Mechanism Design Without Revenue Equivalence*

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Abstract

We characterize incentive compatible mechanisms in quasi-linear environments where the envelope theorem and revenue equivalence fail due to non-convex and non-differentiable valuations. Despite these obstacles, we obtain a characterization based on the familiar Mirrlees representation of the indirect utility and a monotonicity condition on the allocation rule. These conditions pin down the *range* of possible payoffs as a function solely of the allocation rule, thus providing a revenue inequality. We use our characterization in two economic applications where standard techniques based on revenue equivalence fail. We find a budget-balanced efficient mechanism in a public goods setting, and we characterize the optimal selling mechanism when the buyer has loss-averse preferences à la Kőszegi and Rabin (2006).

Keywords: Incentive Compatibility, Revenue Equivalence, Integral Monotonicity, Subderivative Correspondence, Budget Balancedness, Revenue Maximization.

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1 Introduction

Revenue equivalence states that, under certain conditions, two (dominant strategy[1]) incentive compatible mechanisms with the same allocation rule generate utilities for the agent and payments to the planner that differ at most by a constant. The revenue equivalence principle is very useful in simplifying mechanism design problems in a variety of settings, as first shown by Myerson (1981). A large literature studies the breadth of conditions under which revenue equivalence holds.[2]

In this paper, we characterize incentive compatibility in the absence of linearity, convexity and differentiability assumptions on the valuation function with respect to types. Without these assumptions, revenue equivalence as traditionally formulated may fail but the range of potential payoffs can still be determined. Moreover, our characterization can be employed analogously to the traditional one in various applications such as efficient, budget-balancing, and revenue-maximizing mechanism design.

The following simple example illustrates these ideas.

Example 1. The allocation set is $\mathscr{X} = [0,1]$. The agent has a quasi-linear utility function $u = v(x,\theta) - \rho$ defined over alternatives $x \in \mathscr{X}$ and monetary payments $\rho \in \mathbb{R}$. Types are private information and lie in $\Theta = [0,1]$. The agent's valuation function $v\colon \mathscr{X} \times \Theta \to \mathbb{R}$ is given by

$$v(x,\theta) \;=\; \begin{cases} \theta x, & \text{if } x \leq \theta; \\ 2\theta^2 \;-\; \theta x, & \text{if } x > \theta. \end{cases}$$

Think of x as the quantity traded of some good. Our agent has positive marginal utility for the first θ units, and negative marginal utility for additional amounts. Note that for each x, the function $\theta \mapsto v(x,\theta)$ is neither convex nor fully differentiable on Θ. In particular, its partial derivative with respect to θ fails to exist whenever $\theta = x$.

The efficient allocation rule $X^*\colon \Theta \to \mathscr{X}$ selects $X^*(\theta) = \theta$. Clearly X^* is implementable, indeed by a constant payment rule $p \equiv 0$. In this case, the direct mechanism (X, p) generates zero revenue and an *indirect utility function* for the agent given by

$$U(\theta) \;=\; v(X^*(\theta),\theta) - p(\theta) \;=\; \theta^2.$$

However, revenue equivalence fails. For example, consider the alternative payment rule p' defined by $p'(\theta) \;=\; \theta^2/2$, all $\theta \in \Theta$. To see that the mechanism (X^*, p') is

[1]We study single-agent, dominant strategy incentive compatible mechanisms. This is for expositional clarity. It is straightforward to generalize our results to many agents and to Bayesian incentive compatibility.

[2]Berger, Müller, and Naeemi (2010) build on the envelope theorem of Milgrom and Segal (2002) to establish revenue equivalence when the type space is a convex subset of a multi-dimensional Euclidean space and valuations are convex or differentiable in types. See also Krishna and Maenner (2001), Jehiel, Moldovanu, and Stacchetti (1996, 1999), Williams (1999), Heydenreich, Müller, Uetz, and Vohra (2009) and Chung and Olszewski (2007). Holmström (1979) and Carbajal (2010) develop an alternative method to revenue equivalence for efficient allocation rules.

incentive compatible, fix a type $\theta \in \Theta$. Reporting $\hat\theta \in \Theta$ generates payoffs equal to

$$v(X^*(\hat\theta),\theta) \;-\; p'(\hat\theta) \;=\; \begin{cases} \theta\hat\theta \;-\; \hat\theta^2/2, & \text{if } \hat\theta \le \theta; \\ 2\theta^2 \;-\; \theta\hat\theta \;-\; \hat\theta^2/2, & \text{if } \hat\theta > \theta. \end{cases}$$

The above expression is maximized when $\hat\theta = \theta$. In this mechanism, type $\theta = 0$ obtains zero utility, just as in (X^*, p), and yet (X^*, p') generates strictly positive revenues and an indirect utility function U' given by $U'(\theta) = \theta^2/2$. □

We shall see that despite the failure of revenue equivalence, the full set of incentive compatible payment rules can be characterized by versions of standard conditions. Applied to Example 1, we can show that U is an indirect utility function arising from an incentive compatible mechanism with the allocation rule X^* if, and only if,

$$U(\theta) \;=\; U(0) \;+\; \int_0^\theta s(\tilde\theta)\, d\tilde\theta, \qquad \text{for all } \theta \in \Theta, \tag{1}$$

for some function $\theta \mapsto s(\theta)$ such that

$$s \text{ is an integrable selection from the correspondence } S(\theta) = [\theta, 3\theta] \tag{2}$$

satisfying, for all $\theta, \hat\theta \in \Theta$,

$$v(X^*(\theta),\theta) - v(X^*(\theta),\hat\theta) \;\ge\; \int_{\hat\theta}^{\theta} s(\tilde\theta)\, d\tilde\theta \;\ge\; v(X^*(\hat\theta),\theta) - v(X^*(\hat\theta),\hat\theta). \tag{3}$$

Equation 1 is the familiar *Mirrlees representation* of incentive compatible payoffs. Equation 2 generalizes the standard envelope theorem derivation of the integrand $\theta \mapsto s(\theta)$. In models with differentiable valuations, $s(\theta)$ would be given by the partial derivative of $v(X^*(\theta),\tilde\theta)$ with respect to its second argument evaluated at $\tilde\theta = \theta$. This requirement is modified to allow $s(\theta)$ to range between the right and left subderivatives[3] of $v(X^*(\theta),\tilde\theta)$ at $\tilde\theta = \theta$, which in this case are θ and 3θ, respectively. Finally, as is always the case, some form of monotonicity of the allocation rule is necessary for incentive compatibility. Here Equation 3, the *integral monotonicity condition*, turns out to be necessary and, together with the previous two conditions, sufficient.[4]

Our main result, Theorem 1, extends this characterization to a general mechanism design setting with utilities that are quasi-linear in money, a convex, possibly multi-dimensional type space, and a measurable space of outcomes. In place of differentiability, convexity, or absolute continuity, we assume that the valuation function satisfies a uniform Lipschitz condition with respect to types. This condition, along with an

[3] Mathematical concepts and results are presented in Section 2.2.

[4] This condition is also studied in Berger, Müller, and Naeemi (2010) who characterize incentive compatibility in settings where revenue equivalence holds.

additional technical restriction, allows us to employ integrable selections of the subderivative correspondence (cf. Equation 2) in place of the usual envelope derivation. In multi-dimensional settings, a closed-path integrability condition appears in our characterization of incentive compatible mechanisms, in addition to the three conditions mentioned above.

In Section 5 we apply our characterization result to two settings: an efficient public good provision problem, and the design of an optimal selling mechanism when buyers are loss-averse. It is well understood that when revenue equivalence holds, the efficient allocation rule is implementable if and only if it is implementable via a Vickrey-Clarke-Groves (VCG) payment rule. We consider a location model for a public good where revenue equivalence fails and show that, in this case, efficient allocations can be implemented with a budget-balanced payment rule, and yet no VCG mechanism has a balanced budget. In the buyer-seller model, the buyer has loss-averse preferences à la Kőszegi and Rabin (2006) which, due to a kink on the valuation at the reference point, fails standard requirements of differentiability or convexity. Nonetheless, our characterization result applies and allows us to expand Myerson's (1981) techniques to this situation, thus reformulating the seller's problem in terms of virtual valuations. In the optimal selling mechanism, a range of intermediate types purchase their reference quantity so that the flexibility of pricing rules afforded by our characterization plays an important role. Compared to the case without loss aversion, some of these intermediate types have their quantities distorted downward to exploit the loss aversion of even higher types, whose incentive constraints are thereby slackened and their payments correspondingly increased. Similarly, the loss aversion of lower intermediate types is exploited by making them pay a premium to increase their quantity from zero to their reference points.

In Section 4 we derive a *revenue inequality* in the spirit of the standard revenue equivalence principle. Our Theorem 2 shows that, after normalizing the utility of the "lowest type", the difference in equilibrium payoffs generated by two incentive compatible mechanisms with the same allocation rule is bounded, with bounds depending solely on the allocation rule. Using our version of the Mirrlees representation of indirect utilities, we are able to show that any payment rule used to implement an allocation rule has a simple representation based on the subderivative correspondence. It follows that the set of normalized equilibrium payoffs is convex: if two payment rules implement an allocation rule and generate indirect utility functions U and U', respectively, then for every convex combination U^λ of U and U' there exists a payment function that implements the allocation rule and generates an indirect utility equal to U^λ.

In Section 6, we relate our approach to previous work and show that, with the inclusion of a minor measurability hypothesis on the right and left subderivatives of the valuation function, our characterization result is obtained whenever the valuation is either convex, concave, or differentiable with respect to types. We also show that revenue equivalence is restored when valuations are convex or differentiable, or whenever the allocation set is finite. Section 7 ends this paper with some concluding remarks.

1.1 Related Literature

Integral monotonicity is connected to the variety of monotonicity conditions studied in the literature since Rochet (1987), who characterized incentive compatibility via cyclic monotonicity. Saks and Yu (2005), Bikhchandani, Chatterji, Lavi, Mu'alem, Nisan, and Sen (2006), and Ashlagi, Braverman, Hassidim, and Monderer (2010) investigate weaker forms of monotonicity in environments with finitely many allocations. For infinite allocation sets, Archer and Kleinberg (2008) extend the insights of Jehiel, Moldovanu, and Stacchetti (1999) and characterize implementable allocation rules via weak monotonicity (plus a closed-path integrability condition) in environments with multi-dimensional convex type spaces and valuations that are linear in types. In independent work, Berger, Müller, and Naeemi (2010) study environments with convex or differentiable valuations and obtain a characterization of incentive compatibility using integral monotonicity (which they call path monotonicity).

We recently became aware of independent work by Kos and Messner (2010) and Rahman (2010) who also characterize incentive compatibility in general settings. Rahman (2010) uses linear programming and duality to characterize incentive compatible allocations in terms of detectable deviations. Kos and Messner (2010) takes a perspective that is more similar to ours, studying the extremal transfer rules that implement an allocation. Relative to both of these papers, we impose additional structure (notably Lipschitz continuity) and in return obtain characterizations which are more immediately suitable for applications. We demonstrate two such applications in Section 5.

2 Preliminaries

2.1 The design setting

We consider a single-agent mechanism design setting; extensions to multi-agent settings are straightforward. An outcome (x, ρ) is composed of an alternative x that belongs to the *allocation set* $\mathscr{X}$, and a real number ρ representing some quantity of a perfectly divisible commodity (money). Our agent has *quasi-linear preferences*, so that

$$u(x, \theta, \rho) \;=\; v(x, \theta) \;-\; \rho$$

represents the agent's utility when $x \in \mathscr{X}$ is selected and amount $\rho \in \mathbb{R}$ is paid, given her privately known type θ. We denote the agent's *type space* by Θ and refer to $v \colon \mathscr{X} \times \Theta \to \mathbb{R}$ as the *valuation function*.

The following assumptions are made throughout this paper.

(A1) The pair $(\mathscr{X}, \mathcal{M})$ is a measurable space ($\mathcal{M}$ denotes a σ-algebra of subsets of $\mathscr{X}$).

(A2) The type space Θ is a convex, bounded subset of $\mathbb{R}^k$ $(k \geq 1)$.

(A3) For every $x \in \mathscr{X}$, the function $\theta \mapsto v(x,\theta)$ is Lipschitz continuous on Θ: there is a positive number $\ell(x)$ such that $|v(x,\theta) - v(x,\hat{\theta})| \leq \ell(x)\,\|\theta - \hat{\theta}\|$, for all $\theta, \hat{\theta} \in \Theta$. Moreover, $\{\, \ell(x) \mid x \in \mathscr{X} \,\}$ is bounded above, with $\ell = \sup\{\, \ell(x) \mid x \in \mathscr{X} \,\} < +\infty$.

(A1) does not impose any burdensome restriction on the allocation set, which could be finite or infinite. The convexity of the type space in (A2) is a standard assumption and is satisfied in several economic applications.[5] (A3) is used to prove the Lipschitz continuity of the indirect utility function. It is not possible to dispense with the boundedness assumption on the set of Lipschitz constants when $\mathscr{X}$ is infinite.[6] In addition, (A3) allows us to work with the right and left subderivatives of the valuation function with respect to types.

We consider direct mechanisms of the form (X, p), where the function $X\colon \Theta \to \mathscr{X}$ is called the *allocation rule* and the function $p\colon \Theta \to \mathbb{R}$ is called the *payment rule*. X is said to be *implementable* if there exists a payment rule p such that truth-telling is an dominant strategy for the agent; i.e.,

$$v(X(\theta),\theta) \;-\; p(\theta) \;\geq\; v(X(\hat{\theta}),\theta) \;-\; p(\hat{\theta}), \qquad \text{for all } \theta, \hat{\theta} \in \Theta. \tag{4}$$

In that case the direct mechanism (X, p) is said to be *incentive compatible* and the function $U\colon \Theta \to \mathbb{R}$ defined by

$$U(\theta) \;\equiv\; v(X(\theta),\theta) \;-\; p(\theta), \qquad \text{all } \theta \in \Theta, \tag{5}$$

is called the *indirect utility generated by* (X, p). We shall restrict our analysis to measurable allocation rules.

2.2 Subderivatives and the integral of a correspondence

We introduce some concepts and results that are later used in our characterization theorem. The reader is referred to Aubin and Frankowska (1990), Hildenbrand (1974) and Rockafellar and Wets (1998) for details.

Fix $x \in \mathscr{X}$, $\hat{\theta} \in \Theta$, and let $\delta \in \mathbb{R}^k$, $\delta \neq \mathbf{0}$, be a directional vector for which $\hat{\theta} + r\delta \in \Theta$ for a sufficiently small scalar r. The right and left subderivatives of the function $\theta \mapsto$

[5] At some notational cost, one could instead assume that Θ is polygonally connected. The boundedness of the type space plays a technical role in some of our results.

[6] The following well-known example illustrates this point. Let $\mathscr{X} = \mathbb{N}$, $\Theta = [0,1]$, and $f\colon \mathbb{N} \times \Theta \to \mathbb{R}$ be defined by

$$f(n,\theta) \;=\; \begin{cases} 1, & \text{if } 0 \leq \theta \leq 1 - \frac{1}{n}; \\ n(1-\theta), & \text{if } 1 - \frac{1}{n} < \theta \leq 1. \end{cases}$$

For each $n \in \mathbb{N}$, $\theta \mapsto f(n,\theta)$ is Lipschitz on Θ with $\ell(n) = n$. However, the function $g\colon \Theta \to \mathbb{R}$ defined by $g(\theta) \;=\; \sup_{n \in \mathbb{N}} f(n,\theta)$, all $\theta \in \Theta$, is clearly discontinuous at $\theta = 1$.

$v(x,\theta)$ evaluated at $\hat{\theta} \in \Theta$ in the direction δ are defined, respectively, as the following lower and upper limits:

$$\overline{\mathrm{d}}v(x,\hat{\theta};\delta) \;\equiv\; \liminf_{r\downarrow 0} \frac{v(x,\hat{\theta}+r\delta)-v(x,\hat{\theta})}{r}; \tag{6}$$

$$\underline{\mathrm{d}}v(x,\hat{\theta};\delta) \;\equiv\; \limsup_{r\uparrow 0} \frac{v(x,\hat{\theta}+r\delta)-v(x,\hat{\theta})}{r}. \tag{7}$$

By (A3), these subderivatives exist and are finite. Note that $\underline{\mathrm{d}}v(x,\hat{\theta};\delta) = -\,\overline{\mathrm{d}}v(x,\hat{\theta};-\delta)$. If $\theta \mapsto v(x,\theta)$ admits one-sided directional derivatives at $\hat{\theta}$, then one can replace the upper and lower limits in (6) and (7) with the usual one-sided limits, although the above notation is maintained.

Given types θ_1, θ_2 in Θ, denote their vector difference by $\delta_1^2 \equiv \theta_2 - \theta_1$. The open line segment connecting θ_1 to θ_2 is the set $L(\theta_1,\theta_2) = \{\, \theta_1 + \alpha\,\delta_1^2 \mid \alpha \in (0,1) \,\}$. By (A2), one has $L(\theta_1,\theta_2) \subseteq \Theta$ for all $\theta_1,\theta_2 \in \Theta$. We shall make implicit use of the function $\alpha \mapsto \theta_1^2(\alpha) \equiv \theta_1 + \alpha\,\delta_1^2$ mapping $(0,1)$ onto $L(\theta_1,\theta_2)$.

Let $\mathcal{B}(0,1)$ denote the Borel σ-algebra of subsets of $(0,1)$. Let $S\colon (0,1) \rightrightarrows \mathbb{R}$ be a correspondence with closed images. Then S is said to be a measurable correspondence if for every open set O in $\mathbb{R}$, the inverse image $S^{-1}(O) = \{\, \alpha \in (0,1) \mid S(\alpha) \cap O \neq \emptyset \,\}$ belongs to $\mathcal{B}(0,1)$. In particular, $\operatorname{dom} S = \{\, \alpha \in (0,1) \mid S(\alpha) \neq \emptyset \,\}$ and its complement are measurable sets. S is said to be integrably bounded if there exists a non-negative (Lebesgue) integrable function g defined on $(0,1)$ such that $S(\alpha) \subseteq [-\,g(\alpha), g(\alpha)]$ for almost all α in $(0,1)$. A selection s of the correspondence S is a function $\alpha \mapsto s(\alpha)$ such that $s(\alpha) \in S(\alpha)$, a.e. in $(0,1)$. By the Measurable Selection Theorem, a measurable correspondence $S\colon (0,1) \rightrightarrows \mathbb{R}$ admits a measurable selection $s\colon (0,1) \to \mathbb{R}$. If in addition S is integrably bounded, then it admits (Lebesgue) integrable selections. In such case, Aumann's (1965) integral of the correspondence S is the non-empty set

$$\left\{ \int_0^1 s(\alpha)\, d\alpha \mid s(\alpha) \in S(\alpha) \text{ a.e. in } (0,1) \right\}.$$

By the Lyapunov's Convexity Theorem, the integral of a closed-valued, measurable and integrably bounded correspondence $S\colon (0,1) \rightrightarrows \mathbb{R}$ is a non-empty closed interval.

3 Characterizing incentive compatible mechanisms

Consider a direct mechanism (X,p). We start with the following observation: if (X,p) is incentive compatible, then its associated indirect utility is Lipschitz continuous on Θ and has two-sided directional derivatives a.e. in the line segment connecting any two types.

Lemma 1. *Assume that (A1) to (A3) are satisfied, and let* $p\colon \Theta \to \mathbb{R}$ *implement the allocation rule* X*. The following statements hold:*

(a) The indirect utility function U *generated by* (X, p) *is Lipschitz continuous on* Θ*.*

(b) For every pair θ_1, θ_2 *of distinct types in* Θ*,* U *admits two-sided directional derivatives in the direction* $\delta_1^2 = \theta_2 - \theta_1$ *a.e. in* $L(\theta_1, \theta_2)$*.*

Proof (a) Consider arbitrary types θ_1, θ_2 in Θ. From Equation 4 and Equation 5 one sees that

$$
\begin{aligned}
U(\theta_2) - U(\theta_1) \;&\leq\; \big\{v(X(\theta_2), \theta_2) - p(\theta_2)\big\} \;-\; \big\{v(X(\theta_2), \theta_1) - p(\theta_2)\big\} \\
&=\; v(X(\theta_2), \theta_2) \;-\; v(X(\theta_2), \theta_1) \;\leq\; \ell(X(\theta_2))\, \|\theta_2 - \theta_1\| \;\leq\; \ell\, \|\theta_2 - \theta_1\|;
\end{aligned}
$$

where the last two inequalities follow from (A3). Reversing the roles of θ_1 and θ_2, one readily concludes that $|U(\theta_2) - U(\theta_1)| \leq \ell \|\theta_2 - \theta_1\|$, as desired.

(b) Fix distinct types θ_1, θ_2 in Θ. Define the function μ on $(0, 1)$ by $\mu(\alpha) = U(\theta_1^2(\alpha))$, where $\theta_1^2(\alpha) \equiv \theta_1 + \alpha\, \delta_1^2 \in L(\theta_1, \theta_2)$. It is readily seen that μ is Lipschitz continuous. Indeed, for any $0 < \alpha, \alpha' < 1$, one has:

$$
|\mu(\alpha) - \mu(\alpha')| \;=\; |U(\theta_1^2(\alpha)) - U(\theta_1^2(\alpha'))| \;\leq\; \ell\, \|\theta_1^2(\alpha) - \theta_1^2(\alpha')\| \;=\; \ell\, \|\delta_1^2\|\, |\alpha - \alpha'|,
$$

with the above inequality following from part (a). Thus, μ is Lipschitz and therefore absolutely continuous and differentiable a.e. in $(0, 1)$. In particular, if μ is differentiable at $\alpha \in (0, 1)$, then we deduce:

$$
D\mu(\alpha) \;=\; \lim_{r \to 0} \frac{\mu(\alpha + r) - \mu(\alpha)}{r} \;=\; \lim_{r \to 0} \frac{U(\theta_1^2(\alpha) + r\, \delta_1^2) - U(\theta_1^2(\alpha))}{r} \;=\; DU(\theta_1^2(\alpha); \delta_1^2),
$$

with last term above denoting the two-sided directional derivative of U at $\theta_1^2(\alpha)$ in the direction δ_1^2. ■

Milgrom and Segal (2002) obtained an analogue of Lemma 1-(a) under the alternative hypothesis of absolute continuity of $v(x, \cdot)$ on a one-dimensional type space, plus an integral bound condition on the derivative of v with respect to types. The extension of the absolute continuity concept to multi-dimensional settings is not straightforward.[7] On the other hand, Lipschitz continuity extends naturally to multi-dimensional domains and allows us to work with right and left subderivatives.

We mention that, in multi-dimensional settings, Lemma 1-(b) does not imply that U is fully differentiable a.e. in the line connecting θ_1 and θ_2. In fact, there may be many (piecewise) smooth paths between θ_1 and θ_2 for which U is nowhere fully differentiable

[7] For instance, it is possible to construct a convex function on a plane that fails to be absolutely continuous; see Friedman (1940) for details.

in such paths.[8] What Lemma 1-(b) states is that the indirect utility generated by an incentive compatible mechanism admits two-sided directional derivatives in the direction δ_1^2 a.e. in $L(\theta_1, \theta_2)$. We use this fact to derive a relationship between the right and left subderivatives of the valuation function at equilibrium points. Define the *right* and *left subderivative functions between* θ_1 *and* θ_2, respectively, by

$$\bar{s}(\theta_1^2(\alpha)) \ \equiv \ \bar{\mathrm{d}}v(X(\theta_1^2(\alpha)), \theta_1^2(\alpha); \delta_1^2), \qquad \text{all } \alpha \in (0,1), \tag{8}$$

and

$$\underline{s}(\theta_1^2(\alpha)) \ \equiv \ \underline{\mathrm{d}}v(X(\theta_1^2(\alpha)), \theta_1^2(\alpha); \delta_1^2), \qquad \text{all } \alpha \in (0,1). \tag{9}$$

The following technical assumption is required in our characterization of incentive compatible mechanisms.

(M) *Given an allocation rule $X \colon \Theta \to \mathscr{X}$, for every pair of distinct types $\theta_1, \theta_2 \in \Theta$, the right and left subderivative functions $\alpha \mapsto \bar{s}(\theta_1^2(\alpha))$ and $\alpha \mapsto \underline{s}(\theta_1^2(\alpha))$ between θ_1 and θ_2 are $\mathcal{B}(0,1)$-measurable.*

Our assumption (M), which is stated in terms of the valuation function and the allocation rule, may be easily verified in some circumstances (cf. Example 1, where for types $\theta \in \Theta = [0,1]$, $\bar{\mathrm{d}}v(X^*(\theta), \theta) = \theta$ and $\underline{\mathrm{d}}v(X^*(\theta), \theta) = 3\theta$). In Section 6 we show that (M) is satisfied in settings commonly employed in economic applications.

Using Equation 8 and Equation 9, we construct the *subderivative correspondence* $S\left(\theta_1^2(\cdot)\right) : (0,1) \rightrightarrows \mathbb{R}$ *between* θ_1 *and* θ_2 as follows:

$$S\left(\theta_1^2(\alpha)\right) \ \equiv \ \left\{ r \in \mathbb{R} \mid \bar{s}(\theta_1^2(\alpha)) \ \leq \ r \ \leq \ \underline{s}(\theta_1^2(\alpha)) \right\}, \qquad \alpha \in (0,1). \tag{10}$$

$S\left(\theta_1^2(\alpha)\right)$ is empty-valued if $\bar{s}(\theta_1^2(\alpha)) > \underline{s}(\theta_1^2(\alpha))$. Whenever the opposite inequality holds, $S\left(\theta_1^2(\alpha)\right)$ contains all the real numbers between the right and left subderivative of v with respect to types in the direction δ_1^2, evaluated at $\left(X(\theta_1^2(\alpha)), \theta_1^2(\alpha)\right)$.[9] The subderivative correspondence between θ_1 and θ_2 is said to be *regular* if it is non empty-valued a.e. in $(0,1)$, closed-valued, measurable and integrably bounded.

Lemma 2. *Assume that (A1) to (A3) and (M) are satisfied. If X is implementable, then for all $\theta_1, \theta_2 \in \Theta$, $\theta_1 \neq \theta_2$, the subderivative correspondence $S\left(\theta_1^2(\cdot)\right)$ between θ_1 and θ_2 is regular.*

Proof Suppose that $p \colon \Theta \to \mathbb{R}$ implements X. Fix arbitrary types $\theta_1, \theta_2 \in \Theta, \theta_1 \neq \theta_2$. For every $\theta_1^2(\alpha) \in L(\theta_1, \theta_2)$, for any scalar r sufficiently small, the indirect utility U

[8] See Krishna and Maenner (2001) for an example.

[9] Observe $S\left(\theta_2^1(\alpha)\right) = -\,S\left(\theta_1^2(1-\alpha)\right)$, all $\alpha \in (0,1)$.

generated by (X,p) satisfies

$$
\begin{aligned}
U(\theta_1^2(\alpha)+r\,\delta_1^2)\;-\;U(\theta_1^2(\alpha)) \;\geq\; & v(X(\theta_1^2(\alpha)),\theta_1^2(\alpha)+r\,\delta_1^2)\;-\;p(\theta_1^2(\alpha)) \\
& -\;v(X(\theta_1^2(\alpha)),\theta_1^2(\alpha))\;+\;p(\theta_1^2(\alpha)) \\
=\; & v(X(\theta_1^2(\alpha)),\theta_1^2(\alpha)+r\,\delta_1^2)\;-\;v(X(\theta_1^2(\alpha)),\theta_1^2(\alpha)).
\end{aligned}
$$

If $r>0$ then it follows from the above expression that

$$\frac{v(X(\theta_1^2(\alpha)),\theta_1^2(\alpha)+r\,\delta_1^2)-v(X(\theta_1^2(\alpha)),\theta_1^2(\alpha))}{r}\;\leq\;\frac{U(\theta_1^2(\alpha)+r\,\delta_1^2)-U(\theta_1^2(\alpha))}{r}, \tag{11}$$

whereas if $r<0$ we have instead

$$\frac{U(\theta_1^2(\alpha)+r\,\delta_1^2)-U(\theta_1^2(\alpha))}{r}\;\leq\;\frac{v(X(\theta_1^2(\alpha)),\theta_1^2(\alpha)+r\,\delta_1^2)-v(X(\theta_1^2(\alpha)),\theta_1^2(\alpha))}{r}. \tag{12}$$

By Lemma 1-(b), U admits two-sided directional derivatives in the direction δ_1^2 a.e. in $L(\theta_1,\theta_2)$. Thus, taking the lower limit as $r\downarrow 0$ in (11) and the upper limit as $r\uparrow 0$ in (12), we infer that a.e. in $(0,1)$ the following holds:

$$\overline{s}(\theta_1^2(\alpha))\;\leq\;DU(\theta_1^2(\alpha);\delta_1^2)\;\leq\;\underline{s}(\theta_1^2(\alpha)). \tag{13}$$

This shows that $S\left(\theta_1^2(\alpha)\right)\neq\emptyset$ a.e. in $(0,1)$, as desired.

$S\left(\theta_1^2(\cdot)\right)$ is closed-valued by definition. To show that it is measurable, define the correspondence $T\colon(0,1)\rightrightarrows\mathbb{R}$ by $T(\alpha)=\{\overline{s}(\theta_1^2(\alpha))\}\cup\{\underline{s}(\theta_1^2(\alpha))\}$ when Equation 13 is satisfied, and $T(\alpha)=\emptyset$, otherwise. Since the set $\{\,\alpha\in(0,1)\mid T(\alpha)=\emptyset\}$ has zero measure, we deduce from (M) that T is a measurable correspondence, and so is its convex hull $\operatorname{conv}(T)=S\left(\theta_1^2(\cdot)\right)$.[10] Further, notice that by (A2) Θ is bounded and that for every $x\in\mathscr{X}$, (A3) implies that $|\overline{d}v(x,\theta_1^2(\alpha);\delta_1^2)|\leq\ell\|\delta_1^2\|$ and similarly $|\underline{d}v(x,\theta_1^2(\alpha);\delta_1^2)|\leq\ell\|\delta_1^2\|$, for all $\alpha\in(0,1)$. Hence, $S\left(\theta_1^2(\cdot)\right)$ is integrably bounded. ■

The integral of the regular correspondence $S\left(\theta_1^2(\cdot)\right)$ is a non-empty, closed interval. In particular, $\overline{s}(\theta_1^2(\cdot))$ and $\underline{s}(\theta_1^2(\cdot))$ are integrable selections, with the inequalities

$$\int_0^1\overline{s}(\theta_1^2(\alpha))\,d\alpha\;\leq\;\int_0^1 s(\theta_1^2(\alpha))\,d\alpha\;\leq\;\int_0^1\underline{s}(\theta_1^2(\alpha))\,d\alpha$$

valid for any integrable selection $s(\theta_1^2(\cdot))$ of $S\left(\theta_1^2(\cdot)\right)$. We use this fact in our characterization result. Given any subset $\{\theta_1,\theta_2,\theta_3\}$ of Θ, denote $\delta_n^m\equiv\theta_m-\theta_n$ for $n,m=1,2,3$. It is understood that $\overline{s}(\theta_n^m(\cdot))=\underline{s}(\theta_n^m(\cdot))\equiv 0$ whenever $\theta_n=\theta_m$.

[10] Here we use two facts: (i) the union of measurable correspondences is measurable; (ii) the convex hull of a measurable correspondence is also measurable. See the references given in Section 2.2.

Theorem 1. *Assume (A1) to (A3) are satisfied. Suppose that (M) holds for $X\colon \Theta \to \mathscr{X}$. The following statements are then equivalent.*

(a) The allocation rule $X\colon \Theta \to \mathscr{X}$ is implementable, with indirect utility function U.

(b) For every subset $\{\theta_1, \theta_2, \theta_3\}$ of Θ, the subderivative correspondence $S\big(\theta_n^m(\cdot)\big)$ between θ_n and θ_m, $n, m = 1, 2, 3$, is regular and admits an integrable selection $\alpha \mapsto s(\theta_n^m(\alpha))$ that satisfies the integral monotonicity condition*:*

$$v(X(\theta_m), \theta_m) - v(X(\theta_m), \theta_n) \;\geq\; \int_0^1 s(\theta_n^m(\alpha))\, d\alpha \;\geq\; v(X(\theta_n), \theta_m) - v(X(\theta_n), \theta_n),$$

and the Mirrlees representation of its indirect utility U*:*

$$U(\theta_m) \;-\; U(\theta_n) \;=\; \int_0^1 s(\theta_n^m(\alpha))\, d\alpha.$$

Moreover, these selections satisfy the closed-path integrability condition*:*

$$\int_0^1 s(\theta_1^2(\alpha))\, d\alpha \;+\; \int_0^1 s(\theta_2^3(\alpha))\, d\alpha \;+\; \int_0^1 s(\theta_3^1(\alpha))\, d\alpha \;=\; 0.$$

Proof $(a) \implies (b)$ Fix a subset $\{\theta_1, \theta_2, \theta_3\}$ of Θ. From Lemma 2, one has that the correspondence $S\big(\theta_n^m(\cdot)\big)$ is regular, for $n, m = 1, 2, 3$. Let $\theta_n^m(\alpha) = \theta_n + \alpha\, \delta_n^m$ for $\alpha \in (0, 1)$. From Lemma 1-(b), the function μ_n^m defined on the unit interval by $\mu_n^m(\alpha) = U(\theta_n^m(\alpha))$ is absolutely continuous, with $D\mu_n^m(\alpha) = DU(\theta_n^m(\alpha); \delta_n^m)$ for almost all $\alpha \in (0, 1)$. Therefore, one has

$$\mu_n^m(1) - \mu_n^m(0) \;=\; U(\theta_m) - U(\theta_n) \;=\; \int_0^1 DU(\theta_n^m(\alpha); \delta_n^m)\, d\alpha.$$

This expression is combined with Equation 13 to obtain

$$\int_0^1 \overline{s}(\theta_n^m(\alpha))\, d\alpha \;\leq\; U(\theta_m) \;-\; U(\theta_n) \;\leq\; \int_0^1 \underline{s}(\theta_n^m(\alpha))\, d\alpha. \tag{14}$$

To obtain the Mirrlees representation, use the convexity of the integral of the subderivative correspondence $S\big(\theta_n^m(\cdot)\big)$, which gives us an integrable selection $s(\theta_n^m(\cdot))$ such that

$$\int_0^1 s(\theta_n^m(\alpha))\, d\alpha \;=\; U(\theta_m) \;-\; U(\theta_n). \tag{15}$$

From the proof of Lemma 1-(a), we notice that

$$v(X(\theta_m), \theta_m) \;-\; v(X(\theta_m), \theta_n) \;\geq\; U(\theta_m) \;-\; U(\theta_n) \;\geq\; v(X(\theta_n), \theta_m) \;-\; v(X(\theta_n), \theta_n).$$

This expression is combined with Equation 15 to obtain the integral monotonicity condition.

Clearly, we have $(U(\theta_2) - U(\theta_1)) + (U(\theta_3) - U(\theta_2)) + (U(\theta_1) - U(\theta_3)) = 0$. Therefore, using the selection $s(\theta_n^m(\cdot))$ from the subderivative correspondence $S\big(\theta_n^m(\cdot)\big)$ for each respective case, we immediately obtain the closed-path integrability condition.

$(b) \Longrightarrow (a)$ Fix a type $\theta_0 \in \Theta$. Define the payment rule $p\colon \Theta \to \mathbb{R}$ by

$$p(\theta_1) \;=\; v(X(\theta_1), \theta_1) \;-\; \int_0^1 s(\theta_0^1(\alpha))\, d\alpha, \qquad \text{for all } \theta_1 \in \Theta\,.$$

Here $s(\theta_n^m(\cdot))$ are integrable selections of the regular subderivative correspondences $S\big(\theta_n^m(\cdot)\big)$ for which the assumptions of the theorem are satisfied. We claim that (X, p) is incentive compatible. Indeed, for any $\theta_1, \theta_2 \in \Theta$, the payment difference is

$$\begin{aligned} p(\theta_2) - p(\theta_1) \;&=\; v(X(\theta_2), \theta_2) - v(X(\theta_1), \theta_1) + \int_0^1 s(\theta_0^1(\alpha))\, d\alpha + \int_0^1 s(\theta_2^0(\alpha))\, d\alpha \\ &=\; v(X(\theta_2), \theta_2) \;-\; v(X(\theta_1), \theta_1) \;-\; \int_0^1 s(\theta_1^2(\alpha))\, d\alpha, \end{aligned}$$

where the first equality follows from $s(\theta_2^0(\alpha)) \;=\; -\,s(\theta_0^2(1-\alpha))$, and the last equality follows from the closed-path integrability condition. Using this expression, we deduce from the integral monotonicity condition that

$$\begin{aligned} &\big\{v(X(\theta_1), \theta_1) \;-\; p(\theta_1)\big\} \;-\; \big\{v(X(\theta_2), \theta_1) \;-\; p(\theta_2)\big\} \\ &\quad=\; v(X(\theta_1), \theta_1) \;-\; v(X(\theta_2), \theta_1) \;+\; p(\theta_2) \;-\; p(\theta_1) \\ &\quad=\; v(X(\theta_2), \theta_2) \;-\; v(X(\theta_2), \theta_1) \;-\; \int_0^1 s(\theta_1^2(\alpha))\, d\alpha \;\geq\; 0. \end{aligned}$$

Hence, it follows that $v(X(\theta_1), \theta_1) - p(\theta_1) \geq v(X(\theta_2), \theta_1) - p(\theta_2)$. Since θ_1 and θ_2 were arbitrarily chosen, this shows that p implements X, as desired. ■

The closed-path integrability condition of our characterization theorem is trivially satisfied with one-dimensional type spaces. We notice that one can replace the global conditions in part (b) of Theorem 1 with their local versions, an approach that was introduced by Archer and Kleinberg (2008) for the case of linear valuations. This relies on the fact that for all $\theta_1, \theta_2, \theta_3$ in Θ, the convex hull of the closure of the line segments $L(\theta_n, \theta_m)$ $(n, m \;=\; 1, 2, 3)$ is a compact subset of $\mathbb{R}^k$. Thus, one can replace an open cover of any such set with a finite subcover to obtain the desired conditions. This is formally stated in the next proposition, whose proof can be adapted from the arguments of Lemmas 3.2 and 3.5 in Archer and Kleinberg (2008).

Proposition 1. *Assume (A1) to (A3) are satisfied. Assume in addition that (M) is satisfied for the allocation rule $X\colon \Theta \to \mathscr{X}$. The following are equivalent:*

(a) The allocation rule $X\colon \Theta \to \mathscr{X}$ is implementable.

(b) *For each $\theta_1 \in \Theta$, there exists an open neighborhood O of θ_1 such that for all $\theta_2, \theta_3 \in O \cap \Theta$, for $n, m = 1, 2, 3$, the subderivative correspondence $S\big(\theta_n^m(\cdot)\big)$ is regular and admits an integrable selection $s(\theta_n^m(\cdot))$ satisfying the local integral monotonicity condition:*

$$v(X(\theta_m), \theta_m) - v(X(\theta_m), \theta_n) \;\geq\; \int_0^1 s(\theta_n^m(\alpha))\, d\alpha \;\geq\; v(X(\theta_n), \theta_m) - v(X(\theta_n), \theta_n).$$

Further, the selections $s(\theta_n^m(\cdot))$ satisfy the local closed-path integrability condition:

$$\int_0^1 s(\theta_1^2(\alpha))\, d\alpha \;+\; \int_0^1 s(\theta_2^3(\alpha))\, d\alpha \;+\; \int_0^1 s(\theta_3^1(\alpha))\, d\alpha \;=\; 0.$$

Example 1 (continued). The regular subderivative correspondence for the efficient allocation rule X^* is $S(\theta) = [\theta, 3\theta]$, all $\theta \in \Theta = [0, 1]$. Since $X^*(\theta) = \theta$, integral monotonicity is expressed as

$$\theta^2 + \theta\hat{\theta} - 2\hat{\theta}^2 \;\geq\; \int_{\hat{\theta}}^{\theta} s(\tilde{\theta})\, d\tilde{\theta} \;\geq\; \theta\hat{\theta} - \hat{\theta}^2, \qquad 0 \leq \hat{\theta} < \theta \leq 1.$$

Letting $\hat{\theta} = 0$ in the above expression, one sees that for each type $\theta \in \Theta$, it must be that $\theta^2 \geq \int_0^{\theta} s(\tilde{\theta})\, d\tilde{\theta} \geq 0$. It follows that here *any* integrable selection $\theta \mapsto s(\theta)$ of the subderivative correspondence S satisfying $\theta \leq s(\theta) \leq 2\theta$, all $\theta \in \Theta$, can be employed to construct a payment rule that implements X^*. $\square$

4 A Revenue Inequality

In our general setting, the revenue associated with a given allocation rule may not be uniquely determined up to a constant. However, one important feature that resembles the revenue equivalence principle is preserved: from Theorem 1, it follows that the *range* of indirect utilities is determined by the allocation rule alone. In this section, we focus on normalized payment rules that generate an indirect utility of $u_0 \in \mathbb{R}$ for the "lowest type" $\theta_0 \in \Theta$. One has the following *revenue inequality*.

Theorem 2. *Assume (A1) to (A3) hold and let $X\colon \Theta \to \mathscr{X}$ be an allocation rule for which (M) is satisfied. Suppose that $p\colon \Theta \to \mathbb{R}$ and $p'\colon \Theta \to \mathbb{R}$ implement X and let U and U' denote the indirect utility functions generated by (X, p) and (X, p'), respectively, with $U(\theta_0) = U'(\theta_0) = u_0$. Then for all $\theta_1 \in \Theta$:*

$$|\, U(\theta_1) \;-\; U'(\theta_1)\, | \;\leq\; \int_0^1 \big\{ \underline{s}(\theta_0^1(\alpha)) \;-\; \overline{s}(\theta_0^1(\alpha)) \big\}\, d\alpha. \tag{16}$$

Proof By Theorem 1, both indirect utilities U and U' satisfy Equation 14 for types

$\theta_1 = \theta_m$ and $\theta_0 = \theta_n$. We deduce the following inequalities:

$$\begin{aligned} \int_0^1 \overline{s}(\theta_0^1(\alpha))\, d\alpha \;&\leq\; U(\theta_1) - U(\theta_0) \;\leq\; \int_0^1 \underline{s}(\theta_0^1(\alpha))\, d\alpha, \qquad \text{and} \\ -\int_0^1 \underline{s}(\theta_0^1(\alpha))\, d\alpha \;&\leq\; U'(\theta_0) - U'(\theta_1) \;\leq\; -\int_0^1 \overline{s}(\theta_0^1(\alpha))\, d\alpha. \end{aligned}$$

Since $U(\theta_0) = U'(\theta_0) = u_0$, Equation 16 follows by adding up these expressions. ■

Theorem 2 implies that, given two incentive compatible mechanisms that share an allocation rule and generate equilibrium payoff of u_0 to type θ_0, the difference in the indirect utility functions is pinned down by solely the allocation rule, since X alone determines the right and left subderivative functions of Equation 16.

Example 1 (continued). The constant payment rule $p \equiv 0$ implements X^* and generates an indirect utility U with $U(\theta) = \theta^2$, all $\theta \in \Theta$. In addition, the payment rule p' defined by $p'(\theta) = \theta^2/2$ implements X^* and generates an indirect utility U' with $U'(\theta) = \theta^2/2$, all $\theta \in \Theta$. Immediately, for every θ in Θ,

$$\tfrac{1}{2}\theta^2 \;=\; \left|U(\theta) - U'(\theta)\right| \;\leq\; \int_0^\theta \left\{\underline{s}(\tilde{\theta}) - \overline{s}(\tilde{\theta})\right\} d\tilde{\theta} \;=\; \theta^2. \qquad \square$$

Suppose that (X, p) is an incentive compatible mechanism, with associated indirect utility $U\colon \Theta \to \mathbb{R}$. Using the Mirrlees representation of U given in Theorem 1, we infer that for $\theta_0 \in \Theta$, for all $\theta_1 \in \Theta$:

$$U(\theta_1) \;\equiv\; v(X(\theta_1), \theta_1) \;-\; p(\theta_1) \;=\; \int_0^1 s(\theta_0^1(\alpha))\, d\alpha \;+\; U(\theta_0),$$

where $s(\theta_0^1(\cdot))$ is one (of the possibly many) integrable selection of the subderivative correspondence $S\left(\theta_0^1(\cdot)\right)$ between θ_0 and θ_1 that satisfies integral monotonicity and closed-path integrability. Normalizing payments so that $U(\theta_0) = u_0$, we obtain the following result.

Proposition 2. *Assume (A1) to (A3) and (M) hold. Suppose that (X, p) is incentive compatible, with associated indirect utility function $U\colon \Theta \to \mathbb{R}$ satisfying $U(\theta_0) = u_0$. Then, for every $\theta_1 \in \Theta$, there exists an integrable selection $s(\theta_0^1(\cdot))$ of the subderivative correspondence $S\left(\theta_0^1(\cdot)\right)$ between θ_0 and θ_1 such that*

$$p(\theta_1) \;=\; v(X(\theta_1), \theta_1) \;-\; \int_0^1 s(\theta_0^1(\alpha))\, d\alpha \;-\; u_0. \tag{17}$$

Thus, every payment rule p that implements X can be expressed via Equation 17. Consider now two payment rules that are used to implement X, namely p' and p'', with corresponding selections $s'(\theta_0^1(\cdot))$ and $s''(\theta_0^1(\cdot))$ from subderivative correspondence $S\left(\theta_0^1(\cdot)\right)$ between θ_0 and θ_1, for all $\theta_1 \in \Theta$. Fix $0 \leq \lambda \leq 1$. From the convexity of the integral of the subderivative correspondence, for every $\theta_1 \neq \theta_0$ there exists an integrable selection $\alpha \mapsto s^\lambda(\theta_0^1(\alpha))$ such that

$$\int_0^1 s^\lambda(\theta_0^1(\alpha))\,d\alpha \;=\; \lambda \int_0^1 s'(\theta_0^1(\alpha))\,d\alpha \;+\; (1-\lambda)\int_0^1 s''(\theta_0^1(\alpha))\,d\alpha.$$

Clearly, $s^\lambda(\theta_0^1(\cdot))$ satisfies the integral monotonicity and closed-path integrability conditions. Thus, the payment rule $p^\lambda\colon \Theta \to \mathbb{R}$ defined by $p^\lambda(\theta) = \lambda p'(\theta) + (1-\lambda)p''(\theta)$, all $\theta \in \Theta$, implements X as well.

Corollary 1. *If U' and U'' are indirect utility functions generated by two incentive compatible mechanisms sharing the allocation rule X and assigning $U'(\theta_0) = U''(\theta_0) = u_0$ to $\theta_0 \in \Theta$, then for every $\lambda \in [0,1]$ there exists an incentive compatible mechanism (X, p^λ) such that $U^\lambda = \lambda U' + (1-\lambda)U''$.*

5 Applications

We apply our techniques and characterization result to two economic settings: a public goods problem, where we look for efficient implementation, and a buyer-seller situation with a loss-averse buyer, where we address revenue maximization.

When the revenue equivalence principle is in place, the efficient allocation rule is implementable if, and only if, it is implementable via a Vickrey-Clarke-Groves (VCG) payment scheme. It is well known that with sufficiently rich type spaces, generally VCG payments do not balance the budget ex post.[11] We consider a public goods setting where revenue equivalence fails and show that efficient allocations can be implemented with a budget-balanced payment rule that is, in addition, individually rational, and yet no VCG payment balances the budget.

In the buyer-seller model, the buyer has reference-dependent preferences for some good. In particular, the buyer exhibits loss aversion à la Kőszegi and Rabin (2006) which, due to a kink of the valuation at the reference point, fail standard requirements of differentiability or convexity. Nevertheless, our characterization allows us to follow Myerson's (1981) approach and reformulate the seller's maximization problem in terms of virtual valuations. In the optimal selling mechanism, a range of intermediate types purchase their reference quantity, so that the flexibility of pricing rules afforded by our characterization plays an important role. Compared to the optimal selling mechanism without loss aversion, some of these intermediate types have their quantities distorted downward to exploit the loss aversion of higher types, whose incentive constraints are thereby slackened and whose payments are correspondingly increased. On the other hand, the loss aversion of the lower end of this intermediate type range is exploited by making them pay a premium to increase their quantity from zero to their reference

[11] The classic reference is Green and Laffont (1979).

point. It follows that expected revenue generated by the the optimal selling mechanism under loss aversion is higher than expected revenue generated by its counterpart for the case without loss aversion.

5.1 Providing a public good with a balanced budget

The set of public alternatives is $\mathscr{X} = [0,1]$. There are two ex ante identical agents, A and B, with publicly known identities. Each agent $i \in \{A, B\}$ has a type space $\Theta^i = [0,1]$ and a valuation function $v^i \colon \mathscr{X} \times \Theta^i \to \mathbb{R}$ given by:

$$v^i(x, \theta^i) \;=\; 1 - |x - \theta^i|.$$

Think of $\mathscr{X}$ as the set of possible locations for a hospital, library, or other public facility. Agent i resides at location $\theta^i \in [0,1]$ and pays a linear cost to travel to the public facility, were it not situated at θ^i. We abuse notation and write $\boldsymbol{\theta} = (\theta^i, \theta^j)$ to indicate a type profile where $\theta^i \in \Theta^i$ and $\theta^j \in \Theta^j$, for $i, j \in \{A, B\}$, $i \neq j$. The cost of locating the public facility somewhere in the unit interval is represented by the differentiable cost function $c \colon \mathscr{X} \to \mathbb{R}$, with $0 < c'(x) < 2$ for all $x \in \mathscr{X}$. The efficient allocation rule $\boldsymbol{\theta} \mapsto X^*(\boldsymbol{\theta}) = \min\{\theta^A, \theta^B\}$ selects the location that maximizes social welfare $v^A(x, \theta^A) + v^B(x, \theta^B) - c(x)$.

Fix a report $\theta^j \in \Theta^j$. If $\theta^i \leq \theta^j$ is truthfully reported, then $X^*(\boldsymbol{\theta}) = \theta^i$ and thus the right and left subderivatives of $v^i(x, \cdot)$ with respect to types evaluated at $(X^*(\boldsymbol{\theta}), \theta^i)$ are $\overline{\mathrm{d}}v^i(X^*(\boldsymbol{\theta}), \theta^i) = -1$ and $\underline{\mathrm{d}}v^i(X^*(\boldsymbol{\theta}), \theta^i) = 1$. If $\theta^i > \theta^j$ is reported instead, $X^*(\boldsymbol{\theta}) = \theta^j$ and thus $\overline{\mathrm{d}}v^i(X^*(\boldsymbol{\theta}), \theta^i) = \underline{\mathrm{d}}v^i(X^*(\boldsymbol{\theta}), \theta^i) = -1$. Note that (M) is satisfied. Thus, agent i's subderivative correspondence $S^i(\cdot, \theta^j) \colon \Theta^i \rightrightarrows \mathbb{R}$ is

$$S^i(\theta^i, \theta^j) \;=\; \begin{cases} [-1, 1], & \text{if } \theta^i \leq \theta^j; \\ -1, & \text{if } \theta^i > \theta^j. \end{cases} \tag{18}$$

Clearly, $S^i(\cdot, \theta^j)$ is regular. Agent i's integral monotonicity condition is expressed as follows: for all $\hat{\theta}^i < \theta^i \in \Theta^i$,

$$\begin{aligned} v^i(X^*(\theta^i, \theta^j), \theta^i) - v^i(X^*(\theta^i, \theta^j), \hat{\theta}^i) \;&\geq\; \int_{\hat{\theta}^i}^{\theta^i} s^i(\tilde{\theta}^i, \theta^j)\, d\tilde{\theta}^i \\ &\geq\; v^i(X^*(\hat{\theta}^i, \theta^j), \theta^i) - v^i(X^*(\hat{\theta}^i, \theta^j), \hat{\theta}^i), \end{aligned} \tag{19}$$

for an integrable selection $s^i(\cdot, \theta^j)$ of $S^i(\cdot, \theta^j)$. We claim that *any* selection satisfies Equation 19. Indeed, for $\hat{\theta}^i < \theta^i \leq \theta^j$, (19) becomes

$$\theta^i - \hat{\theta}^i \;\geq\; \int_{\hat{\theta}^i}^{\theta^i} s^i(\tilde{\theta}^i, \theta^j)\, d\tilde{\theta}^i \;\geq\; \hat{\theta}^i - \theta^i,$$

a condition satisfied for any selection $\tilde{\theta}^i \mapsto s^i(\tilde{\theta}^i, \theta^j) \in [-1, 1]$ for all $\hat{\theta}^i \leq \tilde{\theta}^i \leq \theta^i$.

Suppose that instead one has $\hat{\theta}^i \leq \theta^j < \theta^i$. In this case, $X^*(\hat{\theta}^i, \theta^j) = \hat{\theta}^i$ and $X^*(\theta^i, \theta^j) = \theta^j$. Thus, (19) is now

$$(\theta^j - \hat{\theta}^i) - (\theta^i - \theta^j) \;\geq\; \int_{\hat{\theta}^i}^{\theta^j} s^i(\tilde{\theta}^i, \theta^j)\, d\tilde{\theta}^i \;+\; \int_{\theta^j}^{\theta^i} s^i(\tilde{\theta}^i, \theta^j)\, d\tilde{\theta}^i \;\geq\; \hat{\theta}^i - \theta^i,$$

which holds for any selection $s^i(\cdot, \theta^j)$ of $S^i(\cdot, \theta^j)$ such that $\tilde{\theta}^i \mapsto s^i(\tilde{\theta}^i, \theta^j) \in [-1, 1]$ for $\hat{\theta}^i \leq \tilde{\theta}^i \leq \theta^j$ and $\tilde{\theta}^i \mapsto s^i(\tilde{\theta}^i, \theta^j) = -1$ for $\theta^j \leq \tilde{\theta}^i \leq \theta^i$. Finally, if $\theta^j < \hat{\theta}^i < \theta^i$, then $X^*(\hat{\theta}^i, \theta^j) = X^*(\theta^i, \theta^j) = \theta^j$ and both valuation differences in (19) are equal to $\hat{\theta}^i - \theta^i$. Integral monotonicity is satisfied for $\tilde{\theta}^i \mapsto s^i(\tilde{\theta}^i, \theta^j) = -1$, for all types $\hat{\theta}^i \leq \tilde{\theta}^i \leq \theta^i$.

It follows from Theorem 1 and Proposition 2 that any payment rule implementing X^* can be constructed using an integrable selection of the subderivative correspondence in Equation 18. Normalizing payments so that $\theta^i = 0$ obtains payoffs equal to 1, agent i's payments take the form:

$$p^i(\boldsymbol{\theta}) \;=\; v^i(X^*(\boldsymbol{\theta}), \theta^i) \;-\; \int_0^{\theta^i} s^i(\tilde{\theta}^i, \theta^j)\, d\tilde{\theta}^i \;-\; 1 \;=\; -\int_0^{\min\{\theta^i, \theta^j\}} s^i(\tilde{\theta}^i, \theta^j)\, d\tilde{\theta}^i. \tag{20}$$

Hence, there exists a payment rule $p = (p^A, p^B)\colon \Theta^A \times \Theta^B \to \mathbb{R}^2$ that implements X^* and balances the budget ex post if, and only if, there are selections s^A and s^B of the subderivative correspondences S^A and S^B such that, for every $\boldsymbol{\theta}$,

$$\int_0^{\min\{\theta^A, \theta^B\}} s^A(\tilde{\theta}^A, \theta^B)\, d\tilde{\theta}^A \;+\; \int_0^{\min\{\theta^A, \theta^B\}} s^B(\tilde{\theta}^B, \theta^A)\, d\tilde{\theta}^B \;+\; c(X^*(\boldsymbol{\theta})) \;=\; 0.$$

We stress that single-peak preferences for the public good is not an essential feature of our stylized model. What matters is that, at efficient allocations, the subderivative correspondence of at least one agent is not a singleton, thus revenue equivalence fails. Hence, it may be possible to balance the budget using non VCG payment schemes.

Example 2. Consider a linear cost function $x \mapsto c(x) = x$. We first claim that no VCG payment can be budget-balanced. Indeed, any VCG payment function p^i_g for agent $i \in \{A, B\}$ takes the form $p^i_g(\boldsymbol{\theta}) \;=\; h^i(\theta^j) - v^j(X^*(\boldsymbol{\theta}), \theta^j) + c(X^*(\boldsymbol{\theta})) \;=\; h^i(\theta^j) - 1 + \theta^j$, where h^i is an auxiliary function defined on Θ^j ($j \neq i$). Suppose, to obtain a contradiction, that there are functions h^i on Θ^j and h^j on Θ^i for which the expression

$$p^A_g(\boldsymbol{\theta}) + p^B_g(\boldsymbol{\theta}) - c(X^*(\boldsymbol{\theta})) \;=\; h^A(\theta^B) + h^B(\theta^A) - 2 + \theta^A + \theta^B - \min\{\theta^A, \theta^B\}$$

vanishes at every profile $\boldsymbol{\theta}$. In particular, for $\theta^B = 1$, one has that $h^A(1) + h^B(\theta^A) = 1$ must hold for all $\theta^A \in \Theta^A$. This implies that h^B is constant on Θ^A. A similar argument applies to h^A; thus $h^i(\theta^j) \equiv h^i$ for $i \in \{A, B\}$. Balancing the budget requires that $h^A + h^B - 2 + \theta^A + \theta^B - \min\{\theta^A, \theta^B\} \;=\; 0$ be satisfied for all $\boldsymbol{\theta}$, which is impossible.

We now use the subderivative correspondence of Equation 18 to construct non VCG payments to implement X^*. Consider first the integrable selection s^i_e of S^i $(i = A, B)$ for which $s^i_e(\boldsymbol{\theta}) = -\frac{1}{2}$ for all $\theta^i \le \theta^j$, and $s^i_e(\boldsymbol{\theta}) = -1$ for all $\theta^i > \theta^j$. This selection yields a payment rule $p_e = (p^A_e, p^B_e)$ that implements X^* in dominant strategies, balances the budget ex post, is ex post individually rational, and is egalitarian in that it does not discriminate ex ante between agents. Indeed, replacing s^i_e in Equation 20 we obtain $p^i_e(\boldsymbol{\theta}) = \frac{1}{2}\min\{\theta^i, \theta^j\}$, for all $\boldsymbol{\theta}$ and $i \in \{A, B\}$. Immediately,

$$p^A_e(\boldsymbol{\theta}) + p^B_e(\boldsymbol{\theta}) \;=\; \min\{\theta^A, \theta^B\} \;=\; c(X^*(\boldsymbol{\theta})), \qquad \text{all } \boldsymbol{\theta} \in \Theta^A \times \Theta^B.$$

To verify that (X^*, p_e) is individually rational, notice that agent i's indirect utility is

$$U^i_e(\boldsymbol{\theta}) \;=\; v^i(X^*(\boldsymbol{\theta}), \theta^i) - p^i_e(\boldsymbol{\theta}) \;=\; \begin{cases} 1 - \frac{1}{2}\theta^i, & \text{if } \theta^i \le \theta^j; \\ 1 - \theta^i + \frac{1}{2}\theta^j, & \text{if } \theta^i > \theta^j. \end{cases}$$

Consider now an alternative payment rule $p_d = (p^A_d, p^B_d)$ defined by Equation 20 using the following selections: $s^A_d(\boldsymbol{\theta}) = -1$ for all $\boldsymbol{\theta}$, and $s^B_d(\boldsymbol{\theta}) = 0$ for $\theta^B \le \theta^A$ and $s^B_d(\boldsymbol{\theta}) = -1$ for $\theta^B > \theta^A$. We claim that (X^*, p_d) is incentive compatible, ex post budget-balanced and ex post individually rational, but discriminate against A. Indeed, $p_d = (p^A_d, p^B_d)$ is defined by

$$p^A_d(\boldsymbol{\theta}) \;=\; \min\{\theta^A, \theta^B\}, \qquad \text{and} \qquad p^B_d(\boldsymbol{\theta}) \;=\; 0, \qquad \text{all } \boldsymbol{\theta} \in \Theta^A \times \Theta^B.$$

The reader can verify that (X^*, p_d) is budget-balanced and individually rational, but agent A bears the entire cost of the public good. Indeed, A is uniformly worse off, and B uniformly better off, under the discriminatory regime: $U^A_d(\boldsymbol{\theta}) - U^A_e(\boldsymbol{\theta}) = -\frac{1}{2}\min\{\theta^A, \theta^B\}$, while $U^B_d(\boldsymbol{\theta}) - U^B_e(\boldsymbol{\theta}) = \frac{1}{2}\min\{\theta^A, \theta^B\}$. $\square$

5.2 Selling to a buyer with reference-dependent preferences

A seller and a buyer are negotiating the quantity and price of a homogenous good. The seller produces $x \in \mathscr{X} = [0, \bar{x}]$ at a constant zero marginal cost. The buyer has a privately known type $\theta \in \Theta = [0, 1]$, and his gross valuation for x is given by θx. The buyer has reference-dependent preferences: he enters negotiations with a reference point $y \in \mathscr{X}$ and evaluates deviations from y differently, depending on whether they are gains or losses. Following Kőszegi and Rabin (2006), we model the buyer's valuation for x, given θ and y, as

$$v(x, \theta, y) \;=\; \theta x \,+\, \mu(x, y),$$

where the gain-loss utility $\mu(x, y)$ is given by

$$\mu(x, y) \;=\; \gamma\,[x-y]^{+} \;+\; (\gamma+\lambda)\,[x-y]^{-}$$

and $\gamma \geq 0$ and $\lambda > 0$. Here $[\alpha]^{+} = \max\{\alpha, 0\}$ and $[\alpha]^{-} = \min\{\alpha, 0\}$ denote the positive and negative parts, respectively. Thus, the buyer is loss-averse: losses relative to the reference point have a larger marginal impact than gains.[12] We assume that y is a linear function of the buyer's type, so that $\theta \mapsto y(\theta) = \theta\bar{x}$. In this formulation, the buyer's valuation function $v\colon \mathscr{X} \times \Theta \to \mathbb{R}$ is given by

$$v(x, \theta) \;=\; \begin{cases} \theta x \,+\, \gamma(x - \theta\bar{x}), & \text{if } x \geq \theta\bar{x}; \\ \theta x \,+\, (\gamma+\lambda)(x - \theta\bar{x}), & \text{if } x \leq \theta\bar{x}. \end{cases}$$

The buyer has the option not to participate in the transaction and thereby obtaining zero quantity and making no payment; this gives type θ a reservation value of $v(0, \theta) = -(\gamma+\lambda)\theta\bar{x}$.[13] The seller has a prior belief about θ represented by the distribution function F over domain Θ, with continuous, strictly positive density f. The seller chooses an incentive compatible and individually rational selling mechanism (X, p) to maximize expected payments from the buyer.

Let $X\colon \Theta \to \mathscr{X}$ be a selling rule. One has $\bar{\mathrm{d}}v(X(\theta), \theta) = \underline{\mathrm{d}}v(X(\theta), \theta) = X(\theta) - \gamma\bar{x}$ when $\theta\bar{x} < X(\theta)$, $\bar{\mathrm{d}}v(X(\theta), \theta) = \underline{\mathrm{d}}v(X(\theta), \theta) = X(\theta) - (\gamma+\lambda)\bar{x}$ when $\theta\bar{x} > X(\theta)$, and $\bar{\mathrm{d}}v(X(\theta), \theta) = X(\theta) - (\gamma+\lambda)\bar{x} < X(\theta) - \gamma\bar{x} = \underline{\mathrm{d}}v(X(\theta), \theta)$ when $\theta\bar{x} = X(\theta)$. In this case, (M) is satisfied whenever X is measurable (cf. Section 6.1). The subderivative correspondence $S\colon \Theta \rightrightarrows \mathbb{R}$ is defined by

$$S(\theta) \;=\; \begin{cases} X(\theta) - \gamma\bar{x}, & \text{if } X(\theta) > \theta\bar{x}; \\ X(\theta) - (\gamma+\lambda)\bar{x}, & \text{if } X(\theta) < \theta\bar{x}; \\ \big[X(\theta) - (\gamma+\lambda)\bar{x}, X(\theta) - \gamma\bar{x}\big], & \text{if } X(\theta) = \theta\bar{x}. \end{cases} \tag{21}$$

From Proposition 2, we restrict the analysis to payment rules of the form $p(\theta) = v(X(\theta), \theta) - \int_0^\theta s(\tilde{\theta})\, d\tilde{\theta} - u_0$, with u_0 indicating the payoff of type $\theta = 0$ and $\theta \mapsto s(\theta)$ is an integrable selection of S. Clearly, an optimal payment rule chooses $u_0 = 0$. Thus, the seller's expected profits are

$$\Pi^E \;=\; \int_0^1 v(X(\theta), \theta) f(\theta)\, d\theta \;-\; \int_0^1 \int_0^\theta s(\tilde{\theta})\, d\tilde{\theta}\, f(\theta)\, d\theta.$$

Integrating by parts the second term of the right-hand side of the above expression and

[12] Eisenhuth (2010) considers the optimal auction to many loss-averse buyers. Unlike selling to a single buyer, auctions naturally generate random allocations to individual bidders, and sufficient smoothness in the distribution of types restores differentiability and allows the use of standard techniques.

[13] Thus, the reservation value is type-dependent.

rearranging, we obtain

$$\Pi^E \ = \ \int_0^1 \left[v(X(\theta),\theta) - \frac{1-F(\theta)}{f(\theta)} s(\theta) \right] f(\theta)\, d\theta. \tag{22}$$

The seller's problem is to choose $X\colon \Theta \to \mathscr{X}$ together with an integrable selection of the subderivative correspondence in Equation 21 to maximize expected revenue Π^E, subject to the incentive compatibility and participation constraints. Let us ignore these constraints for the moment. Pointwise maximization of Equation 22 requires choosing the selection $\bar{s}$ with values $\bar{s}(\theta) = X(\theta) - \gamma\bar{x}$ for $X(\theta) > \theta\bar{x}$, and $\bar{s}(\theta) = X(\theta) - (\gamma+\lambda)\bar{x}$ for $X(\theta) \le \theta\bar{x}$. To simplify notation, express the inverse hazard rate as $\phi(\theta) = [1 - F(\theta)]/f(\theta)$; as usual, we assume that ϕ is a decreasing function. Thus, for every $\theta \in \Theta$, virtual surplus $VS(\theta) \equiv v(X(\theta),\theta) - \phi(\theta)\bar{s}(\theta)$ takes the form

$$VS(\theta) \ = \ \begin{cases} \big(\theta - \phi(\theta) + \gamma\big)X(\theta) \ - \ \gamma\big(\theta - \phi(\theta)\big)\bar{x}, & \text{if } X(\theta) > \theta\bar{x}; \\ \big(\theta - \phi(\theta) + \gamma + \lambda\big)X(\theta) \ - \ (\gamma+\lambda)\big(\theta - \phi(\theta)\big)\bar{x}, & \text{if } X(\theta) \le \theta\bar{x}. \end{cases}$$

It follows that $VS(\theta)$ is maximized by choosing a quantity equal to 0, to $\theta\bar{x}$, or $\bar{x}$; which is the case will vary with θ and ϕ. For suitable parameters, i.e., for $1/f(0) > \gamma + \lambda$, there exist cutoff types θ' and θ'', with $0 < \theta' < \theta'' < 1$, such that $\theta - \phi(\theta) + \gamma + \lambda \ge 0$ for all $\theta \ge \theta'$ and $\theta - \phi(\theta) + \gamma + \lambda < 0$ for all $\theta < \theta'$, and similarly $\theta - \phi(\theta) + \gamma \ge 0$ for all $\theta \ge \theta''$ and $\theta - \phi(\theta) + \gamma < 0$ for all $\theta < \theta''$. We distinguish three cases. In the first case, $0 < \theta < \theta'$ and the optimal solution is therefore $X^o(\theta) = 0$. In the second case, $\theta' \le \theta < \theta''$ and the optimal solution is $X^o(\theta) = \theta\bar{x}$. The analysis of the third case, when $\theta'' \le \theta \le 1$, is slightly more complicated as the optimal selling rule depends on which of the two feasible solutions, $\bar{x}$ and $\theta\bar{x}$, generates higher virtual surplus. Taking differences, we obtain

$$VS(\theta|\bar{x}) - VS(\theta|\theta\bar{x}) \ = \ (\theta - \phi(\theta) + \gamma)(1-\theta)\bar{x} \ - \ \lambda\phi(\theta)\bar{x}. \tag{23}$$

For purposes of exposition we now specialize to an example where this difference can easily be signed and a simple expression for the optimal allocation rule can be derived.

Example 3. Let $\gamma = 0$, $0 < \lambda < 1$ so that only losses relative to the reference point enter μ, and suppose that types are distributed uniformly on $\Theta = [0,1]$. In this case $\theta' = \frac{1-\lambda}{2}$, $\theta'' = \frac{1}{2}$, and the difference in Equation 23 equals $(2\theta - 1 - \lambda)(1-\theta)\bar{x}$. Thus, the optimal selling rule is

$$X^o(\theta) \ = \ \begin{cases} 0, & \text{if } 0 \le \theta < \frac{1-\lambda}{2}; \\ \theta\bar{x}, & \text{if } \frac{1-\lambda}{2} \le \theta < \frac{1+\lambda}{2}; \\ \bar{x}, & \text{if } \frac{1+\lambda}{2} \le \theta \le 1. \end{cases} \tag{24}$$

Notice that X^o is strictly increasing for intermediate types. Contrast this with the case of no loss aversion ($\lambda = 0$), where the optimal selling rule gives 0 to every type $\theta < \frac{1}{2}$, and $\bar{x}$ to every type $\theta \geq \frac{1}{2}$. The integrable selection $\bar{s}$ is therefore

$$\bar{s}(\theta) \;=\; \begin{cases} -\lambda\bar{x}, & \text{if } 0 \leq \theta < \frac{1-\lambda}{2}; \\ (\theta-\lambda)\bar{x}, & \text{if } \frac{1-\lambda}{2} \leq \theta < \frac{1+\lambda}{2}; \\ \bar{x}, & \text{if } \frac{1+\lambda}{2} \leq \theta \leq 1. \end{cases}$$

It is not difficult to verify that the integral monotonicity condition holds. For instance, for $0 \leq \hat{\theta} < \frac{1-\lambda}{2} \leq \theta < \frac{1+\lambda}{2}$, we have $v(X^o(\theta),\theta) - v(X^o(\theta),\hat{\theta}) = \theta(\theta-\hat{\theta})\bar{x}$, while $v(X^o(\hat{\theta}),\theta) - v(X^o(\hat{\theta}),\hat{\theta}) = -\lambda(\theta-\hat{\theta})\bar{x}$, and

$$\int_{\hat{\theta}}^{\theta} \bar{s}(\tilde{\theta})\, d\tilde{\theta} \;=\; -\lambda(\theta-\hat{\theta})\bar{x} \,+\, \tfrac{1}{2}\left(\theta^2 - \left(\tfrac{1-\lambda}{2}\right)^2\right)\bar{x}.$$

From these expression the integral monotonicity follows readily. Using Theorem 1 we deduce that X^o is implementable. The optimal selling mechanism is (X^o, p^o), where

$$p^o(\theta) \;=\; \begin{cases} 0, & \text{if } 0 \leq \theta < \frac{1-\lambda}{2}; \\ \frac{\bar{x}}{2}\left(\theta^2 + 2\lambda\theta + (\frac{1-\lambda}{2})^2\right), & \text{if } \frac{1-\lambda}{2} \leq \theta < \frac{1+\lambda}{2}; \\ \frac{\bar{x}}{2}\left(\lambda^2 + \lambda + 1\right), & \text{if } \frac{1+\lambda}{2} \leq \theta \leq 1. \end{cases}$$

The reader can verify using the Mirrlees representation that the optimal selling mechanism (X^o, p^o) generates an indirect utility function U^o equal to

$$U^o(\theta) \;=\; \begin{cases} -\lambda\theta\bar{x}, & \text{if } 0 \leq \theta < \frac{1-\lambda}{2}; \\ -\lambda\theta\bar{x} + \frac{\bar{x}}{2}\left(\theta^2 - (\frac{1-\lambda}{2})^2\right), & \text{if } \frac{1-\lambda}{2} \leq \theta < \frac{1+\lambda}{2}; \\ -\lambda\theta\bar{x} + \frac{\bar{x}}{2}\left(2\theta(1+\lambda) - \lambda^2 - \lambda - 1\right), & \text{if } \frac{1+\lambda}{2} \leq \theta \leq 1. \end{cases}$$

It is immediate to conclude that the participation constraints are satisfied. $\square$

6 Discussion

We return to the single agent environment described in Section 2.1 to relate our techniques and results to recent developments in the literature.

6.1 When is (M) satisfied?

In some applications our assumption (M) is readily verified. More importantly, (M) will follow directly from the structure of the design problem in commonly used environments. Consider, as Archer and Kleinberg (2008), a setting with a bounded allocation set $\mathscr{X} \subseteq \mathbb{R}^k$, a bounded convex type space $\Theta \subseteq \mathbb{R}^k$, and valuation that are linear in types, so that $\theta \mapsto v(x,\theta) = x \cdot \theta$ for each allocation x. Notice that (A1) to (A3) are

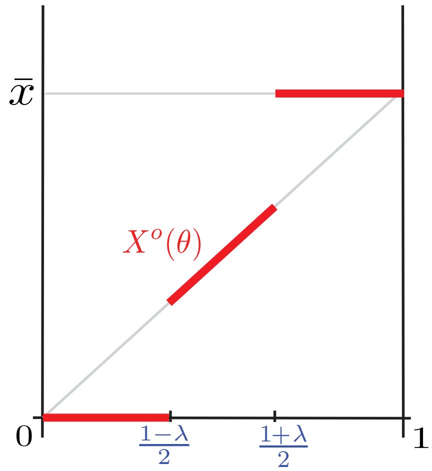

Figure 1: the optimal selling rule X^o

satisfied. Let $X\colon \Theta \to \mathscr{X}$ be any measurable allocation rule. One readily sees that the derivative of v with respect to θ evaluated at $(X(\theta), \theta)$ is given by $Dv(X(\theta), \theta) = X(\theta)$, therefore (M) is satisfied and our characterization theorem applies. Berger, Müller, and Naeemi (2010) deal with an arbitrary allocation set, a convex type space in $\mathbb{R}^k$, and valuations that are convex functions of types. Directional derivatives of convex (and concave) functions are sufficiently well-behaved for our characterization result to be applicable. We require the following preliminary result, which is taken from Hildenbrand (1974, p. 42). Here $\mathcal{M} \otimes \mathcal{B}([0,1])$ denotes the product σ-algebra of $\mathcal{M}$ and $\mathcal{B}([0,1])$.

Proposition 3. *Let $(\mathscr{X}, \mathcal{M})$ be a measurable space and $f\colon \mathscr{X} \times [0,1] \to \mathbb{R}$ be a bounded real-valued function. Suppose the following conditions hold:*

(a) For every $\alpha \in [0,1]$, the function $x \mapsto f(x,\alpha)$ is $\mathcal{M}$-measurable.

(b) For every $x \in \mathscr{X}$, the function $\alpha \mapsto f(x,\alpha)$ is right (left) continuous.

Then the function f is $\mathcal{M} \otimes \mathcal{B}([0,1])$-measurable.

Under a mild measurability requirement on the right and left subderivative functions, Theorem 1 covers relevant cases previously studied in the literature.

Proposition 4. *Let (A1) to (A3) be satisfied. Assume that for every type $\theta \in \Theta$ and every directional vector $\delta \in \mathbb{R}^k$, the functions $x \mapsto \overline{\mathrm{d}}v(x,\theta;\delta)$ and $x \mapsto \underline{\mathrm{d}}v(x,\theta;\delta)$ are $\mathcal{M}$-measurable. In addition, assume one of the following conditions hold:*

(a) For every $x \in \mathscr{X}$, $\theta \mapsto v(x,\theta)$ is convex in Θ.

(b) For every $x \in \mathscr{X}$, $\theta \mapsto v(x,\theta)$ is concave in Θ.

(c) For every $x \in \mathscr{X}$, $\theta \mapsto v(x,\theta)$ is continuously differentiable in Θ.

If $X\colon \Theta \to \mathscr{X}$ is a measurable allocation rule, then (M) is satisfied.

Proof We deal with convex valuations; the remaining cases are similarly handled. Fix $\theta_1, \theta_2 \in \Theta$. From (A3) and the convexity of $\theta \mapsto v(x,\theta)$ on Θ, it follows that for every $x \in \mathscr{X}$ the right subderivative $\overline{\mathrm{d}}v(x, \theta_1^2(\alpha); \delta_1^2)$ is equal to the right derivative of the convex real-valued function $\alpha \mapsto w(\alpha; x, \delta_1^2) = v(x, \theta_1^2 + \alpha\,\delta_1^2)$ defined on $(0,1)$. Thus, the function $\alpha \mapsto \overline{\mathrm{d}}v(x, \theta_1^2(\alpha); \delta_1^2)$ is right continuous on $(0,1)$. By assumption, the function $x \mapsto \overline{\mathrm{d}}v(x, \theta_1^2(\alpha); \delta_1^2)$ is $\mathcal{M}$-measurable for every $\alpha \in (0,1)$. Thus, we infer from Proposition 3 that the function $\overline{\mathrm{d}}(\cdot,\cdot;\delta_1^2)$ is $\mathcal{M} \otimes \mathcal{B}(0,1)$-measurable. Since X is a measurable allocation rule, it is seen that the function $\alpha \mapsto (X(\theta_1^2(\alpha)), \theta_1^2(\alpha))$ is $\mathcal{B}(0,1)$-measurable, hence we obtain the measurability of $\alpha \mapsto \overline{s}(\theta_1^2(\alpha)) = \overline{\mathrm{d}}v(X(\theta_1^2(\alpha)), \theta_1^2(\alpha); \delta_1^2)$ by noticing that the composition of measurable functions is also measurable. The argument for $\underline{s}(\theta_1^2(\cdot))$ is similar, except that in this case one uses the left derivative of the respective convex scalar function, which is left continuous on $(0,1)$. ■

Suppose that we partition $\mathscr{X}$ into measurable sets $\{\mathscr{X}_1, \mathscr{X}_2, \mathscr{X}_3\}$, such that for every $x \in \mathscr{X}_1$ (respectively, $\mathscr{X}_2, \mathscr{X}_3$), the function $\theta \mapsto v(x,\theta)$ is convex in Θ (respectively, concave, continuously differentiable). One can use Proposition 4 to show that (M) is also satisfied in this type of mixed models.

6.2 Back to Revenue Equivalence

Our revenue inequality (Theorem 2) establishes a precise range for the value of the difference between two indirect utilities associated with the allocation rule X. If for all types θ_1, θ_2, the subderivative correspondence $S\left(\theta_1^2(\cdot)\right)$ is single-valued almost everywhere, then revenue equivalence is restored.

Proposition 5. *Assume that (A1) to (A3) and (M) are satisfied. Let $p\colon \Theta \to \mathbb{R}$ and $p'\colon \Theta \to \mathbb{R}$ be two payment rules that implement $X\colon \Theta \to \mathscr{X}$ and generate indirect utility functions U and U', respectively, satisfying $U(\theta_0) = U'(\theta_0) = u_0$. If for every pair of types $\theta_1, \theta_2 \in \Theta$, one has $\underline{s}(\theta_1^2(\alpha)) = \overline{s}(\theta_1^2(\alpha))$ for almost all $\alpha \in (0,1)$, then $U = U'$.*

Proof Suppose that for all θ_1, θ_2 in Θ, it is the case that $\underline{s}(\theta_1^2(\alpha)) = \overline{s}(\theta_1^2(\alpha))$ a.e. in $(0,1)$. Readily from Equation 16, it follows $U(\theta_1) = U'(\theta_1)$, for all $\theta_1 \in \Theta$. ■

We obtain the following corollary.

Corollary 2. *Under the assumptions of Proposition 5, $U = U'$ whenever one of the following conditions hold.*

(a) For every $x \in \mathscr{X}$, $\theta \mapsto v(x,\theta)$ is convex in Θ.

(b) For every $x \in \mathscr{X}$, $\theta \mapsto v(x,\theta)$ is differentiable in Θ.

(c) The allocation set $\mathscr{X}$ is finite.

Proof (a) Suppose $\theta \mapsto v(x,\theta)$ is convex. Then, for all $\hat{\theta} \in \Theta$ and all $\delta \in \mathbb{R}^k$,

$$\underline{\mathrm{d}}v(x,\hat{\theta};\delta) = \lim_{r\uparrow 0} \frac{v(x,\hat{\theta}+r\delta) - v(x,\hat{\theta})}{r} \leq \lim_{r\downarrow 0} \frac{v(x,\hat{\theta}+r\delta) - v(x,\hat{\theta})}{r} = \overline{\mathrm{d}}v(x,\hat{\theta};\delta).$$

Since Equation 13 requires that at almost all equilibrium points $(X(\theta),\theta)$ the reverse inequality holds, one sees that the condition of Proposition 5 is in place.

(b) Immediate.

(c) Let θ_1, θ_2 be arbitrary and define the function $\alpha \mapsto \hat{X}(\alpha) \equiv X(\theta_1^2(\alpha))$ on $(0,1)$. Since $\mathscr{X}$ is finite, $\hat{X}((0,1)) = \{x_1,\ldots,x_N\}$, for some $N \in \mathbb{N}$. Thus, the sets $E_1,\ldots,E_N$, with $E_n = \hat{X}^{-1}(\{x_n\})$ for each $n = 1,\ldots,N$, constitute a collection of pairwise disjoint, measurable subsets of $(0,1)$ whose union is $(0,1)$. Without loss of generality, we assume that each E_n has non zero measure. Then, for each $n = 1,\ldots,N$, the function $\alpha \mapsto \nu_n(\alpha) \equiv v(x_n,\theta_1^2(\alpha))$ is Lipschitz continuous on E_n and therefore differentiable a.e. in E_n (by Rademacher-Stepanoff Theorem). It follows that $\underline{s}(\theta_1^2(\alpha)) = \overline{s}(\theta_1^2(\alpha))$ almost everywhere on the unit interval $(0,1)$, as desired. ■

Note that Corollary 2-(c) does not require the valuation function to be linear, convex or differentiable in types: for the finite allocation sets, assumptions (A1) to (A3) and (M) suffice to obtain revenue equivalence.

6.3 Weak monotonicity versus integral monotonicity

An allocation rule $X\colon \Theta \to \mathscr{X}$ satisfies the *weak monotonicity* condition if for every pair of types $\theta, \hat{\theta}$ in Θ, one has

$$v(X(\theta),\theta) - v(X(\theta),\hat{\theta}) \geq v(X(\hat{\theta}),\theta) - v(X(\hat{\theta}),\hat{\theta}).$$

Weak monotonicity has been shown to characterize implementation when the type space is convex, the valuation function linear in types, and the allocation set either finite or infinite (the latter case requiring the inclusion of the closed-path integrability condition).[14] Clearly, integral monotonicity implies weak monotonicity. However, there are examples where weak monotonicity does not imply integral monotonicity, even when $\mathscr{X}$ is finite and the valuation function $v(x,\cdot)$ is piecewise linear in types (Berger, Müller, and Naeemi, 2010).

On the other hand, Berger, Müller, and Naeemi (2010) discovered that a single-crossing property is sufficient to obtain the equivalence between weak monotonicity

[14] See Jehiel, Moldovanu, and Stacchetti (1999), Bikhchandani, Chatterji, Lavi, Mu'alem, Nisan, and Sen (2006), Saks and Yu (2005), Archer and Kleinberg (2008), and Vohra (2009).

and integral monotonicity, under the assumption that valuations are either convex or differentiable in types. As the next proposition shows, their insight extends to our general setting. Say that the valuation function $v\colon \mathscr{X} \times \Theta \to \mathbb{R}$ satisfies *increasing differences* if for all $x, y \in \mathscr{X}$, all $\theta_1, \theta_2 \in \Theta$, and every $\theta_1^2(\alpha) \in L(\theta_1, \theta_2)$, $v(x, \theta_2) - v(y, \theta_2) \geq v(x, \theta_1^2(\alpha)) - v(y, \theta_1^2(\alpha))$ implies that $v(x, \theta_1^2(\alpha)) - v(y, \theta_1^2(\alpha)) \geq v(x, \theta_1) - v(y, \theta_1)$.

Proposition 6. *Assume that (A1) to (A3) and (M) are satisfied for the allocation rule $X\colon \Theta \to \mathscr{X}$. Suppose in addition that $v\colon \mathscr{X} \times \Theta \to \mathbb{R}$ satisfies increasing differences. Then for every θ_1, θ_2 in Θ, the subderivative correspondence $S\left(\theta_1^2(\cdot)\right)$ is regular and admits an integrable selection $\alpha \mapsto s(\theta_1^2(\alpha))$ such that X satisfies integral monotonicity if and only if X satisfies weak monotonicity.*

Proof Clearly the integral monotonicity condition implies that X is weakly monotone. Conversely, suppose that X satisfies weak monotonicity. Fix arbitrary types θ_1, θ_2. From Lemmas 3 and 5 of Berger, Müller, and Naeemi (2010) it follows that the increasing differences property of the valuation function together with weak monotonicity imply that X is cyclically monotone on the closure of $L(\theta_1, \theta_2)$. Therefore, the restriction of X to the closure of $L(\theta_1, \theta_2)$ is implementable. From Theorem 1, we conclude that the subderivative correspondence $S\left(\theta_1^2(\cdot)\right)$ is regular and admits an integrable selection $\alpha \mapsto s(\theta_1^2(\alpha))$ for which integral monotonicity is satisfied. ■

We stress the fact that, without additional assumptions, Proposition 6 does not imply that there exists a unique selection for which integral monotonicity holds. Thus, while weak monotonicity may be more readily verified in applications, we are nonetheless interested in finding the subderivative correspondence between types and selections consistent with the integral monotonicity condition, since these objects provide the (possibly many) payment rules that can be used for implementation.

7 Concluding remarks

In this paper we present a characterization of (dominant strategy) incentive compatible mechanisms in quasi-linear settings where the envelope theorem and the revenue equivalence principle fail, due to non-convexity and non-differentiability of the valuation function with respect to types. We base our characterization result on the integral monotonicity and the Mirrlees representation of the indirect utilities (plus the standard closed-path integrability condition). Our framework allows for the allocation set to be finite or infinite and the type space of the agent to be a convex subset of a multidimensional Euclidean space. We work with uniformly Lipschitz valuations and impose a measurability requirement on the left and right subderivatives of the valuation function with respect to types.

These conditions pin down the range of payoff differences generated by two incentive compatible mechanisms with the same allocation rule. As a consequence, we obtain a revenue inequality that generalizes the standard revenue equivalence principle: given an implementable allocation rule and normalized payments that assign the same utility to the "lowest type", the difference in equilibrium payoffs generated by any two payment rules is bounded by the allocation rule alone, even though it may be not vanish.

Our results contribute to the already extensive literature on implementation in several ways. First, our environment is less restrictive than those considered in previous work. In particular, we do not assume linearity, convexity or differentiability of the valuation with respect to types, but any such hypothesis, in addition to a mild measurability condition, will imply our assumption (M). Second, our characterization result exploits techniques that differ from the approaches currently employed in the literature, thus allowing us to handle a broader set of problems with richer payment schemes. Third, our approach opens new aspects of institutional design for study. We have provided two applications to illustrate this point. In particular, in certain environments it may be possible to use several payment rules (which differ by more than just a constant) to implement an efficient or revenue maximizing allocation rule. While such schemes may share desirable properties (e.g., budget balancedness, individual rationality), other (un)desirable features could be introduced. We leave a thorough study of these aspects of institutional design for future research.

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