cycle: Case I ($a = 0.12$, $\beta/(1-\eta) = 0.76$)

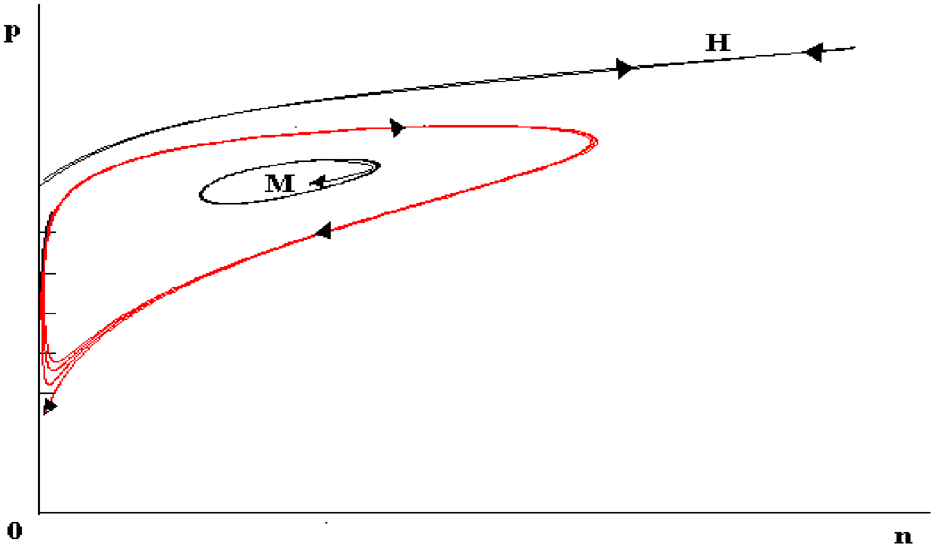

Figure 8: Unstable Cycle: Case II ($a = 0.13$, $\beta/(1-\eta) = 0.44$)