Web Appendix for Critical Types

Jeffrey C. Ely * Marcin Peski †

September 24, 2010

In this web appendix we demonstrate the construction of a regular type. The idea is to construct hierarchies out of "stacking" finite truncations of other hierarchies on top of each other. We illustrate with a simple example with 2 players. Let $\bar{p} = (p_1, p_2, \ldots)$ be an infinite sequence of numbers $p_n \in [0,1]$. Consider a hierarchy $u_{\bar{p}}$ according to which player i believes that the state of nature is equal to $\omega = +1$ with probability p_1 and that player $-i$ believes that the state is equal to $\omega = +1$ with probability p_2 and that $-i$ believes that i believes that the state is equal to $\omega = +1$ with probability p_3 and so on. Now, take two sequences $\bar{p}$ and $\bar{p}'$. For each k, we can construct a hierarchy in which the first k-order beliefs are given by $u_{\bar{p}}$, and after that, the beliefs are described by $u_{\bar{p}'}$: formally, let

$$u_{\bar{p}} \wedge^k u_{\bar{p}'} = u_{\left(p_1, p_2, \ldots, p_k, p'_1, p'_2, \ldots\right)}.$$

We say that the hierarchy $u_{\bar{p}} \wedge^k u_{\bar{p}'}$ is constructed by "stacking" $u_{\bar{p}'}$ on the k-th level of $u_{\bar{p}}$. We can extend the "stacking" operation to general hierarchies u and u'. Additionally, we can consider sequences of hierarchies u^1,u^2,... and levels k_1,k_2,... and construct a new hierarchy by first "stacking" u^2 on the k_1-level of u^1, then add u^3 at the $(k_1 + k_2)$-level of the previously constructed hierarchy, and so on. We use this operation to construct a hierarchy without non-trivial common beliefs. To see how it works, start with hierarchy $u_{(1,1,\ldots)}$. This hierarchy exhibits a common belief in a non-trivial event, namely common knowledge that state is equal to $\omega = +1$. We can eliminate such common belief by "stacking" hierarchy $u_{(0,0,\ldots)}$ on the top of $u_{(1,1,\ldots)}$. So constructed hierarchy $\hat{u} = u_{(1,1,\ldots,1,0,0,\ldots)}$ exhibits common belief in the event that "both players either know (i.e., belief with full probability) that the state is equal to $\omega + 1$ or they know that the state is equal to $\omega = -1$." The latter common belief can be eliminated by "stacking" the hierarchy $u_{\left(\frac{1}{2},\frac{1}{2},\ldots\right)}$ on the top of

*Department of Economics, Northwestern University, jeffely@northwestern.edu.
†Department of Economics, University of Texas at Austin, mpeski@gmail.com

$\hat{u}$. We can continue in this fashion by constructing a hierarchy that does not exhibit a common belief in any event of form "both players believe that the state is equal to $\omega = +1$ with probability $p \in A$" for some closed proper subset $A \subset [0,1]$. Such a hierarchy may still exhibit non-trivial common beliefs. (In fact, any hierarchy of the form $u_{\bar{p}}$ exhibits common knowledge of the event that "opponents beliefs are independent of the state of nature.") More generally, we use the fact that there exists a countable dense subset Q of the universal type space. Find an enumeration $\{u_m, k_m\}$ of $Q \times \mathbb{N}$ where $\mathbb{N}$ is the set of natural numbers. Now we construct a single hierarchy by "stacking" hierarchies u_1,u_2,. . .. at levels k_1,$k_1 + k_2$,. . . . The resulting type u^* is regular. This follows from two observations. First, any set W that is common belief for u^* must include Q. This is because every hierarchy in Q is believed with probability 1 at some level of the hierarchy. Next, since Q is dense, there is no closed, proper subset that includes Q; thus, by our characterization u^* is regular.

1 Example of a Regular Hierarchy

1.1 "Stacking"

We begin by formally defining a "stacking" operation. The basic idea was described in the introduction. For each i, let T_i^c be disjoint copies of space $U_i(\Omega)$ for each $c = 1,2,\ldots$ Define

$$T_i = \bigcup_c T_i^c,$$

and let

$$u_i^c : T_i^c \to U_i(\Omega)$$

be the isomorphism. We construct a belief mapping $\mu_i : T_i \to \Delta(\Omega \times T_{-i})$: for each $t_i \in T_i^c$, let

$$\mu_i(t_i) = \left(\Delta\left(\mathrm{id}_\Omega \times u_{-i}^{c+1}\right)^{-1}\right)\left(u_i^c(t_i)\right),$$

where we use the fact that player i's hierarchy and its belief is denoted with the same symbol $u_i^c(t_i) \in U_i(\Omega) \simeq \Delta(\Omega \times U_{-i}(\Omega))$. Thus, type t_i has beliefs concentrated on the set T_{-i}^{c+1}, $\mu_i(t_i)\left(\Omega \times T_{-i}^{c+1}\right) = 1$. Then, (T_i, μ_i) is a type space, and, of course, the hierarchy of beliefs of type $t_i \in U_i^c$ is equal to $u_i^c(t_i)$. Next, choose any hierarchy profile $v = (v_1, ..., v_N) \in \times U_i(\Omega)$ and $k \geq 1$. We construct a modified type space $\left(T_j^{v,k}, \mu_j^{v,k}\right)$.

For each i, let $T_i^{v,k}$ be a copy of T_i, and let

$$\lambda_i^{v,k} : T_i^{v,k} \to T_i,$$

be the isomorphisms. Let

$$\mu_i^{v,k}(t_i) = \left(\Delta\left(\mathrm{id}_\Omega \times \lambda_{-i}^{v,k}\right)^{-1}\right)\mu_i\left(\lambda_i^{v,k}(t_i)\right), \text{ for each } t_i \in \left(\lambda_i^{v,k}\right)^{-1}\left(T_j^c\right), \text{ and } c \neq k,$$
$$\mu_i^{v,k}(t_i) = \left(\Delta\left(\mathrm{id}_\Omega \times \lambda_{-i}^{v,k}\right)^{-1}\right)\mu_i\left(\lambda_i^{v,k}\left(\left(u_i^k\right)^{-1}(v_i)\right)\right), \text{ for each } t_i \in \left(\lambda_i^{v,k}\right)^{-1}\left(T_j^k\right).$$

For each $t_i \in T_i^{v,k}$, let $\phi(t_i) \in U_i(\Omega)$ be the hierarchy of type u_i. Then, for $c = k$ and each $t_i \in T_i^{c,v,k}$, $\phi(t_i) = v$. Finally, for each hierarchy $u_i \in U_i(\Omega)$, hierarchy profile $v \in \times U_i(\Omega)$ and $k \geq 1$, define

$$u_i \hat{}^k v := \phi\left(\left(u_i^1 \circ \lambda_i^{v,k}\right)^{-1} u_i\right).$$

We say that hierarchies v are stacked up on the kth level of beliefs of hierarchy u_i: the kth-level beliefs are erased and replaced by hierarchies v.

Lemma 1. *For each hierarchy $u_i \in U_i(\Omega)$, hierarchy profile $v \in \times U_i(\Omega)$, $k \geq 1$, each $p > 0$, and $W^v = \times_i (U_i(\Omega) \setminus \{v_i\})$,*

$$u_i \hat{}^k v \notin C_i^p(W^v).$$

Proof. We show by induction on $0 \leq c < k$, that for each player j, and each $t_j \in T_j^{v,k-c}$, $\phi(t_j) \notin C_i^p(W^v)$. Indeed, if $c = 0$, then $\phi(t_j) = v_j \notin W_j^v \supseteq C_i^p(W^v)$. Suppose that the claim holds for $c \geq 0$ and $c < k-1$. Take any j and $t_j \in T_j^{v,k-c-1}$. Then,

$$\mu_j^{v,k}(t_j)\left\{\forall_{j' \neq j}\phi\left(t_{j'}\right) \in C_i^p(W^v)\right\} = 0,$$

which implies that $\phi(t_j) \notin C_i^p(W^v)$. □

1.2 Regular Hierarchy

Let $Q \subseteq U(\Omega)$ be a countable and dense set of hierarchy profiles. Fix mappings $q : \mathbb{N} \to Q$ and $k : \mathbb{N} \to \mathbb{N}$ such that the mapping

$$m \to (q(m), l(m))$$

is a bijection between $\mathbb{N}$ and $Q \times \mathbb{N}$. Let $k(m) = \sum_{m' \leq m} l(m')$. Fix any hierarchy $u_i \in U_i(\Omega)$ and consider a sequence of hierarchies,

$$u_i^0 = u_i, \text{ and } u_i^m = u_i^{m-1 \wedge k(m)} q(m) \text{ for each } m \geq 1.$$

Each hierarchy u_i^m has the same first $k(m)$-th order beliefs as hierarchy u_i^m. Therefore, hierarchies u_i^m converge in the product topology, and let u_i^* be their limit. Then, for each m,

$$u_i^* = u_i^{m-1 \wedge k(m)} v^m,$$

where $v^m = \left(v_1^m, ..., v_N^m\right)$ and for each player j, hierarchy v_j^m has the same first $l(m)$th order beliefs as hierarchy $q(m)$. Take any proper closed product set $W = \times_j W_j \subset U(\Omega)$. Because $U_j(\Omega) \setminus W_j$ is open, there exists hierarchy profile $q = (q_1, ..., q_N) \in Q$ such that for each j, $q_j \notin W_j$. Additionally, there is l such that for each hierarchy profile $v = (v_1, ..., v_i)$ so that hierarchies v_j have the same first lth order beliefs as q_j, $v_j \notin W_j$. Let m be such that $(q, l) = (q(m), l(m))$. By Lemma 1,

$$u_i^* \notin C^p\left(W^{v^m}\right)$$

for any $p > 0$. Because any closed proper W could have been chosen, hierarchy u_i^* is regular by Theorem 1.