Testing Continuity of a Density via $g$-order statistics in the Regression Discontinuity Design

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Abstract

In the regression discontinuity design (RDD), it is common practice to assess the credibility of the design by testing the continuity of the density of the running variable at the cut-off, e.g., McCrary (2008). In this paper we propose a new test for continuity of a density at a point based on the so-called $g$-order statistics, and study its properties under a novel asymptotic framework. The asymptotic framework is intended to approximate a small sample phenomenon: even though the total number $n$ of observations may be large, the number of effective observations local to the cut-off is often small. Thus, while traditional asymptotic analysis in RDD requires a growing number of observations local to the cut-off as $n \to \infty$, our framework allows for the number $q$ of observations local to the cut-off to be fixed as $n \to \infty$. The new test is easy to implement, asymptotically valid under weaker conditions than those used by competing methods, exhibits finite sample validity under stronger conditions than those needed for its asymptotic validity, and has favorable power properties against certain alternatives. In a simulation study, we find that the new test controls size remarkably well across designs. We finally apply our test to the design in Lee (2008), a well-known application of the RDD to study incumbency advantage.

KEYWORDS: Regression discontinuity design, $g$-ordered statistics, sign tests, continuity, density.

JEL classification codes: C12, C14.

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1 Introduction

The regression discontinuity design (RDD) has been extensively used in recent years to retrieve causal treatment effects - see Lee and Lemieux (2010) and Imbens and Lemieux (2008) for exhaustive surveys. The design is distinguished by its unique treatment assignment rule where individuals receive treatment when an observed covariate, known as the running variable, crosses a known cut-off. Such an assignment rule allows nonparametric identification of the average treatment effect (ATE) at the cut-off, provided that potential outcomes have continuous conditional expectations at the cut-off (Hahn et al., 2001). The credibility of this identification strategy along with the abundance of such discontinuous rules have made RDD increasingly popular in empirical applications.

While the continuity assumption that is necessary for nonparametric identification of the ATE at the cut-off is fundamentally untestable, researchers routinely assess the plausibility of their RDD by exploiting two testable implications of a stronger identification assumption proposed by Lee (2008). We can describe the two implications as follows: (i) individuals have imprecise control over the running variable, which translates into the density of the running variable being continuous at the cut-off; and (ii) the treatment is locally randomized at the cut-off, which translates into the distribution of all observed baseline covariates being continuous at the cut-off. The practice of judging the reliability of RDD applications by assessing either of the two above stated implications (commonly referred to as manipulation, or falsification, or placebo tests) is ubiquitous in the empirical literature. Indeed, Table 4 surveys RDD empirical papers in four leading applied economic journals during the period 2011-2015. Out of 62 papers, 43 of them include some form of manipulation, falsification, or placebo test.

This paper proposes a novel test for the null hypothesis on the first testable implication, i.e., the density of the running variable is continuous at the cut-off. The new test has a number of distinctive attractive properties relative to existing methods. First, the test does not require consistent nonparametric estimators of densities and simply exploits the fact that a certain functional of order statistics of the data is approximately binomially distributed under the null hypothesis. Second, our test controls the limiting null rejection probability under fairly mild conditions that, in particular, do not require existence of derivatives of the density of the running variable. In addition, our test is valid in finite samples under stronger, yet plausible, conditions. Third, the asymptotic validity of our test holds under two alternative asymptotic frameworks; one in which the number \( q \) of observations local to the cut-off is fixed as the sample size \( n \) diverges to infinity, and another one where \( q \) is allowed to grow as \( n \) diverges to infinity. Importantly, both frameworks require the same mild assumptions. Fourth, our test is arguably simple to implement as it only involves computing order statistics, a constant critical value, and a single tuning parameter. This contrasts

\footnote{It is important to emphasize that the null hypothesis we test in this paper is neither necessary nor sufficient for identification of the ATE at the cut-off. See Section 2 for a discussion on this.}
with existing alternatives that require local polynomial estimation of some order and either bias
correction or under-smoothed bandwidth choices. Finally, we have developed a companion Stata
package to facilitate the adoption of our test.\footnote{The Stata package rdcont can be downloaded from \url{http://sites.northwestern.edu/iac879/software/}.}

The construction of our test is based on the simple intuition that, when the density of the
running variable is continuous at the cut-off, the fraction of units under treatment and control
local to the cut-off should be roughly the same. This means that the number of treated units,
out of the \( q \) observations closest to the cut-off, is approximately distributed as a binomial random
variable with sample size \( q \) and probability \( \frac{1}{2} \). In order to formalize this intuition, we exploit and
develop properties of the so-called \( g \)-order statistics (see, e.g., Kaufmann and Reiss, 1992; Reiss,
1989) and employ two asymptotic frameworks that aim at capturing the small sample nature of the
problem. In the first framework, the number of observations \( q \) local to the cut-off is fixed as \( n \to \infty \).
This framework is similar to the one in Canay and Kamat (2018), who in turn exploit results from
Canay, Romano and Shaikh (2017), and is the one that we prefer for the testing problem under
consideration. However, it is worth noting that the hypotheses we test, the test statistic, the critical
value, and most of the formal arguments are different from those in Canay and Kamat (2018) or
Canay, Romano and Shaikh (2017). In the second framework, we let \( q \) to slowly diverge to infinity
with \( n \), in the sense that we require \( \frac{q^{3/2}}{n} \to 0 \). The asymptotic framework where \( q \to \infty \) as \( n \to \infty \)
is similar to the one in McCrary (2008); Otsu et al. (2013); Cattaneo et al. (2017a), among others,
and is in line with more traditional asymptotic arguments in non-parametric tests.

From a technical standpoint, this paper has several contributions relative to the existing liter-
ature. To start, our results exhibit two important differences relative to Canay and Kamat (2018)
that go beyond the difference in the null hypotheses. First, we do not study our test as an approx-
imate randomization test but rather as an approximate two-sided sign-test. This not only requires
different analytical tools, but also by-passes some of the challenges that would arise if we were to
characterize our test as an approximate randomization test; see Remark 4.2 for a discussion on
this. In addition, our approach in turn facilitates the analysis for the second asymptotic framework
where \( q \to \infty \). Second, we develop results on \( g \)-order statistics as important intermediate steps
towards our main results. Some of them may be of independent interest; e.g., Theorem 4.1. In
addition, relative to the results in McCrary (2008); Otsu et al. (2013); Cattaneo et al. (2017a); our
test does not involve consistent estimators of density functions to either side of the cut-off and does
not require conditions involving existence of derivatives of the density of the running variable local
to the cut-off. Finally, we note that related binomial tests have been recently presented in the RDD
context by Cattaneo et al. (2016), Cattaneo et al. (2017b), and Frandsen et al. (2017); see Remark
3.3 for a detailed description. While these papers rely on finite sample arguments to justify their
test construction for the hypothesis of local randomization, here we provide a rigorous asymptotic
analysis under the two aforementioned asymptotic frameworks for the hypothesis of continuity of
a density. To the best of our knowledge, the formal asymptotic results we present are original to this paper.

The remainder of the paper is organized as follows. Section 2 introduces the notation and describes the null hypothesis of interest. Section 3 defines \( q \)-order statistics, formally describes the test we propose, and discusses all aspects related to its implementation including a data dependent way of choosing \( q \). Section 4 presents the main formal results of the paper, dividing those results according to the two alternative asymptotic frameworks we employ. In Section 5, we examine the relevance of our asymptotic analysis for finite samples via a simulation study. Finally, Section 6 implements our test to reevaluate the validity of the design in Lee (2008) and Section 7 concludes. The proofs of all results can be found in the Appendix.

2 Setup and notation

Let \( Y \in \mathbb{R} \) denote the (observed) outcome of interest for an individual or unit in the population and \( A \in \{0, 1\} \) denote an indicator for whether the unit is treated or not. Further denote by \( Y(1) \) the potential outcome of the unit if treated and by \( Y(0) \) the potential outcome if not treated. As usual, the (observed) outcome and potential outcomes are related to treatment assignment by the relationship

\[
Y = Y(1)A + Y(0)(1 - A).
\]

The treatment assignment in the (sharp) RDD follows a discontinuous rule,

\[
A = I\{Z \geq \bar{z}\},
\]

where \( Z \in \mathbb{Z} \equiv \text{supp}(Z) \) is an observed scalar random variable known as the running variable and \( \bar{z} \) is the known threshold or cut-off value. For convenience we normalize \( \bar{z} = 0 \), which is without loss of generality as we can always redefine \( Z \) as \( Z - \bar{z} \). This treatment assignment rule allows us to identify the average treatment effect (ATE) at the cut-off; i.e.,

\[
E[Y(1) - Y(0)|Z = 0].
\]

In particular, Hahn et al. (2001) establish that identification of the ATE at the cut-off relies on the discontinuous treatment assignment rule and the assumption that

\[
E[Y(1)|Z = z] \quad \text{and} \quad E[Y(0)|Z = z] \quad \text{are both continuous in} \ z \ \text{at} \ z = 0.
\]

Reliability of the RDD thus depends on whether the mean outcome for units marginally below the cut-off identifies the true counterfactual for those marginally above the cut-off.

The continuity assumption in (2) is arguably weak, but fundamentally untestable. In practice, researchers routinely employ two specification checks in RDD that, in turn, are implications of a
stronger sufficient condition proposed by Lee (2008, Condition 2b). The first check involves testing whether the distribution of pre-determined characteristics (conditional on the running variable) is continuous at the cut-off. See Shen and Zhang (2016) and Canay and Kamat (2018) for a recent treatment of this problem. The second check involves testing the continuity of the density of the running variable at the cut-off, an idea proposed by McCrary (2008). This second check is particularly attractive in settings where pre-determined characteristics are not available or where these characteristics are likely to be unrelated to the outcome of interest. Formally, we can state this hypothesis testing problem as

\[ H_0 : f_Z^+(0) = f_Z^-(0) \text{ vs. } H_1 : f_Z^+(0) \neq f_Z^-(0) , \] (3)

where

\[ f_Z^+(0) \equiv \lim_{\epsilon \to 0} \frac{1}{\epsilon} P\{Z \in [0, \epsilon)\} \text{ and } f_Z^-(0) \equiv \lim_{\epsilon \to 0} \frac{1}{\epsilon} P\{Z \in (-\epsilon, 0)\} , \] (4)

provided these limits exist. In RDD empirical studies, the aforementioned specification checks are often implemented (with different levels of formality) and referred to as falsification, manipulation, or placebo tests (see Table 4 for a survey).

In this paper we propose a test for the null hypothesis of continuity in the density of the running variable \( Z \) at the cut-off \( \bar{z} = 0 \), i.e., (3). The new test has three attractive features compared to existing approaches (see, e.g., McCrary, 2008; Otsu et al., 2013; Cattaneo et al., 2017a). First, it does not require commonly imposed smoothness conditions on the density of \( Z \), as it does not involve non-parametric estimation of such a density. Second, it exhibits finite sample validity under certain (stronger) easy to interpret conditions. Finally, our test involves a single tuning parameter (a feature shared by the approach proposed by Cattaneo et al., 2017a) as opposed to multiple ones in McCrary (2008). We discuss these features further in Section 4.

Remark 2.1. Gerard et al. (2016) study the consequences of discontinuities in the density of \( Z \) at the cut-off. In particular, the authors consider a situation in which manipulation occurs only for a subset of participants and use the magnitude of the discontinuity of \( f(z) \) at \( z = 0 \) to identify the proportion of always-assigned units among all units close to the cut-off. Using this setup, Gerard et al. (2016) show that treatment effects in RDD are not point identified but that the model still implies informative bounds.

Remark 2.2. It is important to emphasize that a running variable with a continuous density is neither necessary nor sufficient for identification of the average treatment effect at the cut-off. For a discussion of this and some intuitive examples, see Lee (2008) and McCrary (2008).

3 A test based on \( g \)-ordered statistics

Let \( P \) be the distribution of \( Z \) and \( Z^{(n)} = \{Z_i : 1 \leq i \leq n\} \) be a random sample of \( n \) i.i.d. observations from \( P \). Let \( q \) be a small (relative to \( n \)) positive integer and \( g : Z \to R \) be a
measurable function such that \( g(Z) \) has a continuous distribution function. For any \( z, z' \in Z \) define \( \leq_g \) as

\[
  z \leq_g z' \quad \text{if} \quad g(z) \leq g(z') .
\]

The ordering defined by \( \leq_g \) is called a \( g \)-ordering on \( Z \). The \( g \)-order statistics \( Z_{g,(i)} \) corresponding to \( Z^{(n)} \) have the property

\[
  Z_{g,(1)} \leq_g \cdots \leq_g Z_{g,(n)} ,
\]

see, e.g., Reiss (1989, Section 2.1) and Kaufmann and Reiss (1992).

To construct our test statistic, we use the sign of the \( q \) values of \( \{Z_i : 1 \leq i \leq n\} \) that are induced by the \( q \) smallest values of \( \{g(Z_i) = |Z_i| : 1 \leq i \leq n\} \). That is, for \( Z_{g,(1)}, \ldots, Z_{g,(q)} \), let

\[
  A_{g,(j)} \equiv I\{Z_{g,(j)} \geq 0\} \quad \text{for} \quad 1 \leq j \leq q ,
\]

and

\[
  S_n \equiv \sum_{j \leq q} A_{g,(j)} .
\]

The test statistic of our test only depends on the data via \( S_n \) and is defined as

\[
  T_q(S_n) \equiv \sqrt{q} \left| \frac{1}{q} S_n - \frac{1}{2} \right| .
\]

In order to describe the critical value of our test it is convenient to recall that the cumulative distribution function (CDF) of a binomial random variable with \( q \) trials and probability of success \( \frac{1}{2} \) is given by

\[
  \Psi_q(b) \equiv \frac{1}{2^q} \sum_{x=0}^{\lfloor b \rfloor} \binom{q}{x} I\{0 \leq b \leq q\} + I\{b > q\} ,
\]

where \( \lfloor x \rfloor \) is the largest integer not exceeding \( x \). Using this notation the critical value for a significance level \( \alpha \in (0,1) \) is given by

\[
  c_q(\alpha) \equiv \sqrt{q} \left( \frac{1}{2} - \frac{b_q(\alpha)}{q} \right) ,
\]

where \( b_q(\alpha) \) is the unique value in \( \{0, 1, \ldots, \lfloor \frac{q}{2} \rfloor\} \) satisfying

\[
  \Psi_q(b_q(\alpha) - 1) \leq \frac{\alpha}{2} < \Psi_q(b_q(\alpha)) .
\]

The test we propose is then given by

\[
  \phi(S_n) = \begin{cases} 
  1 & \text{if } T_q(S_n) > c_q(\alpha) \\
  a_q(\alpha) & \text{if } T_q(S_n) = c_q(\alpha) \\
  0 & \text{if } T_q(S_n) < c_q(\alpha) 
\end{cases}
\]
where
\[ a_q(\alpha) \equiv 2^{q-1} \left( \frac{q}{b_q(\alpha)} \right)^{-1} [\alpha - 2\Psi_q(b_q(\alpha) - 1)] . \] (12)

Intuitively, the test \( \phi(S_n) \) exploits the fact that, under the null hypothesis in (3), the distribution of the treatment assignment should be locally the same to either side of the cut-off. That is, local to the cut-off the treatment assignment behaves as purely randomized under the null hypothesis, and so the fraction of units under treatment and control should be similar.

**Remark 3.1.** The test in (11) is possibly randomized. The non-randomized version of the test that rejects when \( T_q(S_n) > c_q(\alpha) \) is also asymptotically level \( \alpha \) by Theorem 4.2. In our simulations, and for our data dependent choice of \( q \) that we describe in the next section, the randomized and non-randomized versions perform similarly. \( \blacksquare \)

**Remark 3.2.** The value of \( b_q(\alpha) \in \{0, 1, \ldots, \lfloor \frac{q}{2} \rfloor \} \) solving (10) is well-defined and unique for all \( q \geq 1 \) and \( \alpha \in (0, 1) \). To see this, let
\[ q^*(\alpha) \equiv 1 - \frac{\log \alpha}{\log 2} . \] (13)
When \( q < q^*(\alpha) \), (10) uniquely holds for \( b_q(\alpha) = 0 \). In this case, \( \phi(S_n) \) in (11) is purely randomized. When \( q \geq q^*(\alpha) \), the uniqueness of the solution is guaranteed by \( \Psi_q(\cdot) \) being strictly increasing over \( \{0, 1, \ldots, \lfloor \frac{q}{2} \rfloor \} \), \( \Psi_q(0) = \frac{1}{2q} \), and \( \Psi_q(\frac{q}{2}) \geq \frac{1}{2} \). In this case, \( \phi(S_n) \) in (11) deterministically rejects with positive probability. This shows that in order for the non-randomized version of the test to be non-trivial (see Remark 3.1), \( q \) needs to be larger than \( q^*(\alpha) \). In order to better appreciate these magnitudes, note that for \( \alpha = 5\% \) this requires \( q \geq 6 \) while for \( \alpha = 1\% \) this requires \( q \geq 8 \). Similarly, and given \( b_q(\alpha) \), the value of \( a_q(\alpha) \) in (12) is also uniquely defined and taking values in \( [0, 1) \) by the same properties of \( \Psi_q(\cdot) \). \( \blacksquare \)

Given \( q \), the implementation of our test proceeds in the following five steps.

**Step 1.** Find the \( q \) observations closest to the cut-off, i.e., \( Z_{g,(1)}, \ldots, Z_{g,(q)} \).

**Step 2.** Count the number of non-negative observations in \( Z_{g,(1)}, \ldots, Z_{g,(q)} \), i.e., \( S_n \) as in (6).

**Step 3.** Compute test statistic \( T_q(S_n) \) as in (7), \( c_q(\alpha) \) as in (9), and \( a_q(\alpha) \) as in (12).

**Step 4.** Compute the p-value of the non-randomized version of the test as
\[ p_{\text{value}} = 2 \min \{ \Psi_q(S_n), \Psi_q(q - S_n) \} . \] (14)

**Step 5.** Reject the null hypothesis in (3) using \( \phi(S_n) \) in (11). If a non-randomized test is preferred, reject the null hypothesis if \( p_{\text{value}} < \alpha \).
**Remark 3.3.** As it follows from Theorems 4.1 and 4.2 in Section 4, the test \( \phi(S_n) \) is an approximate two-sided sign test (or approximate binomial test). As we mentioned in the introduction, related binomial tests have been recently presented in the RDD context by Cattaneo et al. (2016), Cattaneo et al. (2017b), and Frandsen et al. (2017). The first two papers use a binomial test based on the number of “successes” in a window \( Z \in [-h, h] \) for a varying bandwidth \( h \). The authors propose to vary \( h \) until a “break-down” window size \( h^* \) is found, which is defined as the largest window such that the minimum \( p \)-value of the binomial test is larger than \( \alpha \) for all nested (smaller) windows. The justification provided for the validity of such a test involves a finite sample argument: under the hypothesis of “local randomization/random assignment” in \([-h^*, h^*]\), a binomial test with probability \( \pi \) is exact (Cattaneo et al., 2017b, p. 650, note that \( \pi = \frac{1}{2} \) is the most natural choice in the absence of additional information). Frandsen et al. (2017) deals with discrete running variables and tests a different null hypothesis than ours, but his construction also involves quantiles from a binomial distribution. Contrary to these papers, our goal is not to test a “local random assignment” hypothesis or to deal with discrete running variables, but rather to test the continuity hypothesis in (3). As a result of this, we cannot exploit finite sample arguments and rather need to provide a rigorous asymptotic analysis of the test we propose under an arguably weak assumption (i.e., Assumption 4.1). The formal results in Theorems 4.1, 4.2, and 4.3 are novel to this paper and, to the best of our knowledge, they provide the first formal results about approximate two-sided sign tests for the hypothesis in (3) in the RDD (or any other) framework. ■

### 3.1 Data dependent rule for \( q \)

In this section we discuss the practical considerations involved in the implementation of our test, highlighting how we addressed these considerations in the companion Stata package. The only tuning parameter of our test is the number \( q \) of observations closest to the cut-off and so we propose a data dependent way to choose \( q \) that combines a rule of thumb with a local optimization. We call this data-dependent rule the “informed rule of thumb” and its computation requires the two steps described below. For the sake of clarity, in this section we do not use the normalization \( \bar{z} = 0 \). Additional computational details are presented in Appendix C.

In Section 4.2 we show under Assumption 4.1 that the value of \( q \) that sets the worse case asymptotic bias equal to the standard deviation (in the asymptotic framework where \( q \to \infty \)) is given by

\[
q = n^{2/3} \left( \frac{4 f_Z^2(\bar{z})}{C_P} \right)^{2/3},
\]

where \( C_P \) is the Lipschitz constant in Assumption 4.1(i’). Since the results in Theorem 4.3 also require \( q^{3/2}/n \to 0 \), we propose to start with an initial rule of thumb where \( f_Z(\bar{z}) \) and \( C_P \) are...
computed under the assumption $Z \sim N(\mu, \sigma^2)$ and the rate is set to $n^{1/2}$. This leads to

$$q^{\text{rot}} = n^{1/2} \left( \frac{4 \phi_{\mu, \sigma}^2(\bar{z})}{\phi_{\mu, \sigma}(\mu + \sigma)} \right)^{2/3},$$

where we used that $C_P = \phi_{\mu, \sigma}'(\mu + \sigma) = \frac{1}{\sigma} \phi_{\mu, \sigma}(\mu + \sigma)$ when $Z \sim N(\mu, \sigma^2)$ and $\phi_{\mu, \sigma}(-)$ and $\phi_{\mu, \sigma}'(-)$ denote the density of a normally distributed random variable with mean $\mu$ and variance $\sigma^2$ and its derivative. This initial rule of thumb is location and scale invariant and, by definition, is inversely proportional to the asymptotic bias of the test in the asymptotic framework where $q \to \infty$. The constant multiplying $n^{1/2}$ in (15) is fairly intuitive too. First, it captures the idea that a steeper density at the cut-off should be associated with a smaller value of $q$. Intuitively, the steeper the density, the more it resembles a density that is discontinuous (Figure 1.(c) illustrates this in Section 5). Since the maximum slope is determined by the Lipschitz constant, the rule is inversely proportional to that. Second, it also captures the idea that $q$ should be small if the cut-off is a point of low density. Intuitively, when $f_Z(\bar{z})$ is low, the $q$ closest observations to $\bar{z}$ are likely to be “far” from $\bar{z}$ (Figure 1.(a) with $\mu = -2$ illustrates this in Section 5). One could alternatively replace the normality assumption with a non-parametric estimator of $f_Z(\bar{z})$ but it is unfortunately impossible to choose $C_P$ adaptively for testing (3) (see, e.g., Low, 1997; Armstrong and Kolesár, 2018). Since any data dependent rule for $q$ will require a reference for $C_P$, we prefer to prioritize its simplicity and use normality for both $f_Z(\bar{z})$ and $C_P$.

The second step involves a local maximization of the asymptotic null rejection probability of the non-randomized version of the test. In particular, based on Theorem 4.2 we propose

$$q_{\text{frot}} = \arg\max_{q \in \mathcal{N}(q_{\text{rot}})} \Psi_q(b_q(\alpha) - 1),$$

where $\Psi_q(\cdot)$ is the CDF defined in (8), $b_q(\alpha)$ is defined in (10), and $\mathcal{N}(q_{\text{rot}})$ is a discrete neighborhood of $q_{\text{rot}}$ that we describe in Appendix C. This second step is important for the performance of the non-randomized version of the test (see Remark 3.1) as $\Psi_q(b_q(\alpha) - 1)$ is not monotonic in $q$, see Figure 3. In practice, we replace $\mu$ and $\sigma$ with sample analogs to deliver a feasible informed rule of thumb that we denote by $\hat{q}_{\text{frot}}$.

**Remark 3.4.** The recommended choice of $q$ in (17) can be interpreted as the under-smoothed version of the rule that captures a bias-variance trade-off, where we impose normality to compute unknown constants (see discussion in Section 4.2). It also exploits the shape of the limiting null-rejection probability of the non-randomized version of the test to derive a better choice of $q$. However, even though $q_{\text{frot}}$ is motivated by a root-mean-square error (RMSE) optimal choice of $q$, it is an under-smoothed rule of thumb that is not optimal in a formal sense. ■
4 Asymptotic framework and formal results

In this section we derive the asymptotic properties of the test in (11) using two alternative asymptotic frameworks. The first one, and the most interesting from our point of view, is one where \( q \) is fixed as \( n \to \infty \). This framework is similar to that in Canay and Kamat (2018) and intends to capture a situation where only few of the observations available to the researcher contain good information about the properties “at” the cut-off. The second one is in line with more traditional arguments and requires \( q \to \infty \) and \( q^{3/2} / n \to 0 \) as \( n \to \infty \).

There are three main features of our results that are worth highlighting: (i) our test exhibits similar properties under both asymptotic frameworks, (ii) the implementation of the test does not depend on which asymptotic framework one has in mind, and (iii) all formal results require the same, arguably weak, assumption. We start by introducing this assumption.

Assumption 4.1. The distribution \( P \) satisfies

(i) \( \exists \delta > 0 \) such that \( Z \) has a continuous density on \( (-\delta, 0) \cup (0, \delta) \).

(i') \( \exists \delta > 0 \) and \( C_P \in (0, \infty) \) such that

\[
|f_Z^+(z) - f_Z^+(0)| \leq C_P|z| \text{ for } z \in (0, \delta) \text{ and } |f_Z^-(z) - f_Z^-(0)| \leq C_P|z| \text{ for } z \in (-\delta, 0).
\]

(ii) \( f_Z^-(0) + f_Z^+(0) > 0 \).

Assumptions 4.1(i) and 4.1(i') each impose different degrees of smoothness on the density of \( Z \) local to the cut-off \( \bar{z} = 0 \). Indeed, Assumption 4.1(i') strengthens Assumption 4.1(i) by replacing the requirement of continuity on \( (-\delta, 0) \cup (0, \delta) \) with that of Lipschitz continuity on the same region. In the formal results that follow, we use Assumption 4.1(i) in the asymptotic framework where \( q \) is fixed as \( n \to \infty \) and Assumption 4.1(i') in the asymptotic framework where \( q \to \infty \) as \( n \to \infty \). Both assumptions allow for the distribution of \( Z \) to be discontinuous, both at the cut-off \( \bar{z} = 0 \) and outside a neighborhood of the cut-off. As a result, they allow us to study alternative hypotheses with a mass point at the cut-off. More importantly, they do not require the density of \( Z \) to be differentiable anywhere. This is in contrast to McCrary (2008), who requires three continuous and bounded derivatives of the density of \( Z \) (everywhere except possibly at \( \bar{z} = 0 \)), and Cattaneo et al. (2017a) and Otsu et al. (2013), who require the density of \( Z \) to be twice continuously differentiable local to the cut-off (in the case of a local-quadratic approximation). Assumption 4.1(ii) rules out a situation where \( f_Z^-(0) = f_Z^+(0) = 0 \), which is implicitly assumed away in McCrary (2008) and Otsu et al. (2013) and is weaker than assuming a positive density of \( Z \) in a neighborhood of the cut-off as in Cattaneo et al. (2017a). In Section 5 we explore the sensitivity of our results to violations of these conditions.
4.1 Results for fixed $q$

In this section we present two main results. The first result, Theorem 4.1, describes the asymptotic properties of $S_n$ in (6) when $q$ is fixed as $n \to \infty$. This result about $g$-order statistics with $g(\cdot) = |\cdot|$ represents an important milestone in proving the asymptotic validity of our test. The second result, Theorem 4.2, exploits the result in Theorem 4.1 to show that the test in (11) controls the limiting rejection probability under the null hypothesis.

**Theorem 4.1.** Let Assumption 4.1(i)-(ii) hold and let $q \geq 1$ be fixed. If $P\{Z = 0\} = 0$, then

$$S_n \xrightarrow{d} S \sim \text{Bi}(q, \pi_f)$$

as $n \to \infty$, where $\text{Bi}(q, \pi_f)$ denotes the Binomial distribution with $q$ trials and probability of success

$$\pi_f \equiv \frac{f^+_Z(0)}{f^-_Z(0) + f^+_Z(0)}.$$

If $P\{Z = 0\} > 0$, then $S_n = q$ with probability one.

Theorem 4.1, although fairly intuitive, does not follow from standard arguments. The case when $P\{Z = 0\} > 0$ is relatively simple, so we focus our discussion on the case $P\{Z = 0\} = 0$. First, the random variables $\{A_{g,j} : 1 \leq j \leq q\}$ are not necessarily i.i.d. by virtue of being indicators of $g$-order statistics. In general, they are neither independent nor identically distributed. Second, applying results from the literature on $g$-order statistics (e.g., Kaufmann and Reiss, 1992, Theorem 1) requires $g(Z) = |Z|$ to have a continuous distribution function everywhere on its domain. Under Assumption 4.1(i) this is only true in $(0, \delta)$, and mass points are allowed, both at zero and outside $(0, \delta)$. In the proof of Theorem 4.1 we use a smoothing transformation of $Z$ as an intermediate step and then accommodate the results in Kaufmann and Reiss (1992, Theorem 1) to reach the desired conclusion.

The following result, which heavily relies on Theorem 4.1, is the main result of this section and characterizes the asymptotic properties of our test.

**Theorem 4.2.** Let Assumption 4.1(i)-(ii) hold and let $q \geq 1$ be fixed. Then, under $H_0$ in (3),

$$\lim_{n \to \infty} E[\phi(S_n)] = 2\Psi_q(b_q(\alpha) - 1) + \frac{a_q(\alpha)}{2^{q-1}} \left( \frac{q}{b_q(\alpha)} \right) = \alpha.$$

In addition, under $H_1$ in (3), $\lim_{n \to \infty} E[\phi(S_n)] > \alpha$.

Theorem 4.2 shows that $\phi(S_n)$ behaves asymptotically, as $n \to \infty$, as the two-sided sign-test in an experiment with $S \sim \text{Bi}(q, \pi)$ and the hypotheses $H_0 : \pi = \frac{1}{2}$ versus $H_1 : \pi \neq \frac{1}{2}$. This test is not only among the oldest significance tests in statistics (see, e.g., Arbuthnott (1710)), but it is also the uniformly most powerful test among the class of unbiased test for such hypothesis testing problem; see Lehmann and Romano (2005, Section 4.2) and Lemma B.5 in the appendix.
Remark 4.1. Theorem 4.2 shows that \( \lim_{n \to \infty} E[\phi(S_n)] > \alpha \) under \( H_1 \) in (3). It is worth noticing that \( \lim_{n \to \infty} E[\phi(S_n)] = 1 \) for any alternative hypothesis with a mass point at the cut-off, i.e., a distribution such that \( P\{Z = 0\} > 0 \). ■

Remark 4.2. The test \( \phi(S_n) \) could be alternatively characterized as an “approximate” randomization test, see Canay et al. (2017) for a general description of such tests. However, such a characterization would make the analysis of the formal properties of the test more complicated and, in particular, the results in Canay et al. (2017) would not immediately apply due to two fundamental challenges. First, Assumption 3.1(iii) in Canay et al. (2017) is immediately violated in our setting. Second, such an approach would require an asymptotic approximation to the joint distribution of \( \{A_{g,j} : 1 \leq j \leq q\} \), which in turn would require a strengthening of Lemma B.4. Our proof approach avoids both of these technicalities by directly exploiting the binary nature of \( \{A_{g,j} : 1 \leq j \leq q\} \) and by simply approximating the distribution of \( S_n \), which is a scalar, as in Theorem 4.1. ■

Remark 4.3. It is possible to show that \( \phi(S_n) \) in (11) is level \( \alpha \) in finite samples whenever the distribution of \( Z \) is continuous and symmetric at the cut-off. In this case, the fundamental result in Lemma B.4 holds for \( S_n \) with \( P\{Z > 0 \mid |Z| < r\} = \frac{1}{2} \) for any \( r > 0 \), and the proof of Theorem 4.2 can in turn be properly modified to show \( E[\phi(S_n)] = \alpha \) for all \( n \geq 1 \). ■

4.2 Results for large \( q \)

In this section we study the properties of \( \phi(S_n) \) in (11) in an asymptotic framework where \( q \) diverges to infinity as \( n \to \infty \). This asymptotic framework is in line with traditional non-parametric arguments and so our results depend on the assumed smoothness of the density of \( Z \) and the rate at which \( q \) is allowed to grow. Importantly, the results in this section follow from Assumption 4.1(i')-(ii) so, accounting for the differences between Assumptions 4.1(i) and 4.1(i'), the result below shows that the asymptotic properties of our test under small and large \( q \) require similar, and arguably mild, conditions.

Theorem 4.3. Let Assumption 4.1(i')-(ii) hold and let \( q \) be such that \( q \to \infty \) and \( \frac{q^{3/2}}{n} \to 0 \) as \( n \to \infty \). Then,

\[
\sqrt{q} \left( \frac{1}{q} S_n - \pi_f \right) \overset{d}{\to} N(0, \pi_f(1 - \pi_f)),
\]

provided \( P\{Z = 0\} = 0 \), where \( \pi_f \) is as in Theorem 4.1. Moreover, the following holds for \( \alpha \in (0, 1) \):

(a) Under \( H_0 \) in (3), \( \lim_{n \to \infty} E[\phi(S_n)] = \alpha. \)

(b) Under \( H_1 \) in (3), \( \lim_{n \to \infty} E[\phi(S_n)] = 1. \)
(c) Under a sequence of alternative distributions local to \( H_0 \) satisfying \( \sqrt{q}(\pi_f - \frac{1}{2}) \to \Delta \neq 0 \),

\[
\lim_{n \to \infty} E[\phi(S_n)] = P\{\zeta + 2\Delta \gg 2\alpha/2\} > \alpha ,
\]

where \( \zeta \sim N(0, 1) \) and \( z\alpha/2 \) is the \((1 - \frac{\alpha}{2})\)-quantile of \( \zeta \).

Theorem 4.3, although fairly intuitive again, does not follow from standard arguments. In particular, given that the random variables \( \{A_{\pi(j)} : 1 \leq j \leq q\} \) are neither independent nor identically distributed, the result does not follow from a simple application of the central limit theorem. We instead adapt Kaufmann and Reiss (1992, Theorem 1) and prove the result using first principles and the normal approximation to the binomial distribution.

Given the result in Theorem 4.3, we can provide some insight on the properties of the data dependent rule for choosing \( q \) that we described in Section 3.1. Under \( H_0 \) and Assumption 4.1(i')-(ii), the results in Armstrong and Kolesár (2019) imply that

\[
\frac{\sqrt{q}(1 - \pi_f)}{\sqrt{\pi_f(1 - \pi_f)}} = \zeta_n + B_{n,q} + o_p(1) ,
\]

where \( \zeta_n \overset{d}{\to} \zeta \sim N(0, 1) \) and \( B_{n,q} \) is a bias term satisfying

\[
|B_{n,q}| \leq \frac{q^{3/2}}{n} \frac{C_P}{4f_Z(0)^2} . \quad (18)
\]

Denote by \( t^* \) the right-hand side of (18) and note that this can be interpreted as the ratio of bias to standard deviation (sd). We can then solve for \( q \) to obtain

\[
q^* = n^{2/3}(t^*)^{2/3} \left( \frac{4f_Z(0)^2}{C_P} \right)^{2/3} . \quad (19)
\]

This derivation shows that the requirement \( q^{3/2}/n \to 0 \) is analogous to under-smoothing as this is the rate condition that removes the worst-case asymptotic bias.\(^3\) This immediately gives two alternative interpretations to the data dependent rule \( q^{irot} \) in (17) (see Armstrong and Kolesár, 2019, Section 4.2, for a more detailed description). First, leaving the normal approximation aside, \( q^{irot} \) is equivalent to \( q^* \) with a bias-sd ratio \( t^* \) given by

\[
t^* = \left( \frac{n^{1/2}}{n^{2/3}} \right)^{3/2} \Rightarrow t^* = 0.12 \text{ for } n = 5,000 .
\]

This suggests that if one accounts for this bias, the size of \( \phi(S_n) \) for \( \alpha = 5\% \) would approximately be \( P\{|\zeta + 0.12| > z\alpha/2\} = 5.16\% \), and so \( q^{irot} \) makes the bias negligible when \( n = 5,000 \). Second, suppose that the Lipschitz constant provided by the normal approximation in \( q^{irot} \) is too small for

\(^3\)A previous version of this paper did not include Assumption 4.1(i') and the requirement \( q^{3/2}/n \to 0 \), which is required to control the asymptotic bias term. We thank Tim Armstrong for pointing this out to us.
the true constant $C_P$ in Assumption 4.1(i'). To be concrete, suppose that $C_P/\phi'_{\mu,\sigma}(\mu + \sigma) = a > 1$. It follows that accounting for the approximation error in the Lipschitz constant, $q^{irot}$ is equivalent to $q^*$ with a bias-sd ratio $t^*$ given by

$$t^* = \left(\frac{a}{n^{2/3}}\right)^{3/2} \Rightarrow t^* = 0.36 \text{ for } n = 5,000 \text{ and } a = 3.$$

This suggests that the size of $\phi(S_n)$ for $\alpha = 5\%$ would approximately be $P(|\zeta + 0.36| > z_{\alpha/2}) = 6.38\%$ when $n = 5,000$. In other words, even if the true constant $C_P$ is three times larger than the approximation used by $q^{irot}$, $\phi(S_n)$ would still exhibit mild over-rejection under the null hypothesis. The price we pay for this robustness under the null hypothesis (in terms of performance and mild requirements) is power under the alternative hypothesis, a feature that we explore in the simulations of Section 5.

**Remark 4.4.** It may be tempting to use the first part of Theorem 4.3 to consider a variation of the test we propose; namely the test that rejects $H_0$ when $T_q(S_n) > \frac{1}{2}z_{\alpha/2}$ and $z_{\alpha/2}$ is the $(1 - \frac{a}{2})$-quantile of a standard normal random variable. However, we do not recommend this variation as it provides no theoretical advantages over $\phi(S_n)$ in the asymptotic framework where $q \to \infty$, and it is not formally justified in the asymptotic framework where $q$ is fixed (in particular, such a variation will not inherit the properties discussed in Remark 4.3).

**Remark 4.5.** As pointed out by a referee, in the asymptotic framework where $q \to \infty$, the test statistic $T_q(S_n)$ can be shown to be proportional to a Wald-type statistic

$$W_n = |\hat{f}_Z(h_n) - \hat{f}_Z(-h_n)|,$$

where $\hat{f}_Z(z)$ is a non-parametric kernel density estimator of $f_Z$ (implemented with a uniform on $[-1, 1]$ kernel and bandwidth $h_n$). Under some conditions it will follow that $W_n$ is asymptotically normal and a test for $H_0$ could be constructed by using the quantile of a normal distribution (possibly by additionally estimating the asymptotic variance). One could go a step further and use the bound on the bias term $B_{n,q}$ to construct a test that explicitly accounts for the asymptotic bias of the test following the approach proposed by Armstrong and Kolesár (2019). Since our goal is to describe the properties of $\phi(S_n)$ as $q \to \infty$ and the equivalence between these approaches and our test do not hold in the framework where $q$ is fixed, we do not pursue this strategy here.

## 5 Simulations

In this section, we examine the finite-sample performance of the test we propose in this paper with a simulation study. Instead of just presenting designs where our proposed test excels relative to competing ones, we present an array of data generating processes that hopefully illustrate the
relative strengths and weaknesses of the test we propose. The data for the study is simulated as an
i.i.d. sample from the following designs, where below Beta\((a,b)\) denotes the Beta distribution with
parameters \((a,b)\).

**Design 1:** For \(\mu \in \{-2, -1, 0\}\), \(Z \sim N(\mu, 1)\).

**Design 2:** For \(\lambda \in \{\frac{1}{3}, 1\}\),

\[
Z \sim \begin{cases} 
V_1 \text{ with prob. } \lambda \\
V_2 \text{ with prob. } (1 - \lambda)
\end{cases},
\]
where \(V_1 \sim 2\text{Beta}(2, 4) - 1\) and \(V_2 \sim 1 - 2\text{Beta}(2, 8)\).

**Design 3:** For \((\lambda_1, \lambda_2, \lambda_3) = (0.4, 0.1, 0.5)\),

\[
Z \sim \begin{cases} 
V_1 \text{ with prob. } \lambda_1 \\
V_2 \text{ with prob. } \lambda_2 \\
V_3 \text{ with prob. } \lambda_3
\end{cases},
\]
where \(V_1 \sim N(-1, 1)\), \(V_2 \sim N(-0.2, 0.2)\), and \(V_3 \sim N(3, 2.5)\).

**Design 4:** For \(\kappa \in \{0.05, 0.10, 0.25\}\), the density of \(Z\) is given by

\[
f_Z(z) = \begin{cases} 
0.75 & \text{if } z \in [-1, -\kappa] \\
0.75 - \frac{1}{2\kappa}(z + \kappa) & \text{if } z \in [-\kappa, \kappa] \\
0.25 & \text{if } z \in [\kappa, 1]
\end{cases}.
\]

**Design 5:** For \(\kappa \in \{0.05, 0.10, 0.25\}\), the density of \(Z\) is given by

\[
f_Z(z) = \begin{cases} 
0.25 & \text{if } z \in [-1, -\kappa] \\
0.50 & \text{if } z \in [-\kappa, \kappa] \\
0.75 & \text{if } z \in [\kappa, 1]
\end{cases}.
\]

**Design 6:** We first non-parametrically estimate the density of the running variable in Lee
(2008, see Section 6 for details) and then take i.i.d. draws from such a density.

Design 1 in Figure 1(a) is the canonical normal case and, by Remark 4.3, our test is expected
to control size in finite samples when \(\mu = 0\) but not when \(\mu \in \{-2, -1\}\). Indeed, \(\mu = -2\) is
a challenging case due to the low probability of getting observations to the right of the cut-off.
Design 2 in Figure 1(b) is taken from Canay and Kamat (2018). Design 3 in Figure 1(c) is a
parametrization of the taxable income density in Saez (2010, Figure 8). This design exhibits a
spike (almost a kink) to the left of the cut-off which is essentially a violation of the smoothness assumptions required by McCrary (2008) and Cattaneo et al. (2017a). It also exhibits a steep density at the cut-off, which also makes it a difficult case in general. Similar to Design 3, Design 4 in Figure 1(b) also illustrates the difficulty in distinguishing a discontinuity from a very steep slope; see Kamat (2017) for a formal discussion. Here we can study the sensitivity to the slope by changing the value of $\kappa$. Design 5 in Figure 1(e) requires $\delta$ in Assumption 4.1(a) to be such that $\delta < \kappa$ in order for our approximations to be accurate, but as opposed to Design 4, it is locally symmetric around the cut-off. As $\kappa$ gets smaller, we expect our test to perform worse if $q$ is not chosen carefully. Finally, Design 6 in Figure 1(f) draws data i.i.d. from the non-parametric density estimator of the running variable in Lee (2008), i.e., $Z$ is the difference in vote shares between Democrats and Republicans.

We consider sample sizes $n \in \{1,000; 5,000\}$, a nominal level of $\alpha = 10\%$, and perform 10,000 Monte Carlo repetitions. Designs 1 to 6 satisfy the null hypothesis in (3). We additionally consider the same models under the alternative hypothesis by randomly changing the sign of observations in the interval $z \in [0, 0.1]$ with probability $Pr = 0.2 - 2z$. We report results for the following tests.

**AS-NR** and **AS-R**: the approximate sign-test we propose in this paper in its two versions. The randomized version (AS-R) in (11) and the non-randomized version (AS-NR) that rejects when $p_{value}$ in (14) is below $\alpha$, see Remark 3.1. We include the randomized version in order
Table 1: Rejection probabilities (in %) under $H_0$ and $H_1$ across Designs 1-6 and for $n = 1,000$. 

<table>
<thead>
<tr>
<th>Design</th>
<th>$H_0$ AS-NR $q$</th>
<th>$H_0$ AS-R $q$</th>
<th>$H_0$ McC $q$</th>
<th>$H_0$ CJM $q$</th>
<th>$H_1$ AS-NR $q$</th>
<th>$H_1$ AS-R $q$</th>
<th>$H_1$ McC $q$</th>
<th>$H_1$ CJM $q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>D1: $\mu = 0$</td>
<td>4.1 6.4 6.0 9.4 9.6 9.0 8.1</td>
<td>5.5 9.9 9.6 13.6 13.8 11.2 9.0</td>
<td>11.9 84.4 99.8 8.2</td>
<td>14.0 86.1 99.8 9.4</td>
<td>12.2 12.8 18.3 15.8</td>
<td>16.0 15.4 19.0</td>
<td>10.8 10.7 7.3</td>
<td>12.5 10.7 7.2</td>
</tr>
<tr>
<td>D1: $\mu = 1$</td>
<td>4.2 8.0 12.5 10.7 10.8 12.2 8.7</td>
<td>5.9 12.8 18.3 15.8</td>
<td>11.4 10.8 7.3</td>
<td>14.0 86.1 99.8 9.4</td>
<td>12.5 10.7 7.2</td>
<td>16.0 15.4 19.0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>D1: $\mu = 2$</td>
<td>4.2 8.0 12.5 10.7 10.8 12.2 8.7</td>
<td>5.9 12.8 18.3 15.8</td>
<td>11.4 10.8 7.3</td>
<td>14.0 86.1 99.8 9.4</td>
<td>12.5 10.7 7.2</td>
<td>16.0 15.4 19.0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>D2: $\lambda = \frac{1}{3}$</td>
<td>4.3 6.8 7.5 10.2 10.3 10.9 9.5</td>
<td>6.6 15.8 22.4 18.3</td>
<td>10.6 10.7 8.0</td>
<td>18.4 17.1 10.3</td>
<td>16.0 15.4 19.0</td>
<td>18.4 17.1 10.3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>D2: $\lambda = 1$</td>
<td>4.3 6.8 7.5 10.2 10.3 10.9 9.5</td>
<td>6.6 15.8 22.4 18.3</td>
<td>10.6 10.7 8.0</td>
<td>18.4 17.1 10.3</td>
<td>16.0 15.4 19.0</td>
<td>18.4 17.1 10.3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>D3</td>
<td>4.8 18.5 40.1 25.2 25.5 99.9 85.8</td>
<td>7.7 27.7 51.1 35.6</td>
<td>25.5 99.9 85.8</td>
<td>36.0 100.0 87.7</td>
<td>19.7 19.2 11.9</td>
<td>19.7 19.2 11.9</td>
<td></td>
<td></td>
</tr>
<tr>
<td>D4: $\kappa = 0.25$</td>
<td>4.1 8.6 12.5 11.4 11.6 12.4 9.5</td>
<td>6.7 17.0 25.2 19.6</td>
<td>11.6 65.1 93.0 51.2</td>
<td>51.3 92.2 70.2</td>
<td>19.7 19.2 11.9</td>
<td>19.7 19.2 11.9</td>
<td></td>
<td></td>
</tr>
<tr>
<td>D4: $\kappa = 0.1$</td>
<td>4.6 17.3 42.7 16.6 16.8 47.9 24.4</td>
<td>8.2 32.0 60.9 28.7</td>
<td>11.6 65.1 93.0 51.2</td>
<td>51.3 92.2 70.2</td>
<td>19.7 19.2 11.9</td>
<td>19.7 19.2 11.9</td>
<td></td>
<td></td>
</tr>
<tr>
<td>D4: $\kappa = 0.05$</td>
<td>6.6 49.2 86.3 36.1 36.2 85.2 60.9</td>
<td>11.6 65.1 93.0 51.2</td>
<td>51.3 92.2 70.2</td>
<td>19.7 19.2 11.9</td>
<td>19.7 19.2 11.9</td>
<td>19.7 19.2 11.9</td>
<td></td>
<td></td>
</tr>
<tr>
<td>D5: $\kappa = 0.25$</td>
<td>4.1 6.4 6.4 9.7 9.9 20.8 12.8</td>
<td>5.6 10.0 9.8 13.9</td>
<td>14.0 13.7 20.2</td>
<td>13.9 15.3 21.1</td>
<td>13.9 15.3 21.1</td>
<td>13.9 15.3 21.1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>D5: $\kappa = 0.1$</td>
<td>4.0 6.3 6.5 9.8 9.8 18.0 33.0</td>
<td>5.8 9.8 10.1 13.7</td>
<td>13.9 15.3 21.1</td>
<td>13.9 15.3 21.1</td>
<td>13.9 15.3 21.1</td>
<td>13.9 15.3 21.1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>D5: $\kappa = 0.05$</td>
<td>4.1 7.6 35.0 10.3 10.4 40.6 25.7</td>
<td>6.2 13.7 53.7 15.2</td>
<td>15.3 58.7 32.3</td>
<td>15.3 58.7 32.3</td>
<td>15.3 58.7 32.3</td>
<td>15.3 58.7 32.3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>D6</td>
<td>4.0 6.3 6.4 9.7 9.9 9.0 10.9</td>
<td>5.8 10.5 10.8 14.7</td>
<td>14.9 15.1 11.4</td>
<td>14.9 15.1 11.4</td>
<td>14.9 15.1 11.4</td>
<td>14.9 15.1 11.4</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 1: Rejection probabilities (in %) under $H_0$ and $H_1$ across Designs 1-6 and for $n = 1,000$. 

to illustrate the differences between the randomized and non-randomized versions of the test. 
The tuning parameter $q$ is set to 

$$q \in \{20, 50, 75, \hat{q_{irot}}\},$$

where $\hat{q_{irot}}$ is the feasible informed rule of thumb described in Section 3 and Appendix C.

McC: the test proposed by McCrary (2008). We implement this test using the function DCdensity from the R package rdd, with the default choices for the bandwidth parameter and kernel type.

CJM: the test proposed by Cattaneo et al. (2017a). We implement this test using the rddensity function from the R package rddensity. We use jackknifed standard errors and bias correction, as these are the default choices in the paper.

Tables 1 and 2 report rejection probabilities under the null and alternative hypotheses for the six designs we consider and for sample sizes of $n = 1,000$ and $n = 5,000$, respectively. We start by discussing the results under the null hypothesis. AS-NR delivers rejection probabilities under the null hypothesis closer to the nominal level than those delivered by McC and CJM in most of the designs. The two empirically motivated designs (Designs 3 and 6) illustrate this feature clearly. Designs 4 and 5 also show big differences in performance, both in cases where AS-NR delivers rejection rates equal to the nominal level (Design 5) and McC and CJM severely over-reject; as well as in cases where all tests over-reject (Design 4, $\kappa = 0.05$) but AS-NR is relatively closer to the nominal level. A particularly difficult case for AS-NR is Design 1 with $\mu = -2$, where the
Table 2: Rejection probabilities (in %) under $H_0$ and $H_1$ across Designs 1-6 and for $n = 5,000$.

<table>
<thead>
<tr>
<th>Design</th>
<th>$\mu = 0$</th>
<th>$\mu = 1$</th>
<th>$\mu = 2$</th>
<th>$\lambda = 1/3$</th>
<th>$\lambda = 1$</th>
<th>$\lambda = 1$</th>
<th>All $\kappa$</th>
<th>All $\kappa$</th>
</tr>
</thead>
<tbody>
<tr>
<td>D1:</td>
<td>4.2</td>
<td>6.7</td>
<td>6.4</td>
<td>10.3</td>
<td>10.4</td>
<td>9.2</td>
<td>9.3</td>
<td>6.0</td>
</tr>
<tr>
<td>D2:</td>
<td>4.0</td>
<td>7.0</td>
<td>7.1</td>
<td>10.6</td>
<td>10.8</td>
<td>11.0</td>
<td>10.2</td>
<td>5.8</td>
</tr>
<tr>
<td>D3:</td>
<td>4.3</td>
<td>6.1</td>
<td>6.4</td>
<td>9.3</td>
<td>9.5</td>
<td>11.0</td>
<td>8.5</td>
<td>6.3</td>
</tr>
<tr>
<td>D4:</td>
<td>4.3</td>
<td>6.6</td>
<td>6.5</td>
<td>11.1</td>
<td>11.2</td>
<td>12.3</td>
<td>10.0</td>
<td>6.4</td>
</tr>
<tr>
<td>D5:</td>
<td>4.3</td>
<td>6.9</td>
<td>6.8</td>
<td>10.2</td>
<td>10.4</td>
<td>50.6</td>
<td>44.4</td>
<td>6.4</td>
</tr>
<tr>
<td>D6:</td>
<td>4.3</td>
<td>6.6</td>
<td>6.7</td>
<td>9.8</td>
<td>9.9</td>
<td>45.6</td>
<td>30.6</td>
<td>6.2</td>
</tr>
</tbody>
</table>

Table 3: Mean values of $\hat{q}_{\text{irot}}$ in Designs 1-6 and for $n = 1,000$ and $n = 5,000$.

<table>
<thead>
<tr>
<th>D1</th>
<th>D1</th>
<th>D1</th>
<th>D2</th>
<th>D2</th>
<th>D3</th>
<th>D4</th>
<th>D5</th>
<th>D6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu = 0$</td>
<td>$\mu = 1$</td>
<td>$\mu = 2$</td>
<td>$\lambda = 1/3$</td>
<td>$\lambda = 1$</td>
<td>All $\kappa$</td>
<td>All $\kappa$</td>
<td>53.0</td>
<td>37.0</td>
</tr>
<tr>
<td>$n = 1000$</td>
<td>53.0</td>
<td>37.0</td>
<td>8.5</td>
<td>37.0</td>
<td>37.0</td>
<td>51.7</td>
<td>40.4</td>
<td>39.8</td>
</tr>
<tr>
<td>$n = 5000$</td>
<td>147.0</td>
<td>54.1</td>
<td>18.0</td>
<td>119.0</td>
<td>62.0</td>
<td>119.0</td>
<td>119.0</td>
<td>119.0</td>
</tr>
</tbody>
</table>

probability of getting observations to the right of the cut-off is below 2%. This showcases the satisfactory performance of our data dependent rule $\hat{q}_{\text{irot}}$, which takes the lowest value in that particular design. Tables 1 and 2 also show negligible differences between the randomized (AS-R) and non-randomized (AS-NR) versions of our test, consistent with our discussion in Remark 3.1.

To describe the performance of the different tests under the alternative hypothesis, we focus on designs where the rejection under the null hypothesis is close to the nominal level for all tests. In those cases, we see that AS-NR has competitive power, although it is rarely the test with the highest rejection under the alternative hypothesis. For $n = 1,000$, AS-NR indeed delivers the highest rejection probability under the alternative hypothesis in Design 1 for all values of $\mu$, Design 2 with $\lambda = 1$, and Design 4 with $\kappa = 0.25$. In the rest of the cases McC exhibits the highest power, sometimes followed by AS-NR and sometimes followed by CJM. For $n = 5,000$ McC performs relatively better, but only in Design 2 with $\lambda = 1$ is CJM leading to higher rejection rates than AS-NR. Overall, the best design in terms of power for AS-NR is Design 1, while the worst one is Design 5 with $\kappa = 0.05$.

Table 3 shows the mean values of $\hat{q}_{\text{irot}}$ across simulations for all designs and sample sizes. As
described in Section 3, $\hat{q}_{\text{rot}}$ takes into account both the slope and the magnitude of the density at the cut-off. As a result, $\hat{q}_{\text{rot}}$ is relatively high in designs with flat density at the cut-off and high $f_Z(0)$ (e.g., Design 1 with $\mu = 0$) and relatively low in designs with steep slopes or low $f_Z(0)$ (e.g., Design 1 with $\mu = -2$ or Design 2 with $\lambda = 1$). To gain further insight on the sensitivity of our test to the choice of $q$, Figure 2 displays the rejection probabilities of AS-NR and AS-R as a function on $q$ in two types of designs. In the top row we illustrate two designs where the rejection probability is mostly insensitive to the choice of $q$ (Design 1 with $\mu = 0$ and Design 6). These are designs where the density is rather flat around the cut-off and so increasing $q$ does not deteriorate the performance of our test. In the bottom row we illustrate two designs where the rejection probability is highly sensitive to the choice of $q$ (Design 1 with $\mu = -1$ and Design 3). These are designs that feature a steep density at the cut-off (also low in Design 1), and so increasing $q$ very quickly deteriorates the performance of the test under the null hypothesis. The data dependent rule $\hat{q}_{\text{rot}}$ is displayed in each case with a vertical dashed line and seems to be doing a good job at choosing relatively smaller values in the sensitive cases.

We conclude this section by highlighting how one could compare the results in Tables 1 and 2...
for a fix value of $q$ to appreciate the results in Section 4.1. For example, taking $q = 75$, the rejection probability in Design 1 with $\mu = -2$ and Design 3 are 99.8 and 40.1, respectively, when $n = 1,000$. The same numbers when $n = 5,000$ are 26 and 8.4, respectively, which are closer to the nominal level as predicted by our results.

6 Empirical Illustration

In this section we briefly reevaluate the validity of the design in Lee (2008). Lee studies the benefits of incumbency on electoral outcomes using a discontinuity constructed with the insight that the party with the majority wins. Specifically, the running variable $Z$ is the difference in vote shares between Democrats and Republicans at time $t$; see Figure 1(f) for a graphical illustration of the density of $Z$. The assignment rule then takes a cutoff value of zero that determines the treatment of incumbency to the Democratic candidate, which is used to study their election outcomes in time $t+1$. The total number of observations is 6,559 with 2,740 below the cutoff. The dataset is publicly available at http://economics.mit.edu/faculty/angrist/data1/mhe.

Lee assessed the credibility of the design in this application by inspecting discontinuities in means of the baseline covariates, but mentions in footnote 19 the possibility of using the test proposed by McCrary (2008). Here, we frame the validity of the design in terms of the hypothesis in (3) and use the newly developed test as described in Section 3, using $\hat{q}_{\text{rot}}$ as our default choice for the number of observations $q$. The new test delivers a $p$-value of 0.55 for $S_n = 73$ out of $\hat{q}_{\text{rot}} = 138$ observations. The null hypothesis of continuity is therefore not rejected.

7 Concluding remarks

This paper presents a new tests for testing the continuity of a density at a point in RDD. The test can be interpreted as an approximate two-sided sign test and is based on the so-called $g$-order statistics. We study its properties under a novel asymptotic framework where the number $q$ of observations employed by the test are allowed to be fixed as the sample size $n \to \infty$. Our new test is easy to implement, asymptotically valid under weaker conditions than those used by competing methods, exhibits finite sample validity under stronger conditions than those needed for its asymptotic validity, and delivers competitive power properties in simulations.

A final aspect we would like to highlight of our test is its simplicity. The test only requires to count the number of non-negative observations out of the $q$ observations closest to the cut-off (note that this is all we need to compute the $p$-value in (14)), and does not involve kernels, local polynomials, bias correction, or bandwidth choices. Importantly, we have developed the rdcont Stata package that allow for effortless implementation of the test we propose in this paper.
A Proof of the main results

A.1 Proof of Theorem 4.1

Consider the case $P\{Z = 0\} = 0$ first. In this case, by Assumption 4.1(i) the distribution of $Z$ is continuous in $(-\delta, \delta)$ for some $\delta > 0$. Throughout the proof we repeatedly use $\{Z_i^*: 1 \leq i \leq n\}$ as defined in Lemma B.1, which in turn allow us to apply Kaufmann and Reiss (1992, Theorem 1) later in the proof, when invoking Lemma B.4.

Let $Z^*_{g,(1)}, \ldots, Z^*_{g,(q)}$ denote the $q$ values of $\{Z_i^*: 1 \leq i \leq n\}$ that are induced by the $q$ smallest values of $\{g(Z_i^*) = |Z_i^*|: 1 \leq i \leq n\}$ and let

$$A^*_{g,(j)} \equiv I\{Z^*_{g,(j)} \geq 0\} \text{ for } 1 \leq j \leq q$$

and

$$S^*_n \equiv \sum_{j=1}^q A^*_{g,(j)} . \quad (A-20)$$

Next consider $S_n$ in (6) and note that $S_n$ takes values in $\mathbb{N}_q = \{0, 1, \ldots, q\}$. By the Portmanteau’s theorem, see e.g. van der Vaart and Wellner (1996, Theorem 1.3.4(iii)), it follows that if $S_n \overset{d}{\to} S$ for some random variable $S$, then

$$1 = \liminf_{n \to \infty} P\{S_n \in \mathbb{N}_q\} \leq P\{S \in \mathbb{N}_q\} ,$$

since $\mathbb{N}_q$ is a closed subset of $\mathbb{R}$. Thus, it must be that $S_n$ and $S$ take values in $\mathbb{N}_q$, and so by Durrett (2010, Exercise 3.2.11), convergence in distribution is equivalent to convergence of the probability mass function (pmf) for all $s \in \mathbb{N}_q$. To establish this result, let $p_q(s|\pi)$ be the pmf of a Binomial random variable with $q$ trials and probability of success $\pi \in [0, 1]$, i.e.

$$p_q(s|\pi) = \binom{q}{s} \pi^s (1-\pi)^{q-s} . \quad (A-21)$$

It suffices to show that for any $\eta > 0$, there exists $N$ such that $\forall n \geq N$,

$$|P\{S_n = s\} - p_q(s|\pi_f)| \leq \eta ,$$

with

$$\pi_f = \frac{f^+_Z(0)}{f^+_Z(0) + f^-_Z(0)} . \quad (A-22)$$

To this end, first note that $p_q(s|\pi)$ is continuous in $\pi$ and so there exists $\mu > 0$ such that

$$\sup_{|\pi - \pi_f| \leq \mu} |p_q(s|\pi) - p_q(s|\pi_f)| \leq \frac{\eta}{2} . \quad (A-23)$$

For such $\mu$, we can find $\varepsilon \in (0, \frac{\delta}{2})$ such that

$$\sup_{r \leq \varepsilon} \left|P\{Z^* \geq 0 \mid |Z^*| < r\} - \pi_f\right| = \sup_{r \leq \varepsilon} \left|P\{Z \geq 0 \mid |Z| < r\} - \pi_f\right| \leq \mu , \quad (A-24)$$

where the first equality holds by Lemma B.1(b) and the second equality holds by Lemma B.2. The rest of the argument will make repeated reference to $\varepsilon$ determined by (A-24).
Next consider the following decomposition for \( s \in N_q \),
\[
P\{S_n = s\} = R_{n,1} + R_{n,2} + R_{n,3},
\]
with
\[
R_{n,1} = P\{S_n = s\} - P\{S^*_n = s\},
\]
\[
R_{n,2} = \int_0^\infty P\{S^*_n = s \mid |Z^*_{g,(q+1)}| = r\} dP\{|Z^*_{g,(q+1)}| = r\},
\]
\[
R_{n,3} = \int_0^\infty P\{S^*_n = s \mid |Z^*_{g,(q+1)}| = r\} dP\{|Z^*_{g,(q+1)}| = r\}.
\]
First, Lemma B.3(b) implies that \( R_{n,1} = o(1) \). Second, note that
\[
0 \leq R_{n,2} \leq \int_0^\infty dP\{|Z^*_{g,(q+1)}| = r\} = P\{|Z^*_{g,(q+1)}| \geq \epsilon\} = o(1),
\]
where the last equality follows from Lemma B.3(a). Finally, note that for \( \pi(r) = P\{Z \geq 0 \mid |Z| < r\} \),
\[
R_{n,3} = \int_0^\epsilon p_q(s|\pi(r)) dP\{|Z^*_{g,(q+1)}| = r\}
\]
\[
\geq P\{|Z^*_{g,(q+1)}| \leq \epsilon\} \inf_{r \leq \epsilon} p_q(s|\pi(r))
\]
\[
\geq P\{|Z^*_{g,(q+1)}| \leq \epsilon\} \inf_{|\pi-\pi_f| \leq \mu} p_q(s|\pi)
\]
\[
\geq P\{|Z^*_{g,(q+1)}| \leq \epsilon\} p_q(s|\pi_f) - \frac{\eta}{2},
\]
where the first line follows from Lemma B.4, the third line follows from (A-24), and the fourth line follows from (A-23). By the analogous arguments,
\[
R_{n,3} \leq P\{|Z^*_{g,(q+1)}| \leq \epsilon\} p_q(s|\pi_f) + \frac{\eta}{2} \leq p_q(s|\pi_f) + \frac{\eta}{2}.
\]
(27)
Combining (A-26) and (A-27) we obtain
\[
|R_{n,3} - p_q(s|\pi_f)| \leq \frac{\eta}{2} + (1 - P\{|Z^*_{g,(q+1)}| \leq \epsilon\}) = \frac{\eta}{2} + o(1),
\]
where the equality follows from Lemma B.3(a). We conclude that \( |P\{S_n = s\} - p_q(s|\pi_f)| \leq \frac{\eta}{2} + o(1) \) for all \( s \in N_q \) and this completes the proof for the case where \( P\{Z = 0\} = 0 \).

Now consider the case where \( P\{Z = 0\} > 0 \). Take the following decomposition,
\[
P\{S_n \neq q\} = \hat{R}_{n,1} + \hat{R}_{n,2} + \hat{R}_{n,3},
\]
with
\[
\hat{R}_{n,1} = P\{S_n \neq q\} - P\{S^*_n \neq q\},
\]
\[
\hat{R}_{n,2} = P\{S^*_n \neq q\} \cap \{|Z^*_{g,q+1}| \neq 0\},
\]
\[
\hat{R}_{n,3} = P\{S^*_n \neq q\} \cap \{|Z^*_{g,q+1}| = 0\}.
\]
First, Lemma B.3(b) implies that \( \hat{R}_{n,1} = o(1) \). Second, \( \hat{R}_{n,2} \leq P\{|Z^*_{g,q+1}| \neq 0\} = o(1) \), where the last equality follows from Lemma B.3(c). Finally, \( \{|Z^*_{g,q+1}| = 0\} \) implies that \( \{S^*_n = q\} \) by definition, and so \( \hat{R}_{n,3} = 0 \). We conclude that \( P\{S_n = q\} \to 1 \) as \( n \to \infty \) and this completes the proof. ■
A.2 Proof of Theorem 4.2

By the definition of $\phi(S_n)$ in (11) and the expressions of $T(S_n)$ in (7) and $c_q(\alpha)$ in (9),

$$E[\phi(S_n)] = P\{S_n < b_q(\alpha)\} + P\{S_n > q - b_q(\alpha)\} + a_q(\alpha)(P\{S_n = b_q(\alpha)\} + P\{S_n = q - b_q(\alpha)\}) .$$

Theorem 4.1 shows that $P\{S_n = s\} = P\{S = s\} + o(1)$ for all $s \in \mathbb{N}_q \equiv \{0, 1, \ldots, q\}$, where $S \sim \text{Bi}(q, \pi_f)$ and $\pi_f$ is as in (A-22). It follows from this result and the above display that $E[\phi(S_n)] \to E[\phi(S)]$ as $n \to \infty$, where

$$E[\phi(S)] = P\{S < b_q(\alpha)\} + P\{S > q - b_q(\alpha)\} + a_q(\alpha)(P\{S = b_q(\alpha)\} + P\{S = q - b_q(\alpha)\}) . \quad (A-28)$$

We complete the proof by analyzing each term on the right hand side of (A-28) under $H_0$ and $H_1$ in (3).

Under $H_0$ in (3), $\pi_f = \frac{1}{2}$. In this case, it follows immediately that

$$P\{S < b_q(\alpha)\} + P\{S > q - b_q(\alpha)\} = 2\Psi_q(b_q(\alpha) - 1) ,$$

where we used that $b_q(\alpha) \in \{0, 1, \ldots, \frac{q}{2}\}$ and $P\{S < b\} = P\{S > q - b\}$ for any $b \in \{0, \ldots, \frac{q}{2}\}$ when $\pi_f = \frac{1}{2}$. In addition,

$$a_q(\alpha)(P\{S = b_q(\alpha)\} + P\{S = q - b_q(\alpha)\}) = 2a_q(\alpha)\frac{1}{2^q} \left( \frac{q}{b_q(\alpha)} \right) ,$$

where we used that $\left( \frac{q}{C} \right) = \left( \frac{q}{q - C} \right)$ for any $C \in \{0, \ldots, q\}$. We conclude that, under $H_0$ in (3),

$$\lim_{n \to \infty} E[\phi(S_n)] = 2\Psi_q(b_q(\alpha) - 1) + \frac{a_q(\alpha)}{2^{q-1}} \left( \frac{q}{b_q(\alpha)} \right) = \alpha , \quad (A-29)$$

where the last equality follows by definition of $a_q(\alpha)$.

To claim that $E[\phi(S)] > \alpha$ under $H_1$ in (3), let $S \sim \text{Bi}(q, \pi)$ and consider testing $H_0 : \pi = \frac{1}{2}$ against a simple alternative $H_1 : \pi = \pi^* \neq \frac{1}{2}$. By Lehmann and Romano (2005, Theorem 3.2.1), the most powerful test rejects for large values of the likelihood ratio

$$\frac{\pi^*(1 - \pi^* - S)}{1 - \pi^*} .$$

That is, if $\pi^* > \frac{1}{2}$ the most powerful test rejects for small values of $S$, while if $\pi^* < \frac{1}{2}$ the most powerful test rejects for large values of $S$. Since $E[\phi(S)] = \alpha$ by (A-29), it follows that $\phi(S)$ is most powerful for this simple testing problem. The result then follows from Lehmann and Romano (2005, Corollary 3.2.1) as the alternative $\pi^* \neq \frac{1}{2}$ was arbitrary. ■

A.3 Proof of Theorem 4.3

For $S_n$ as in (6) and $S_n^*$ as in (A-20), let

$$\xi_q(\pi) \equiv \sqrt{q} \left( \frac{1}{q} S_n - \pi \right) \quad \text{and} \quad \xi_q^*(\pi) \equiv \sqrt{q} \left( \frac{1}{q} S_n^* - \pi \right) ,$$

For any $\pi \in (0, 1)$ and for $S \sim \text{Bi}(q, \pi)$, let $\Psi_q(x|\pi)$ denote the cdf of $S$ and let

$$J_q(x|\pi) \equiv P \left\{ \sqrt{q} \left( \frac{1}{q} S - \pi \right) \leq x \right\} .$$
It suffices to show that for any $\eta > 0$, there exists $N$ such that $\forall n \geq N$,

$$
\left| P\{\xi_q(\pi_f) \leq x\} - \Phi\left(\frac{x}{\sqrt{q_f(1-\pi_f)}}\right) \right| \leq \eta.
$$

Let $\{\epsilon_q : q \geq 1\}$ be a sequence in $(0, \delta/2)$ that satisfies $\sqrt{q}\epsilon_q \to 0$ and $\epsilon_q \frac{n}{q} \to \infty$. Since $\frac{q^{3/2}}{n} \to 0$, these conditions occur for all $q$ sufficiently large if we set $\epsilon_q = \frac{1}{q^{1/2}}\left(\log\left(\frac{n}{q^{1/2}}\right)\right)^{-1}$. Consider the following decomposition for $x \in \mathbb{R}$,

$$
P\{\xi_q(\pi_f) \leq x\} = \bar{R}_{n,1} + \bar{R}_{n,2} + \bar{R}_{n,3},
$$

(A-30)

with

$$
\begin{align*}
\bar{R}_{n,1} &\equiv P\{\xi_q(\pi_f) \leq x\} - P\{\xi_q^*(\pi_f) \leq x\} \\
\bar{R}_{n,2} &\equiv \int_{\epsilon_q}^{\infty} P\{\xi_q^*(\pi_f) \leq x \mid |Z_{g,(q+1)}| = r\} dP\{|Z_{g,(q+1)}^*| = r\} \\
\bar{R}_{n,3} &\equiv \int_{0}^{\epsilon_q} P\{\xi_q^*(\pi_f) \leq x \mid |Z_{g,(q+1)}| = r\} dP\{|Z_{g,(q+1)}^*| = r\}.
\end{align*}
$$

First, Lemma B.3(b) implies that $\bar{R}_{n,1} = o(1)$. Second, $\bar{R}_{n,2} = o(1)$ follows from

$$
0 \leq \bar{R}_{n,2} \leq \int_{\epsilon_q}^{\infty} dP\{|Z_{g,(q+1)}^*| = r\} = P\left\{\frac{n}{q}|Z_{g,(q+1)}^*| \geq \epsilon_q \frac{n}{q}\right\} = o(1),
$$

where the last equality follows from Lemma B.6 and $\epsilon_q \frac{n}{q} \to \infty$. Finally, let $\pi_q^+ = \pi_f + \frac{1}{2} \frac{\epsilon_q C_P}{f_Z(0) + f_{\pi_f}(0)}$ and consider the following derivation,

$$
\begin{align*}
\bar{R}_{n,3} &\geq P\{|Z_{g,(q+1)}^*| \leq \epsilon_q\} \inf_{r \leq \epsilon_q} P\left\{S_n^* \leq \sqrt{q_x} + q\pi_f \mid |Z_{g,(q+1)}^*| = r\right\} \\
&= P\{|Z_{g,(q+1)}^*| \leq \epsilon_q\} \inf_{r \leq \epsilon_q} \Psi_q \left(\sqrt{q_x} + q\pi_f \mid \pi(r)\right) \\
&\geq P\{|Z_{g,(q+1)}^*| \leq \epsilon_q\} \Psi_q \left(\sqrt{q_x} + q\pi_f \mid \pi_q^+\right) \\
&= P\{|Z_{g,(q+1)}^*| \leq \epsilon_q\} J_q \left(x - \sqrt{q}(\pi_q^+ - \pi_f) \mid \pi_q^+\right) \\
&= P\{|Z_{g,(q+1)}^*| \leq \epsilon_q\} J_q \left(x - 1 - \frac{\sqrt{q}\epsilon_q C_P}{f_Z(0) + f_{\pi_f}(0)} \mid \pi_q^+\right), \quad (A-31)
\end{align*}
$$

where the first equality uses $\pi(r) = P\{Z > 0 \mid |Z| < r\}$ and follows from Lemma B.4, the second inequality follows from $\Psi_q(x|\pi)$ being decreasing in $\pi$ and $\pi(r) \leq \pi_q^+$ for $r \leq \epsilon_q$ from Lemma B.7, and the last equality follows from the definition of $\pi_q^+$. By an analogous argument,

$$
\begin{align*}
\bar{R}_{n,3} &\leq P\{|Z_{g,(q+1)}^*| \leq \epsilon_q\} \sup_{r \leq \epsilon_q} P\left\{S_n^* \leq \sqrt{q_x} + q\pi_f \mid |Z_{g,(q+1)}^*| = r\right\} \\
&\leq P\{|Z_{g,(q+1)}^*| \leq \epsilon_q\} \Psi_q \left(\sqrt{q_x} + q\pi_f \mid \pi_q^-\right) \\
&\leq P\{|Z_{g,(q+1)}^*| \leq \epsilon_q\} J_q \left(x + 1 - \frac{\sqrt{q}\epsilon_q C_P}{f_Z(0) + f_{\pi_f}(0)} \mid \pi_q^-\right), \quad (A-32)
\end{align*}
$$

where in this case we define $\pi_q^- = \pi_f - \frac{\epsilon_q C_P}{f_Z(0) + f_{\pi_f}(0)}$. To complete the proof, it suffices to show that the right-hand side expressions of (A-31) and (A-32) converge to $\Phi\left(\frac{x}{\sqrt{\pi_f(1-\pi_f)}}\right)$. We only show the result for (A-31), as the result for (A-32) is analogous.
It follows by the Berry-Esseen theorem that
\[ J_q \left( x - \frac{1}{2} \frac{\sqrt{\epsilon_q} C_P}{f_Z(0) + f_Z(0)} \right) \to \pi^+_q = 0. \]
where the convergence follows from \( q \to \infty \) and \( \pi^+_q \to \pi_f \in (0, 1) \). Since \( \sqrt{\epsilon_q} \to 0 \), the continuity of the standard normal cdf implies that
\[ \sqrt{\pi^+_q (1 - \pi^+_q)} \to 0. \]
Finally, Lemma B.6 and \( \epsilon_q \frac{n}{q} \to \infty \) imply that
\[ P\{|Z^*_{g(n+1)}| \leq \epsilon_q\} = P\left\{ \frac{n}{q}|Z^*_{g(n+1)}| \leq \frac{n}{q} \epsilon_q \right\} \to 1. \]
By combining (A-33), (A-34), and (A-35), the desired result follows.

\[ \Box \]

### B Auxiliary Results

**Lemma B.1.** Let \( \delta > 0 \) be as in Assumption 4.1(i)-(ii) and \( \{v_i : 1 \leq i \leq n\} \) be an i.i.d. sample such that \( v_i \sim U(-\frac{\delta}{2}, \frac{\delta}{2}) \) independent of \( Z^{(n)} \). Define the sequence of i.i.d. random variables \( \{Z^*_i : 1 \leq i \leq n\} \) as
\[ Z^*_i = Z_i + v_i I\{|Z_i| \geq \delta\}. \]
The following statements follow from Assumption 4.1(i)-(ii) whenever \( P\{Z = 0\} = 0 \).

(a) The distribution function of \( |Z^*| \) is continuous on \( \mathbb{R} \).

(b) For any \( r \in (0, \frac{\delta}{2}) \),
\[ P\{Z^* \geq 0 \ | |Z^*| < r\} = P\{Z \geq 0 \ | |Z| < r\}. \]  \( \text{(B-36)} \)

(c) For any \( r > 0 \), \( P\{|Z^*| < r\} > 0 \).

**Proof.** Note that the distribution function of \( Z \) is continuous in \(( -\delta, \delta )\) by Assumption 4.1(i) and \( P\{Z = 0\} = 0 \). Then, to prove part (a), let \( E \subset \mathbb{R} \) be a set of zero Lebesgue measure and note that
\[ P\{|Z^*| \in E\} = P\{|Z + vI\{|Z| \geq \delta\}| \in E\} = P\{|Z + vI\{|Z| \geq \delta\}| \in E \cap \{|Z| \geq \delta\}\} + P\{|Z + vI\{|Z| \geq \delta\}| \in E \cap \{|Z| < \delta\}\} = P\{|Z + vI\{|Z| \geq \delta\}| \in E \cap \{|Z| \geq \delta\}\} + P\{|Z| \in E \cap \{|Z| < \delta\}\} \leq P\{|Z + vI\{|Z| \geq \delta\}| \in E \cap \{|Z| \geq \delta\}\} + P\{|Z| \in E \cap \{|Z| < \delta\}\} = 0 \]
where the last equality holds because the distribution function of \( |Z + vI| \) is continuous and \( E \cap (0, \delta) \) is a subset of zero Lebesgue measure in the region where the distribution function of \( |Z| \) is continuous.

For part (b) note that for any \( r \in (0, \frac{\delta}{2}) \), \( |Z^*| < r \) implies that \( Z = Z^* \) and (B-36) follows.
For part (c) use again that \( P(|Z^*| < r) = P(|Z| < r) \) whenever \( r \in (0, \frac{2}{g}) \). In addition, Assumption 4.1(i) implies that the distribution of \( Z \) is continuous in \((-\delta, \delta)\) when \( P(Z = 0) = 0 \). It follows that for any \( 0 < \epsilon < \delta, \)
\[
\frac{1}{\epsilon} P(|Z| < \epsilon) = \frac{1}{\epsilon} P(Z \in [0, \epsilon)) + \frac{1}{\epsilon} P(Z \in (-\epsilon, 0)).
\]
Taking limits as \( \epsilon \downarrow 0 \), using the definitions of \( f^+_Z(0) \) and \( f^-_Z(0) \) in (4), and invoking Assumption 4.1(ii) shows that \( \lim_{\epsilon \downarrow 0} \frac{1}{\epsilon} P(|Z| < \epsilon) = f^+_Z(0) + f^-_Z(0) > 0 \). Thus, there exists \( \epsilon < \delta \) such that \( P(|Z| < \epsilon) > 0 \) for all \( \epsilon \in (0, \delta) \) and so \( P(|Z| < r) > 0 \) for all \( r \in \mathbb{R} \). This completes the proof. ■

**Lemma B.2.** Let Assumption 4.1(i)-(ii) hold, \( \pi_f \) be defined as in (A-22), and assume \( P(Z = 0) = 0 \). Then, for all \( \mu > 0 \), there exists \( \epsilon > 0 \) s.t.
\[
\sup_{\epsilon \leq r} \left| P\{Z \geq 0 \mid |Z| < r\} - \pi_f \right| \leq \mu.
\]

**Proof.** First note that, under Assumption 4.1(ii), the proof of Lemma B.4 shows that \( P(|Z| < r) > 0 \) for all \( r \in \mathbb{R} \). It follows that
\[
P\{Z \geq 0 \mid |Z| < \epsilon\} = \frac{P\{Z \in [0, \epsilon)\}}{P\{Z \in (-\epsilon, \epsilon)\}}
= \frac{\frac{1}{\epsilon} P\{Z \in [0, \epsilon)\}}{\frac{1}{\epsilon} P\{Z \in (-\epsilon, 0)\}}
= \frac{f^+_Z(0)}{f^+_Z(0) + f^-_Z(0)} + \delta_\epsilon,
\]
where \( \delta_\epsilon \to 0 \) as \( \epsilon \to 0 \) and in the last equality we used the definitions of \( f^+_Z(0) \) and \( f^-_Z(0) \) in (4) and Assumption 4.1(ii) again. The result then follows by definition of \( \pi_f \). ■

**Lemma B.3.** Let Assumption 4.1(i)-(ii) hold and \( \frac{2}{n} \to 0 \) as \( n \to \infty \). Then,
\begin{enumerate}
\item[(a)] For any \( \epsilon \in (0, \frac{\delta}{2}) \), \( P\{\limsup_{n \to \infty}\{|Z^*_g(q+1)| \leq \epsilon\} \} = P\{\liminf_{n \to \infty}\{|Z^*_g(q)\} \leq \epsilon\} = 1. \)
\item[(b)] \( P\{\limsup_{n \to \infty}\{|S_n - S^*_n\| \leq \epsilon\} = 1 \), where \( S_n \) is as in (6) and \( S^*_n \) is as in (A-20). \)
\item[(c)] If \( P\{Z = 0\} > 0 \), (a) also holds with \( \epsilon = 0 \). \)
\end{enumerate}

**Proof.** Fix \( \epsilon \in (0, \frac{\delta}{2}) \) arbitrarily and set \( N_n \equiv \sum_{i=1}^n I\{|Z_i| \leq \epsilon\} \). Note that \( N_n \geq q + 1 \) implies that \( Z^*_i = Z_i \) and \( Z^*_g(j) = Z_g(j) \) for at least \( q + 1 \) observations that are within an \( \epsilon \)-neighborhood of zero. It follows that for all these observations, \( A_{g(j)} = A^*_{g(j)} = Z^*_g(j) \leq \epsilon \), and \( Z_g(j) \leq \epsilon \). We conclude that \( N_n \geq q + 1 \) implies
\[
S_n = S^*_n, \quad Z^*_g(q) \leq \epsilon, \quad \text{and} \quad Z_g(q+1) \leq \epsilon.
\]
Results (a)-(b) thus follow from proving that \( P\{\limsup_{n \to \infty}\{|N_n \geq q + 1\} \} = 1 \). In order to show this, note that \( N_n \sim Bi(n, P\{|Z| \leq \epsilon\}) \). Now set \( \mu \equiv \frac{1}{2} P\{|Z| \leq \epsilon\} \), which is strictly positive by the proof of Lemma B.1. It follows that
\[
P\{\liminf_{n \to \infty}\{|N_n \geq q + 1\}\} = P\left\{\liminf_{n \to \infty}\left\{\frac{1}{n} N_n \geq \frac{1}{n} (q + 1)\right\}\right\}
\geq P\left\{\liminf_{n \to \infty}\left\{\frac{1}{n} N_n \geq \mu\right\}\right\} = 1
\]
where the inequality holds for all $n > \frac{1}{2}(q + 1)$, and the last equality follows by the strong law of large numbers, i.e., $N_n/n \xrightarrow{a.s.} 2\mu > 0$. This completes the proof for parts (a)-(b).

Finally, part (c) holds by repeating the same argument for $\epsilon = 0$. The result extends in this case because $\mu \equiv \frac{1}{2}P\{|Z| = 0\} > 0$. ■

**Lemma B.4.** Let Assumption 4.1(i)-(ii) hold and assume $P\{|Z = 0\} = 0$. Fix $r \in (0, \frac{5}{2})$ and $q \in \{1, \ldots, n - 1\}$ arbitrarily. Then, for all $s \in \mathbb{N}_q \equiv \{0, 1, \ldots, q\}$,

$$P\{S_n^* = s \mid |Z_{\tilde{q},(q+1)}^*| = r\} = p_q(s|\pi(r))$$

where $p_q(s|\pi(r))$ is the pmf defined in (A-21) with $\pi(r) \equiv P\{|Z| \geq 0 \mid |Z| < r\}$.

**Proof.** Let $X \equiv (|Z|, A^*)$ with $A^* \equiv I\{|Z| \geq 0\}$ and note that the $g$-order statistics we defined in Section 3 using $g = \cdot$, could be alternatively obtained using $X$ and $\tilde{g}$-order statistics where $\tilde{g}$ is now the projection into the first component of $X$, i.e.

$$\tilde{g}(X) = |Z^*|.$$

In this way, and for this particular choice of $\tilde{g}$, $\tilde{g}$-order statistics on $X$ deliver

$$X_{\tilde{g},(1)} \equiv (|Z^*|_{(1)}, A_{1|1}^*) \leq \tilde{g} (|Z^*|_{(2)}, A_{2|2}^*) \leq \cdots \leq \tilde{g} (|Z^*|_{(n)}, A_{n|n}^*) \equiv X_{\tilde{g},(q)} ,$$

where the random variables $(A_{1|1}^*, \ldots, A_{n|n}^*)$ are called *induced order statistics* or *concomitants* of order statistics, see David and Galambos (1974); Bhattacharya (1974).

Let $\tilde{X}_1, \ldots, \tilde{X}_q$ be i.i.d. bivariate random variables such that $\tilde{X} \overset{d}{=} \{X \mid \tilde{g}(X) < r\}$. Theorem 1 in Kaufmann and Reiss (1992) states that

$$(X_{\tilde{g},(1)}, \ldots, X_{\tilde{g},(q)}) \mid \tilde{g}(X_{\tilde{g},(q+1)}) = r \overset{d}{=} (\tilde{X}_{\tilde{g},(1)}, \ldots, \tilde{X}_{\tilde{g},(q)}) , \tag{B-37}$$

with $\tilde{X}_{\tilde{g},(1)}, \ldots, \tilde{X}_{\tilde{g},(q)}$ being the $\tilde{g}$-order statistics of $\tilde{X}_1, \ldots, \tilde{X}_q$, provided that (i) $\tilde{g}(X)$ has a continuous distribution and (ii) $P\{\tilde{g}(X) < r\} > 0$. Since $\tilde{g}(X) = |Z^*|$ has a continuous distribution by Lemma B.1(a) and $P\{\tilde{g}(X) < r\} = P\{|Z^*| < r\} > 0$ by Lemma B.1(c), we use (B-37) to prove our result.

Next, note that we can re-write $S_n^*$ in (A-20) as a function of $(X_{\tilde{g},(1)}, \ldots, X_{\tilde{g},(q)})$ by using the function $h$ that projects into the second component of $X$, i.e.

$$S_n^* = \sum_{j=1}^q A_{\tilde{g},(j)}^* = \sum_{j=1}^q A_{\tilde{g},(j)} = \sum_{j=1}^q h(X_{\tilde{g},(j)}) ,$$

where in the second equality we used that $A_{\tilde{g},(j)}^* = A_{\tilde{g},(j)}$ by definition. Using this characterization, it follows that

$$P\{S_n^* = s \mid |Z_{\tilde{g},(q+1)}^*| = r\} = P\left\{\sum_{j=1}^q h(X_{\tilde{g},(j)}) = s \mid |\tilde{g}(X_{\tilde{g},(q+1)})| = r\right\}$$

$$= P\left\{\sum_{j=1}^q h(\tilde{X}_{\tilde{g},(j)}) = s\right\}$$

$$= P\left\{\sum_{j=1}^q h(\tilde{X}_j) = s\right\}$$

$$= p_q(s|\pi(r)) ,$$

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where the second equality follows from (B-37), the third equality follows from $\sum_{j=1}^{q} h(\tilde{X}_{g,(j)}) = \sum_{j=1}^{q} h(\tilde{X}_{j})$, and the last equality follows from $h(\tilde{X}_{1}), \ldots, h(\tilde{X}_{q})$ being i.i.d. bivariate random variables such that $h(\tilde{X}) \overset{d}{=} \{h(X) \mid \hat{g}(X) < r\}$ and $h(\tilde{X}) \overset{d}{=} \{I[Z^* \geq 0] \mid |Z^*| < r\}$ being distributed Bernoulli with parameter $\pi(r) = P\{Z^* \geq 0 \mid |Z^*| < r\}$. Since $P\{Z^* \geq 0 \mid |Z^*| < r\} = P\{Z \geq 0 \mid |Z| < r\}$ for $r \in (0, \frac{1}{2})$ by Lemma B.1(b), this completes the proof. 

**Lemma B.5.** Let $S \sim Bi(q, \pi)$ for $q > 1$ and consider testing $H_0 : \pi = \frac{1}{2}$ versus $H_1 : \pi \neq \frac{1}{2}$ at level $\alpha \in (0, 1)$ using the test $\phi(S)$ in (11). It follows that (i) $\phi(S)$ is a level $\alpha$ test, (ii) $\phi(S)$ is unbiased, (iii) $\phi(S)$ is uniformly most powerful unbiased.

**Proof.** This result follows from Lehmann and Romano (2005, Example 4.2.1) after noticing that Equations (4.5) and (4.6) in Lehmann and Romano (2005) reduce to (12) in this paper (which jointly determines $a_q(\alpha)$ and $b_q(\alpha)$) under $H_0 : \pi = \frac{1}{2}$. 

**Lemma B.6.** Let Assumption 4.1(i')-(ii) hold and suppose $q \to \infty$ and $\frac{q}{n} \to 0$ as $n \to \infty$. Then

$$\frac{n}{q} |Z^*_{g,(q)}| \overset{P}{\to} \frac{1}{f^+_Z(0) + f^-_Z(0)}.$$  

**Proof.** For any $\epsilon > 0$, it suffices to show that

$$P\left\{ \frac{n}{q} |Z^*_{g,(q)}| > \frac{1}{f^+_Z(0) + f^-_Z(0)} + \epsilon \right\} \to 0 \quad \text{and} \quad P\left\{ \frac{n}{q} |Z^*_{g,(q)}| < \frac{1}{f^+_Z(0) + f^-_Z(0)} - \epsilon \right\} \to 0.  \quad (B-38)$$

We only show the first result in (B-38), as the second one follows from symmetric arguments. By definition, $|Z^*_{g,(q)}| = |Z^*|_{(q)}$ where $|Z^*|_{(q)}$ denotes the $q$-th order statistic of the absolute value of $Z^*$. Denote by $Q$ the cdf of $|Z^*|$ and by $U_{(q)}$ the $q$-th order statistic of a $U[0, 1]$ distributed random variable. Lemma B.1(a) implies that $Q(\cdot)$ is a continuous cdf. Then, for $\hat{M} = \frac{1}{f^+_Z(0) + f^-_Z(0)} + \epsilon$, note that

$$P\left\{ \frac{n}{q} |Z^*_{g,(q)}| > \hat{M} \right\} = P\left\{ |Z^*|_{(q)} > \frac{q}{n} \hat{M} \right\} = P\left\{ U_{(q)} > Q\left( \frac{q}{n} \hat{M} \right) \right\} = P\left\{ \frac{n^{1/2}}{\epsilon_n} (U_{(q)} - \mu_n) > \frac{n^{1/2}}{\epsilon_n} \left( Q\left( \frac{q}{n} \hat{M} \right) - \mu_n \right) \right\},  \quad (B-39)$$

where $\epsilon_n^2 = \mu_n(1 - \mu_n)$ and $\mu_n = q/(n + 1)$. Letting

$$\gamma_n = \epsilon_n n^{1/2} \left( \frac{q}{n} \hat{M} - \mu_n \right),$$

it follows from (B-39) and Reiss (1989, Eq. (3.1.2) in Lemma 3.1.1) that

$$P\left\{ \frac{n}{q} |Z^*_{g,(q)}| > \hat{M} \right\} \leq \exp\left( -\frac{\gamma_n^2}{3(1 + \gamma_n/(\epsilon_n n^{1/2}))} \right).$$

To complete the proof it suffices to show that the right-hand side expression in the display above converges to zero. To this end, it suffices to show that $\gamma_n \to \infty$ and $\gamma_n/(\epsilon_n n^{1/2})$ converges to a positive constant. In turn, this follows from showing that

$$\epsilon_n n^{1/2} \to \infty \quad \text{and} \quad \frac{1}{\epsilon_n} \left( Q\left( \frac{q}{n} \hat{M} \right) - \mu_n \right) \to \epsilon \left( f^+_Z(0) + f^-_Z(0) \right) > 0,  \quad (B-40)$$

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where the limit of the second expression is positive by Assumption 4.1(ii). To show the first result in (B-40), note that \( q \frac{n^2}{n^2} \rightarrow 0 \) implies \( q \frac{n}{n^2} = \frac{\mu_2(1-\mu_n)}{\mu_n(n-1)} = -\frac{n}{n+1} (1 - \frac{q}{n+1}) \rightarrow 1 \). Combined with \( q \rightarrow \infty \), this then implies that \( e_n^{\frac{1}{2}} \rightarrow \infty \). To show the second result in (B-40), note that \( \frac{q}{n} \rightarrow 0 \) implies that for \( n \) sufficiently large we obtain \( \frac{q}{n} M < \delta \) and so \( Q(\frac{q}{n} M) = \int_0^{\frac{q}{n} M} f_Z^+(z)dz + \int_0^{\frac{q}{n} M} f_Z^-(z)dz \). Since \( Z^* = Z \) on \((\delta, \delta)\). Then,

\[
\frac{1}{c_{n}^2} \left( Q(\frac{q}{n} M) \right) = \frac{q/n}{\mu_n(1-\mu_n)} \left( \frac{n}{q} \int_0^{\frac{q}{n} M} f_Z^+(z)dz + \frac{n}{q} \int_0^{\frac{q}{n} M} f_Z^-(z)dz \right) = \frac{q/n}{\mu_n(1-\mu_n)} \left( \frac{n}{q} \int_0^{\frac{q}{n} M} (f_Z^+(z) - f_Z^-(0))dz + \frac{n}{q} \int_0^{\frac{q}{n} M} (f_Z^+(z) - f_Z^-(0))dz \right) + \frac{q/n}{\mu_n(1-\mu_n)} \left( \frac{n}{q} \int_0^{\frac{q}{n} M} f_Z^+(0)dz + \frac{n}{q} \int_0^{\frac{q}{n} M} f_Z^-(0)dz \right) \rightarrow \overline{M} \left( f_Z^+(0) + f_Z^-(0) \right) - 1 = \epsilon \left( f_Z^+(0) + f_Z^-(0) \right),
\]

where the convergence follows from Assumption 4.1(i') and \( \frac{q}{n} \rightarrow 0 \), which imply that

\[
\left| \frac{n}{q} \int_0^{\frac{q}{n} M} (f_Z^+(z) - f_Z^-(0))dz \right| \leq \frac{C_P M^2 q}{n} \rightarrow 0 \quad \text{and} \quad \left| \frac{n}{q} \int_0^{\frac{q}{n} M} (f_Z^-(z) - f_Z^-(0))dz \right| \leq \frac{C_P M^2 q}{n} \rightarrow 0,
\]

and \( \frac{q/n}{\mu_n(1-\mu_n)} \rightarrow 1 \). ■

**Lemma B.7.** Let Assumption 4.1(i') hold, \( \pi_f \) be as in (A-22), and \( \pi(r) = \mathbb{P}\{Z > 0 \mid |Z| < r\} \). Then, for any \( r \in (0, \delta) \),

\[
|\pi(r) - \pi_f| \leq \frac{r}{2} \frac{C_P}{f_Z^+(0) + f_Z^-(0)}.
\]

**Proof.** Fix \( r \in (0, \delta) \) arbitrarily. Start by re-writing \( \pi(r) \) as follows,

\[
\pi(r) = \frac{1}{r} \int_0^r f_Z^+(z)dz = \frac{1}{r} \int_0^0 f_Z^+(z)dz = \left( \frac{1}{r} \int_0^0 f_Z^+(z)dz + 1 \right)^{-1}
\]

\[
= \left( \frac{f_Z^+(0) + \frac{1}{r} \int_0^0 (f_Z^+(z) - f_Z^-(0))dz}{f_Z^+(0) + \frac{1}{r} \int_0^0 (f_Z^+(z) - f_Z^-(0))dz + 1} \right)^{-1}.
\]

Next, note that Assumption 4.1(i') implies that

\[
\left| \frac{1}{r} \int_0^r (f_Z^+(z) - f_Z^-(0))dz \right| \leq \frac{r}{2} C_P \quad \text{and} \quad \left| \frac{1}{r} \int_0^0 (f_Z^+(z) - f_Z^-(0))dz \right| \leq \frac{r}{2} C_P.
\]

Combining these two derivations we conclude that

\[
\pi(r) \leq \left( \frac{f_Z^+(0) - \frac{r}{2} C_P}{f_Z^+(0) + \frac{r}{2} C_P} + 1 \right)^{-1} = \pi_f + \frac{\frac{r}{2} C_P}{f_Z^+(0) + f_Z^-(0)}.
\]

\[
\pi(r) \geq \left( \frac{f_Z^-(0) + \frac{r}{2} C_P}{f_Z^-(0) + f_Z^+(0)} + 1 \right)^{-1} = \pi_f - \frac{\frac{r}{2} C_P}{f_Z^+(0) + f_Z^-(0)}.
\]

This implies the desired result. ■
C Computational details on the data-dependent rule for $q$

In the simulations of Section 5 and in the companion Stata package, the feasible informed rule of thumb is computed as follows. First, we compute

$$\hat{q}_{\text{rot}} = \max \left\{ q^*(\alpha), n^{1/2} \left( \hat{\sigma} - 4\hat{\sigma}^2 \phi_{\hat{\mu}, \hat{\sigma}} (\hat{\tau}) \right)^{2/3} \right\},$$

where $q^*(\alpha) = 1 - \log \alpha / \log 2$, $\hat{\mu}$ is the sample mean of $\{Z_1, \ldots, Z_n\}$, $\hat{\sigma}^2$ is the sample variance of $\{Z_1, \ldots, Z_n\}$, $\hat{\tau}$ is the cut-off point, and $n$ is the sample size. In principle, the value $\hat{q}_{\text{rot}}$ could be used to implement our test. However, this would ignore the non-monotonicity of the limiting null rejection probability of the non-randomized version of our test, which according to Theorem 4.2, equals $2\Psi_q(b_q(\alpha) - 1)$ with $b_q(\alpha)$ defined in (10). Figure 3 displays $2\Psi_q(b_q(\alpha) - 1)$ for $\alpha = 5\%$ as a function of $q$. The figure shows that $2\Psi_q(b_q(\alpha) - 1)$ takes values very close to $\alpha$ for $q$ as low as 17 (i.e., 4.9%), but could be far from $\alpha$ for $q = 19$ (i.e., 1.9%). We therefore propose an additional layer in the data-dependent way of choosing $q$ that guarantees that such a value delivers a local “peak” of $2\Psi_q(b_q(\alpha) - 1)$ in Figure 3.

![Figure 3: Limiting null rejection probability (in %) of the non-randomized version of the test, $2\Psi_q(b_q(\alpha) - 1)$, as a function of $q$ (red solid line). Nominal level of the test (dotted orange line).](image)

To be concrete, we define $\hat{q}_{\text{frot}}$ as

$$\hat{q}_{\text{frot}} = \arg\max_{q \in \mathcal{N}(\hat{q}_{\text{rot}})} \Psi_q(b_q(\alpha) - 1),$$

where $\mathcal{N}(\hat{q}_{\text{rot}}) \equiv \{ q \in \mathbb{N} : \max\{q^*(\alpha), \hat{q}_{\text{rot}} - [4 \log(\hat{q}_{\text{rot}})]\} \leq q \leq \hat{q}_{\text{rot}} + [4 \log(\hat{q}_{\text{rot}})] \}$. The value of window size $[4 \log(\hat{q}_{\text{rot}})]$ is the minimum number of points that are required to reach a local peak of $2\Psi_q(b_q(\alpha) - 1)$ for values of $\alpha \in \{1\%, 5\%, 10\%\}$ and is such that, for large values of $q_{\text{rot}}$, the window gets larger to improve the chances of getting one of the peaks closer to $\alpha$ as $q_{\text{rot}}$ increases. A smaller window size may not guarantee one actually reaches a local peak. The value $\hat{q}_{\text{frot}}$ defined in (C-42) is the one we use in the simulations of Section 5 and the default value in the companion Rdcont Stata package.
## Surveyed papers on RDD

Table 4 displays the list of papers we surveyed in leading journals that use regression discontinuity designs. For a description on the criteria used to compile the list of papers in Table 4, see Canay and Kamat (2018, Appendix E).

<table>
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<th>Authors (Year)</th>
<th>Journal</th>
<th>Density Test</th>
<th>Mean Test</th>
<th>Authors (Year)</th>
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Table 4: Papers using manipulation/placebo tests from 2011 – 2015.
References

Arbuthnott, J. (1710). An argument for divine providence, taken from the constant regularity observed in the births of both sexes. *Philosophical Transactions*, 27 186–190.


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