# Foundations of Dominant Strategy Mechanisms*

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July 10, 2006

**Abstract**

Wilson (1987) criticizes applied game theory's reliance on common-knowledge assumptions. In reaction to Wilson's critique, the recent literature of mechanism design has adopted the goal of finding *detail-free* mechanisms in order to eliminate this reliance. In practice this has meant restricting attention to simple mechanisms such as dominant strategy mechanisms. However there has been little theoretical foundation for this approach. In particular it is not clear the search for an optimal mechanism that does not rely on common-knowledge assumption would lead to simpler mechanisms rather than more complicated ones. This paper tries to fill the void. In the context of an expected revenue maximizing auctioneer, we investigate some foundations for using simple, dominant-strategy auctions.

**JEL Classification:** C70, D82
**Keywords:** Wilson Doctrine, dominant strategy mechanism, detail-free mechanism

*Thanks to Larry Epstein, Stephen Morris and Balasz Szentes for discussions. Also we thank the Editor and two anonymous referees for helpful suggestions for revising the paper.

†Support from the National Science Foundation under grant #SES 99-85462 is gratefully acknowledged.

# 1 Introduction

In the recent literature of mechanism design, there is a research agenda which is motivated by the so-called *Wilson Doctrine*. Roughly speaking, the Wilson Doctrine refers to the vision, articulated in Wilson (1987), that a good theory of mechanism design should not rely too heavily on assumptions of common knowledge:

> "Game theory has a great advantage in explicitly analyzing the consequences of trading rules that presumably are really common knowledge; it is deficient to the extent it assumes other features to be common knowledge, such as one agent's probability assessment about another's preferences or information. [...] I foresee the progress of game theory as depending on successive reduction in the base of common knowledge required to conduct useful analyses of practical problems. Only by repeated weakening of common knowledge assumptions will the theory approximate reality."

Although there is no clear prescription from Wilson (1987) on how exactly to reduce the dependence on common knowledge assumptions, the methodology on which the literature has converged is to impose strong solution concepts which minimize the impact of any such assumption. To understand the intuitive logic behind this methodology, one can consider the problem of optimal auction design with possibly correlated valuations. The traditional approach proceeds in two steps. In step one, the model is closed by adopting the assumption that the distribution of valuations is common knowledge. Step two then solves the model by searching for the optimal Bayesian incentive compatible selling mechanism. Step one inadvertently imposes strong assumptions on bidders' beliefs, which step two then takes literally. The results are unrealistic and/or undesirable features of the optimal mechanism. To avoid these perverse results without giving up step one, one can try to modify step two. In particular, one can replace Bayesian incentive compatibility with stronger solution concepts that are insensitive to different assumptions on bidders' beliefs.

This is the approach taken by, e.g. Dasgupta and Maskin (2000) and Perry and Reny (2002) who study the design of efficient auctions in interdependent-value settings. To ensure that the auction form does not rely on fine details of the bidders' information, they insist that their designs are ex post incentive compatible. Similarly, when Segal (2003) designs optimal auctions in private-value settings, he also insists that his designs are dominant strategy incentive compatible. Both ex post incentive compatibility and dominant strategy incentive compatibility are stronger solution concepts than Bayesian incentive compatibility.

This methodology appears misdirected. The goal is to eliminate the dependence on simplifying common-knowledge assumptions imposed at step one, but instead of relaxing those assumptions directly, stronger solution concepts are imposed at step two in order to minimize their effect. In this paper, we shall try to provide some foundations for this approach. We focus on private-value auctions and ask whether a profit-maximizing auctioneer who is unwilling to cling to strong common-knowledge assumptions can have a *rational* basis for restricting attention to dominant strategy mechanisms.

The usual argument for imposing stronger solution concepts is that the resulting mechanisms will then be *detail free*: the rules would not have to be tailored to any fine details of the environment in which it is employed. Indeed, *detail-freeness* is the usual interpretation of the Wilson Doctrine. However, *a priori*, it is not apparent at all why detail-free mechanisms would look as simple as the mechanisms prescribed in the above-cited studies. If anything, the established intuition in mechanism design suggests that detail-free mechanisms in general should look very complicated indeed.

To see why, recall that a mechanism designer can in principle ask her agents *anything* that she does not know, and she should do so if the answers are potentially useful. For example, consider an auction and an auctioneer who assumes that the bidders share a common prior $\rho$ about their valuations for the object up for sale. Then it is well-known that the precise rules of the optimal mechanism depend on the value of $\rho$; i.e., it is not detail-free. To eliminate this dependence, the auctioneer must construct a more general mechanism which directly asks the bidders to announce their prior and adjusts outcomes and payments according to the answer.

The mechanism that results will be free of any details about the bidders' *first-order* beliefs; i.e., their beliefs over their valuations. But this mechanism is still predicated on the assumption that these beliefs are common-knowledge. If the auctioneer wishes to remove this and further detail-dependence, she should ask more and more questions of the bidders. Pushing this logic to its extreme, a truly detail-free mechanism would become so complicated that it would entail asking agents to report *everything*; i.e., their whole infinite hierarchies of beliefs.

Of course, the results that can be obtained from a detail-free mechanism are limited by the constraint of incentive compatibility. In particular, to ensure that bidders truthfully announce their priors, the mechanism must provide them with an incentive to do so. Note that this would not be a problem if the auctioneer assumes that this prior is common-knowledge: simply impose a penalty on the agents if their announcements disagree. But a detail-free mechanism must not stake incentive compatibility on such assumptions. Indeed, because the correct incentives for truthfully revealing these first-order beliefs will depend on the bidders' second- and higher-order beliefs, a detail-free mechanism is implementable only if each bidder has an incentive to announce truthfully his complete belief-hierarchy.

It is well-known that dominant strategy mechanisms satisfy this strong form of incentive compatibility; i.e., they are detail-free.[1] However, dominant strategy mechanisms constitute just one special class of detail-free mechanisms and there has previously been no formal justification (in terms of optimality) for the leap from detail-freeness in general to dominant strategy mechanisms in particular.

[1]See Bergemann and Morris (2005) for a modern treatment. The classical reference is d'Aspremont and Gerard-Varet (1979)

In this paper, we shall provide a rationale for using dominant strategy mechanisms which confronts these problems. Our theory is based on the following often-repeated informal motivation.[2] Let $\nu$ denote the distribution of the bidders' valuations. An auctioneer may have confidence in her estimate of $\nu$, perhaps based on data from similar auctions in the past. But she does not have reliable information about the bidders' beliefs (including their beliefs about one another's valuations, their beliefs about these beliefs, etc.), as these are arguably never observed. She can choose *any* detail-free mechanism, including those that allow her to ask the bidders anything about their beliefs that might be relevant. In general such a mechanism may perform well under some specific common-knowledge assumptions but may perform badly if those assumptions turn out to be false. On the other hand, a dominant strategy mechanism secures a fixed expected revenue, independent of any assumption about the bidders' beliefs. If the auctioneer is not sufficiently confident in any such assumption, she may optimally choose a dominant strategy mechanism.

We call this story the *maxmin* foundation of dominant strategy mechanisms, because the auctioneer chooses among mechanisms according to their worst-case performance. Formally, the theorem we are seeking is illustrated in Figure 1. In Figure 1, we (heuristically) plot the performance of arbitrary detail-free mechanisms against different assumptions about bidders' beliefs. The graph of any dominant strategy mechanism—and in particular the graph of the best one among all dominant strategy mechanisms—will be a horizontal line. To establish the maxmin foundation, we would need to show that the graph of any (potentially very complicated) mechanism must dip below the graph of the best dominant strategy mechanism at some point.

Although we believe that Figure 1 captures the intuition of many advocates of dominant strategy mechanisms, it turns out to be very difficult to prove in general. With no restriction on the environment, the set of all detail-free mechanisms is quite rich, and it would be contrary to the spirit of our investigation to impose exogenous restrictions on the complexity of the mechanism.

Instead, in this paper, we introduce a sufficient condition on the distribution of bidders' *valuations* (recall that the auctioneer has confidence in the distribution of bidders' valuations although not in the distribution of bidders' beliefs). The condition generalizes to the case of an arbitrary (possibly correlated) $\nu$ what Myerson (1981) calls "the regular case" in his classical paper on optimal auctions with independent valuations. It is a familiar condition in mechanism design and comfortably assumed in many applications.

In fact, under our condition, we are able to prove a stronger result (see Figure 2): there will be a particular distribution of bidders' beliefs, at which point the graph of *every* (potentially very complicated) detail-free mechanism must dip below the graph of the best dominant strategy mechanism. We say that this distribution *rationalizes* dominant-strategy incentive-compatibility.

[2] See, for example, Segal (2003) sec.VI, who conjectures a result similar to ours.

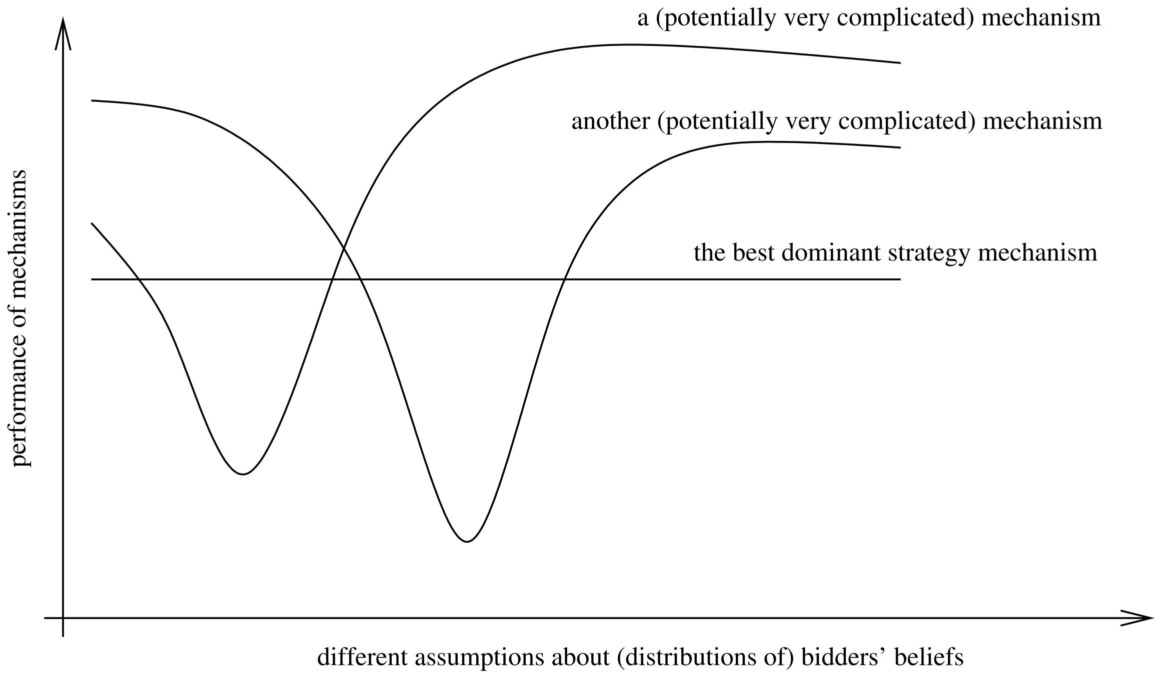


Figure 1: The graph of any mechanism dips below the graph of the best dominant strategy mechanism at some point.

The rationalizing distribution has a simple form. It can be described by a finite type space in which each possible bidder's valuation is represented by a single type. Moreover, when the distribution of bidders' valuations, $\nu$, converges to a product measure, the bidders' beliefs approach those that would obtain if $\nu$ were the common prior. This ties our theorem nicely to the classical result that there exists a dominant strategy mechanism that is optimal among Bayesian mechanisms when valuations are independently distributed.

Clearly Figure 2 implies the maxmin foundation we seek. In addition, Figure 2 is significant in its own right. To expand on this, let us think about the auctioneer in a different, perhaps more standard, context.

Imagine the auctioneer as a Bayesian decision maker. When she needs to choose a mechanism, she forms a subjective belief about bidders' beliefs, and compares different mechanisms by calculating the expected performance with respect to that subjective belief. When we observe that this auctioneer chooses a dominant strategy mechanism, we can ask whether or not such a choice is consistent with Bayesian rationality; i.e., whether or not such a choice is optimal with respect to *some* subjective beliefs. If so, we say that there is a Bayesian foundation for dominant strategy mechanisms. Figure 2 says that, in the regular case, dominant strategy mechanisms have a Bayesian foundation.

Note that the existence of a rationalizing belief (Figure 2) is a stronger requirement than the maxmin foundation (Figure 1). We mentioned previously that we do not know whether the maxmin foundation is valid in general (beyond the regular case). However, we do show

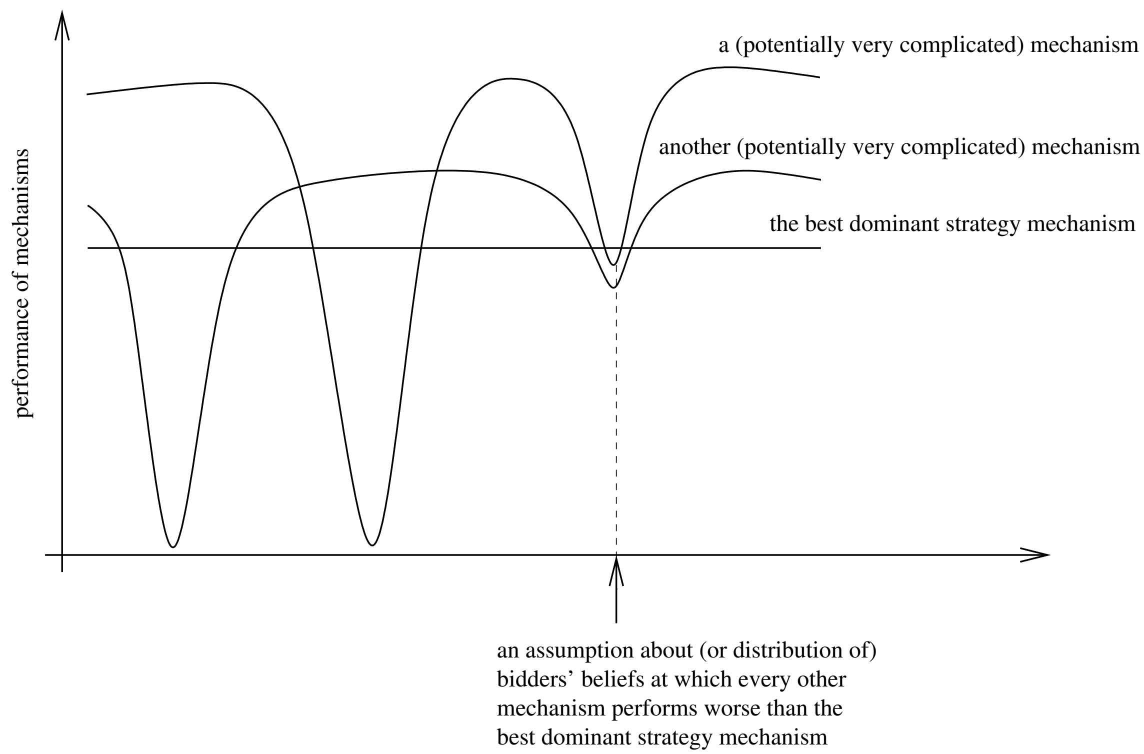


Figure 2: there is a particular point at which the graph of every mechanism dips below the graph of the best dominant strategy mechanism.

by example below that beyond the regular case, a Bayesian foundation need not exist. As a negative result about the rationality of using dominant strategy mechanisms, we view this as particularly strong: for some distributions of valuations, no Bayesian expected-revenue maximizing auctioneer would optimally employ a dominant strategy mechanism, regardless of her beliefs.

Finally, we relate our results to the widely adopted assumption that all agents' beliefs are consistent with a common prior. A tempting conjecture is that such consistency is a necessary condition for a rationalizing belief. After all, it is well known that when agents have inconsistent beliefs there exists a bet which has positive expected value for all. It would seem that the auctioneer could improve upon any dominant strategy mechanism by building into it such a bet.

This intuition is incorrect and the reason uncovers a key insight that is behind our analysis. Any bet between the auctioneer and the bidders must be incentive compatible. Even when beliefs are inconsistent, the incentive compatibility constraint can prevent the auctioneer from profiting from the inconsistency. Indeed, the rationalizing belief we identify in is in general inconsistent with a common prior. The belief is constructed so that incentive compatibility renders any side-bets unprofitable. Because this is a key step in our analysis, following the statement of our result in Section 3 we present a worked example which illustrates it.

Nevertheless the common prior assumption (CPA) is widely adopted in applied analysis, and so it may be of interest to know whether dominant strategy mechanisms can be rationalized by a belief consistent with the CPA. Surprisingly, we show by example in Section 5 that this need not be possible even in the regular case. We do however present a positive result that holds when the valuation distribution is close to independent. In this case there is a CPA belief for the auctioneer against which dominant strategies are approximately optimal: the associated revenue loss relative to the optimal mechanism is vanishingly small.

The rest of the paper is organized as follows. The remainder of this introduction discusses some important related literature. Section 2 presents the model and formalizes the problem. Our main result will be presented and proved in Section 3. Our first example also appears in this section. Section 4 interprets our result in terms of a Bayesian foundation and then presents an example to show that a Bayesian foundation for dominant strategy mechanisms need not exist in general. In Section 5, we present our results related to the CPA and Section 6 then concludes the paper with an observation about the English auction.

## 1.1 Related Literature

This paper is not the first to offer a foundation for dominant strategy mechanisms. Bergemann and Morris (2005) offers an alternative foundation for ex post incentive compatible mechanisms, which in private-value settings are equivalent to dominant strategy mechanisms. The main difference between Bergemann and Morris (2005) and the present paper concerns the type of mechanisms being considered. Bergemann and Morris (2005) focus exclusively on mechanisms in which the outcome can depend only on payoff-relevant data. These mechanisms are naturally suited to study *efficient* design. On the other hand, we are interested here in revenue maximization for a seller. The optimal mechanism for such a designer will almost always depend not just on the valuations, but also on payoff-irrelevant data such as beliefs and higher-order beliefs.[3] This is why the results of Bergemann and Morris (2005) do not apply in our setting.

Neeman (2003) is similar in spirit in that he performs a worst-case assessment of the English auction (which is a dominant strategy mechanism). He compares the revenue generated by the English auction to the benchmark of full-surplus extraction. The ratio of these two values is called "effectiveness." He shows that the effectiveness of the English auction can be fairly high, and in fact close to 1 for a wide variety of distributions of valuations. The benchmark of full-surplus extraction was used despite the fact that this benchmark may not be feasible even for the optimally chosen mechanism,[4] mainly because determining the optimal auction for an environment as general as he considers is a daunting task. One contribution of the present paper is to show how to derive the optimal auction *in the worst-case assumption*

[3]For instance, see the auctions depicted below in Figures 10, 11, 12, 13, 8, and 9.

[4]In fact, Neeman (2004) showed this.

about bidder's beliefs. We are thus able to compare dominant strategy mechanisms with the optimal auction benchmark and show that the optimal dominant strategy auction performs at least as well in the worst case. We discuss another connection with Neeman (2003) in footnote 6 after we introduce the regular case.

# 2 Preliminaries

## 2.1 Notation

If $\{X_i\}_{i=1}^N$ is a collection of sets, then $X$ denotes the Cartesian product $\times_i X_i$, or the set of "profiles" of elements of $\{X_i\}$. We write $X_{-i} = \times_{j \neq i} X_j$. If $x \in X$, then $x_i$ refers to the $i$th co-ordinate, and we use $x_{-i}$ to denote the element of $X_{-i}$ obtained by removing $x_i$. Likewise, if $\{f_i\}_{i \in N}$ is a collection of mappings $f_i : X_i \to Y_i$, then $f_{-i}$ denotes the "product" mapping $f_{-i} : X_{-i} \to Y_{-i}$ where $f_{-i}(x_{-i}) = (f_1(x_1), \ldots, f_{i-1}(x_{i-1}), f_{i+1}(x_{i+1}), \ldots, f_N(x_N))$. If $Y$ is a measurable set, then $\Delta Y$ is the set of all probability measures on $Y$. If $Y$ is a metric space, then we treat it as a measurable space with its Borel $\sigma$-algebra.

## 2.2 The Auction Environment

A single unit of an indivisible object is up for sale. There is a set $N$ of risk-neutral bidders with privately known valuations competing for the object. Each bidder has $M$ possible valuations and for notational simplicity, we suppose that the set $V_i$ of possible valuations is the same for each bidder $i$ and that $V_i = \{v^1, v^2, \ldots, v^M\}$, where $v^m - v^{m-1} = \gamma$ for each $m$ for some $\gamma > 0$.[5] The bidders' valuations are distributed according to a given probability distribution $\nu \in \Delta V$. Note that we are allowing for correlated values and that the independent private value model is included as a special case when $\nu$ is a product measure. We assume that $\nu$ has full-support.

A bidder $i$ with valuation $v_i$ receives expected utility $p_i v_i - t_i$ if $p_i$ is the probability with which he will be awarded the object and if his expected monetary payment is $t_i$. A typical element of $V$ is $v$, and a typical element of $V_{-i}$ is $v_{-i}$.

We consider distributions satisfying a generalized version of Myerson's (1981) regularity condition. Let $F_i(v_i, v_{-i}) = \sum_{\hat{v}_i \leq v_i} \nu(\hat{v}_i, v_{-i})$ denote the cumulative distribution function of $i$'s valuation conditional on the opponents having valuation profile $v_{-i}$. The *virtual valuation* of bidder $i$ at profile $v$ is

$$\gamma_i(v) := v_i - \gamma \frac{1 - F_i(v)}{\nu(v)}.$$

[5]These notational conventions simplify the statements of results and notation, but are entirely innocuous. Assumptions of asymmetry in the bidders' valuation sets, or differing gaps between valuations would not affect any of our results.

**Definition 1** *We say that $\nu$ is regular if the virtual valuations satisfy the single-crossing condition: for each $v$, $i \in \{1, \dots, N\}$, and $j \in \{0, \dots, N\}, j \neq i$,*

$$\gamma_i(v) \geq \gamma_j(v) \implies \gamma_i(\hat{v}_i, v_{-i}) > \gamma_j(\hat{v}_i, v_{-i})$$

*for every $\hat{v}_i > v_i$, where $\gamma_0(\cdot) \equiv 0$ denotes the auctioneer's value for the object.*

Our definition extends Myerson's (1981) regularity condition to correlated $\nu$ but reduces to his original condition when $\nu$ is independent. To see this note that if $\nu$ is independent, then the virtual valuation of bidder $j$ depends only on $v_j$. Thus, the single crossing condition reduces to the requirement that the virtual valuation of each bidder $i$ is increasing.

The regularity condition is stated directly in terms of virtual valuations. A familiar set of sufficient conditions on $\nu$ is given below. First, the *monotone hazard rate condition* is satisfied if for each $i$ and $v_{-i}$, the hazard rate, $h_i(\hat{v}_i|v_{-i}) = \frac{\nu(\hat{v}_i, v_{-i})}{1-F_i(\hat{v}_i, v_{-i})}$, is an increasing function of $\hat{v}_i$. The valuations are *affiliated* if for each pair of profiles $v$, $v'$, $\nu(v \vee v') \cdot \nu(v \wedge v') \geq \nu(v) \cdot \nu(v')$, where $v \vee v'$ is the component-wise maximum and $v \wedge v'$ the component-wise minimum of the two valuation vectors.[6]

We prove the following in Appendix B (see Proposition 3.).

**Proposition 1** *If $\nu$ satisfies both the monotone hazard rate condition and affiliation, then $\nu$ is regular.*

## 2.3 Types

To characterize the (equilibrium) behavior of the bidders who compete in some given auction mechanism, it is not enough to specify the bidders' possible valuations or even the probability distribution from which they are drawn. In addition, we must also specify their beliefs about the valuations of their opponents (called the *first-order* beliefs), their beliefs about one another's first-order beliefs (called the *second-order* beliefs), etc.

The standard approach to modeling the bidders' information is to use a type space. A type space, denoted $\Omega = (\Omega_i, f_i, g_i)_{i \in N}$ is defined by a measurable space of *types* $\Omega_i$ for each player, and a pair of measurable mappings $f_i : \Omega_i \to V_i$, defining the valuations of each type, and $g_i : \Omega_i \to \Delta\Omega_{-i}$, defining each type's belief about the types of the other bidders.

[6] Affiliation is a strong form of positive correlation. In the worst-case analysis of Neeman (2003), the distribution of valuations itself was a free variable. He showed that the worst-case distribution of valuations involves *negative* correlation. It is thus not surprising that we use a condition such as affiliation. Furthermore, our counterexample in Section 4 also involves negative correlation. While the performance measure used in Neeman (2003) is not the same as ours, the similarity between this aspect of the two results suggests some deeper connection.

A type space encodes in a parsimonious way the beliefs and all higher-order beliefs of the bidders. [7] One simple kind of type space is the *naive type space*[8] generated by the valuation distribution $\nu$. In the naive type space, each bidder believes that all bidders' valuations are drawn from the distribution $\nu$, and this is common-knowledge. In the formal notation of type spaces introduced above, this is modeled as follows. For each $v_i \in V_i$, there is a unique type, denoted $\omega^{v_i}$, with the property $f_i(\omega^{v_i}) = v_i$ . The belief $g_i(\omega^{v_i})$ is defined in two steps: first the conditional probability $\nu(\cdot|v_i)$ over $V_{-i}$ is derived from $\nu$, then this is transformed in the natural way into a belief over the other bidders' types, so that the probability $\omega^{v_i}$ assigns to the type profile $\omega^{v_{-i}}$ for the opponents is given by $g_i(\omega^{v_i})\left[\omega^{v_{-i}}\right] = \nu(v_{-i}|v_i)$. We let $\Omega^{\nu}$ denote the naive type space associated with valuation distribution $\nu$.

The naive type space is used almost without exception in auction theory and mechanism design. The cost of this parsimonious model is that it implicitly embeds some strong assumptions about bidders' beliefs, and these assumptions are not innocuous. For example, if the bidders' valuations are independent under $\nu$, then in the naive type space, the bidders' beliefs are commonly known. On the other hand, for a generic $\nu$, it is common-knowledge that there is a one-to-one correspondence between valuations and beliefs. Myerson (1981) characterizes the optimal auction in the independent case and Crémer and McLean (1985) in the other case. Which of these cases holds makes a big difference for the structure and welfare properties of the optimal auction. These and similar issues have been raised in Neeman (2004), Bergemann and Morris (2005), Heifetz and Neeman (2006), Morris (2002), Dekel, Fudenberg, and Morris (forthcoming) and Weinstein and Yildiz (2004). The spirit of the Wilson doctrine is to avoid making such assumptions.

Instead, as explained in the introduction, the common approach is to maintain the naive type space, but try to diminish its adverse effect by imposing stronger solution concepts. To provide foundations for this methodology, we have to return to the fundamentals. Formally, weaker assumptions about bidders' beliefs are captured by larger type spaces. Indeed, we can remove these assumptions altogether by allowing for every conceivable hierarchy of higher-order beliefs. By the results of Mertens and Zamir (1985), there exists a *universal* type space, $\Omega^* = (\Omega_i^*, f_i^*, g_i^*)_{i \in N}$, with the property that, for every valuation $v_i$ and every infinite hierarchy of beliefs $\hat{h}_i$, there is a type of player $i$, $\omega_i$, with valuation $v_i$ and whose hierarchy is $\hat{h}_i$, Moreover, each $\Omega_i^*$ is a compact topological space.[9]

[7]Consider a type $\omega_i \in \Omega_i$. Its first-order belief is a probability distribution over the valuation profiles of $i$'s opponents. We can uncover this probability distribution as follows. The probability type $\omega_i$ assigns to a given valuation profile $v_{-i}$ is $g_i(\omega_i)(\{\omega_{-i} : f_{-i}(\omega_{-i}) = v_{-i}\})$; i.e., the probability $\omega_i$ assigns to the set of types of the opponents' with valuations $v_{-i}$. Next, for any profile $\rho_{-i}$ of first-order beliefs for $i$'s opponents, let $\beta_{-i}(\rho_{-i})$ be the set of types with first order beliefs (as derived previously) $\rho_{-i}$. Then the second-order beliefs of $\omega_i$ assign probability $g_i(\omega_i)\left[f_{-i}^{-1}(v_{-i}) \cap \beta_{-i}(\rho_{-i})\right]$ to the profile $(v_{-i}, \rho_{-i})$ of valuation/first-order belief pairs for the opponents. This procedure can be repeated to compute all higher-order beliefs of each bidder.

[8]This terminology originated in Bergemann and Morris (2005).

[9]See Appendix A for the details on the Mertens and Zamir (1985) construction and how it is applied to our setting. To be precise, the universal type space includes all hierarchies that satisfy a natural *coherency* property. Also, in the MZ universal type space generated by $V$, there would exist types who are uncertain

Another sense in which $\Omega^*$ is "universal" is the following. Certain simple type spaces are essentially "subspaces" of $\Omega^*$ as captured by the following proposition, which will be used in the proof of the main Theorem.

**Lemma 1** *Let $\Omega$ be a type space in which the mapping $\omega_i \to (f_i(\omega_i), h_i(\omega_i))$ is one-to-one. Then there exists subsets $\hat{\Omega}_i \subset \Omega_i^*$ and bijective mappings $m_i : \Omega_i \to \hat{\Omega}_i$ such that*

1. $f_i^*(m_i(\omega_i)) = f_i(\omega_i)$ *for all* $\omega_i \in \Omega_i$,
2. $g_i^*(m_i(\omega_i)) [m_{-i}(\omega_{-i})] = g_i(\omega_i) [\omega_{-i}]$ *for all* $\omega_i \in \Omega_i$ *and* $\omega_{-i} \in \Omega_{-i}$,

*where* $m_{-i}(\omega_{-i}) = (m_1(\omega_1), \ldots, m_{i-1}(\omega_{i-1}), m_{i+1}(\omega_{i+1}), \ldots, m_N(\omega_N))$.

The lemma shows that simple type spaces in which each possible valuation/belief-hierarchy pair is held by exactly one type of each player can be embedded in the universal type space in a way that preserves all of the relevant structure.

When we start with the universal type space, we remove any implicit assumptions about the bidders' beliefs. We can now explicitly model any such assumption as a probability distribution over the bidders' universal types. Specifically, an *assumption* for the auctioneer is a distribution $\mu$ over $\Omega^*$.

In this paper we will mainly deal with two varieties of type spaces, naive type spaces and the universal type space. Once the information of the bidders' has been specified through the choice of type space, the seller's problem is to design a selling procedure in order to maximize revenue. We turn to this in the next subsection.

## 2.4 Mechanisms

An auction mechanism consists of a set $M_i$ of *messages* for each bidder $i$, an allocation rule $p : M \to [0,1]^N$, and a payment function $t : M \to \mathbf{R}^N$. Each bidder will select a message from his set $M_i$, and based on the resulting profile of messages $m$, the object is awarded according to $p(m)$ and payments are exacted according to $t(m)$. Player $i$ receives the object with probability $p_i(m)$ and pays $t_i(m)$ to the seller.

We consider environments in which the seller cannot compel the bidders to participate in the auction, so we require that each $M_i$ includes the *non-participation* message $\emptyset_i$. Selecting $\emptyset_i$ is equivalent to "opting-out" of the auction and so we assume that for any profile $m$ in which $m_i = \emptyset_i$, the allocation and payments rules satisfy $p_i(m) = 0$ and $t_i(m) = 0$. A direct-revelation mechanism for a given type space $\Omega$ is one in which $M_i = \Omega_i \cup \{\emptyset_i\}$.

about their own valuations. Our private-values model corresponds to the subspace of the universal type space in which it is common-knowledge that each bidder knows his own value. (Heifetz and Neeman, 2006, Section 2.2) call this the *private values* universal type space and Bergemann and Morris (2005) the *known own-payoffs* universal type space.

The auction mechanism defines a game-form, which together with the type space constitutes a game of incomplete information. The mechanism design problem is to fix a solution concept and search for the auction mechanism that delivers the maximum revenue for the seller in some outcome consistent with the solution concept. The now-widely adopted approach to implement the Wilson-doctrine and minimize the role of assumptions built into the naive type space is to adopt a strong solution concept which does not rely on these assumptions. In our private-value setting the often-used solution concept for this purpose is dominant strategy equilibrium.

By the revelation principle, an outcome can be implemented in dominant strategy equilibrium if and only if it is dominant strategy incentive compatible.

**Definition 2** *A direct-revelation mechanism $\Gamma$ for type space $\Omega$ is dominant strategy incentive compatible (dsIC) if for each bidder $i$ and type profile $\omega \in \Omega$,*

$$\begin{aligned} &p_i(\omega)f_i(\omega_i) - t_i(\omega) \geq 0, \quad \text{and} \\ &p_i(\omega)f_i(\omega_i) - t_i(\omega) \geq p_i(\hat{\omega}_i, \omega_{-i})f_i(\omega_i) - t_i(\hat{\omega}_i, \omega_{-i}), \end{aligned}$$

*for any alternative type $\hat{\omega}_i \in \Omega_i$.*

**Definition 3** *A dominant strategy mechanism is a dsIC direct-revelation mechanism for the naive type space $\Omega^\nu$. We denote by $\Phi$ the class of all dominant strategy mechanisms.*

When the type space is the naive type space, we have $|\Omega_i^\nu| = |V_i|$, and the incentive compatibility constraints for dsIC depend only on valuations. As a result, an auction mechanism is dsIC with respect to $\Omega^\nu$ if and only if it is dsIC with respect to any other naive type space $\Omega^{\nu'}$. So we can always discuss whether an auction mechanism is a dominant strategy mechanism without referring to any specific naive type space.

To provide a foundation for this indirect approach to implement the Wilson Doctrine, we shall compare it to the direct route of completely eliminating common knowledge assumptions about beliefs. We maintain the standard solution concept of Bayesian equilibrium but now we enlarge the type space all the way to the universal type space. The revelation principle implies that the set of resulting outcomes is equal to those that arise from truth-telling in Bayesian incentive compatible (BIC) direct-revelation mechanisms for the universal type space.

**Definition 4** *A direct-revelation mechanism $\Gamma$ for type space $\Omega = (\Omega_i, f_i, g_i)$ is Bayesian incentive compatible (BIC) if for each bidder $i$ and type $\omega_i \in \Omega_i$,*

$$\begin{aligned} &\int_{\Omega_{-i}} [p_i(\omega)f_i(\omega_i) - t_i(\omega)]g_i(\omega_i)[d\omega_{-i}] \geq 0, \quad \text{and} \\ &\int_{\Omega_{-i}} [p_i(\omega)f_i(\omega_i) - t_i(\omega)]g_i(\omega_i)[d\omega_{-i}] \geq \int_{\Omega_{-i}} [p_i(\hat{\omega}_i, \omega_{-i})f_i(\omega_i) - t_i(\hat{\omega}_i, \omega_{-i})]\, g_i(\omega_i)[d\omega_{-i}] \end{aligned}$$

*for any alternative type $\hat{\omega}_i \in \Omega_i$.*

A mechanism which does not rely on implicit assumptions about higher-order beliefs should be incentive compatible for all belief-hierarchies. In other words, it should be BIC relative to the universal type space.

**Definition 5** *Let $\Psi$ be the class of all BIC direct-revelation mechanism for the universal type space. We say that such a mechanism is detail-free.*

## 2.5 The Auctioneer as a Maxmin Decision Maker

The given valuation distribution, $\nu$, represents the auctioneer's estimate of the bidders' valuations. An assumption $\mu$ about the distribution valuations and beliefs of the bidders is consistent with this estimate if the induced marginal distribution on $V$ is $\nu$. Let $\mathcal{M}(\nu)$ denote the compact subset of such assumptions. For any mechanism $\Gamma$, the $\mu$-expected revenue of $\Gamma$ is defined as $R_\mu(\Gamma) = \int_{\Omega^*} \sum_i t_i(\omega)\, d\mu(\omega)$.

Unlike the standard formulation of the optimal auction design problem, we do not assume that the auctioneer has confidence in the naive type space as her model of bidders' beliefs. Rather the auctioneer considers other assumptions within the set $\mathcal{M}(\nu)$ as possible as well. An auctioneer who chooses an auction that maximizes the worst-case performance solves the *maxmin*[10] problem of

$$\sup_{\Gamma \in \Psi} \inf_{\mu \in \mathcal{M}(\nu)} R_\mu(\Gamma). \tag{1}$$

If the auctioneer uses a dominant strategy mechanism, then her maximum revenue would be:

$$\Pi^D(\nu) := \sup_{\Gamma \in \Phi} R_\nu(\Gamma),$$

where $R_\nu(\Gamma) = \sum_v \nu(v) \sum_i t_i(v)$ for any dominant strategy mechanism $\Gamma \in \Phi$.

As we show in the following lemma, any dominant strategy mechanism can be extended to a revenue equivalent detail-free mechanism. Thus, the optimal dominant strategy revenue is a lower bound for the maxmin value in (1).

**Lemma 2** *The auctioneer can do no worse than the optimal dominant strategy mechanism; i.e.,*

$$\sup_{\Gamma \in \Psi} \inf_{\mu \in \mathcal{M}(\nu)} R_\mu(\Gamma) \geq \Pi^D(\nu). \tag{2}$$

**Proof:** Let $\Gamma = (p, t)$ be any dominant strategy mechanism. It induces a mechanism $\Gamma' = (p', t')$ for the universal type space as follows. For all $\omega \in \Omega^*$, set $p'(\omega) = p(f^*(\omega))$ and $t'(\omega) = t(f^*(\omega))$. In other words, $\Gamma'$ is defined over the universal type space but depends

[10] Another way to think about this formulation of the problem is to view the auctioneer as uncertainty averse. The beliefs of the bidders are ambiguous to the auctioneer and this ambiguity is modeled by supposing that the auctioneer holds all possible priors $\mu$.

only on valuations and not on beliefs. Since for each realized valuation profile, $\Gamma'$ produces the same outcome as the dominant strategy mechanism $\Gamma$, it follows immediately that $\Gamma'$ is BIC. Moreover, for any $\mu \in \mathcal{M}(\nu)$,

$$R_\mu(\Gamma') = \mathbf{E}_\mu t' = \mathbf{E}_\mu t \circ f^* = \sum_{v \in V} t(v) \cdot \mu(\{\omega : f^*(\omega) = v\}) = \sum_{v \in V} t(v)\nu(v) = \mathbf{E}_\nu t = R_\nu(\Gamma).$$

Thus $\inf_{\mu \in \mathcal{M}(\nu)} R_\mu(\Gamma') = R_\nu(\Gamma)$, and the Lemma follows. ■

The maxmin foundation of dominant strategy mechanisms exists when in fact the auctioneer can do *no better* than the optimal dominant strategy mechanism; i.e., when (2) holds with equality. We will show that the maxmin foundation exists for every regular $\nu$. Specifically, we shall prove that, whenever $\nu$ is regular, there will exist an assumption $\mu^* \in \mathcal{M}(\nu)$, under which

$$\Pi^D(\nu) = \sup_{\Gamma \in \Psi} R_{\mu^*}(\Gamma), \tag{3}$$

which implies

$$\Pi^D(\nu) = \sup_{\Gamma \in \Psi} R_{\mu^*}(\Gamma) \geq \inf_{\mu \in \mathcal{M}(\nu)} \sup_{\Gamma \in \Psi} R_\mu(\Gamma) \geq \sup_{\Gamma \in \Psi} \inf_{\mu \in \mathcal{M}(\nu)} R_\mu(\Gamma),$$

so that (2) holds with equality. For this reason, if $\mu^*$ satisfies (3) then we say that $\mu^*$ *rationalizes* the use of dominant strategy mechanisms.

# 3 The Main Result

We can now state the main result.

**Theorem 1** *If $\nu$ is regular, then the use of dominant strategy mechanisms has a maxmin foundation, i.e.*

$$\sup_{\Gamma \in \Psi} \inf_{\mu \in \mathcal{M}(\nu)} R_\mu(\Gamma) = \Pi^D(\nu).$$

The proof of Theorem 1 is in the appendix. Here we shall use a simple example to illustrate the ideas behind the proof. Consider an auction with two bidders, each with two possible valuations. Bidders' valuations are correlated according to the distribution $\nu$ depicted in Figure 3.

The optimal dominant strategy mechanism is depicted in Figure 4. In Figure 4, "$\alpha = i$" is the shorthand for "allocating the object to bidder $i$" (i.e., $p_i = 1$ and $p_{-i} = 0$), and "$\alpha = 0$" means no sale.

We first verify that $\nu$ is regular. Note that the virtual valuation of the high-valuation type is equal to the valuation itself. Thus, the single-crossing condition will be satisfied

| | $v_1 = 4$ | $v_1 = 9$ |
|---|---|---|
| $v_2 = 11$ | 3/10 | 1/10 |
| $v_2 = 5$ | 3/10 | 3/10 |

Figure 3: The distribution $\nu$ of bidders' valuations.

| | $v_1 = 4$ | $v_1 = 9$ |
|---|---|---|
| $v_2 = 11$ | $\alpha = 2, t_1 = 0, t_2 = 11$ | $\alpha = 2, t_1 = 0, t_2 = 11$ |
| $v_2 = 5$ | $\alpha = 0, t_1 = 0, t_2 = 0$ | $\alpha = 1, t_1 = 9, t_2 = 0$ |

Figure 4: The optimal dominant strategy mechanism $\Gamma$.

provided the high valuation of bidder $i$ exceeds the low valuation of bidder $-i$, and this is indeed the case in our example. Hence, according to Theorem 1 there exists an assumption $\mu^*$ consistent with the distribution $\nu$ such that equation (3) holds. To illustrate the issues that are involved, we construct one such assumption below, keeping our exposition informal.

It will suffice to consider assumptions which have a simple form. For each valuation of bidder $i$, there will be exactly one type with that valuation in the support. We write $a_i$ ($b_i$) for the first-order belief held by a high-valuation (low-valuation) type of $i$ that the opponent $-i$ has high valuation. Figure 5 depicts a probability distribution over the four possible profiles of valuation/first-order belief pairs.

| | $b_1 = 2/5$ | $a_1 = 1/4$ |
|---|---|---|
| $a_2 = 1/4$ | 3/10 | 1/10 |
| $b_2 = 2/5$ | 3/10 | 3/10 |

Figure 5: Deriving the assumption $\mu^*$.

Figure 5 uniquely defines an assumption $\mu^*$ as follows. We first derive the belief hierarchies from Figure 5 by induction. For example, for a low-valuation type of bidder 1, the second-order belief assigns probability 2/5 (3/5) to bidder 2 having high (low) valuation *and* holding first-order belief $a_2 = 1/4$ ($b_2 = 2/5$); and a high-valuation (low-valuation) type of bidder 2 has a third-order belief that assigns probability 3/4 (3/5) to bidder 1 having low valuation *and* having such a second-order belief, and so on. Thus, we derive a unique belief-hierarchy for each valuation. The assumption $\mu^*$ is the measure which attaches the probabilities in figure 5 to the resulting four valuation/hierarchy profiles. It is obvious that this assumption $\mu^*$ is consistent with the distribution $\nu$.

Under this assumption $\mu^*$, there are at least two potential ways to improve upon the optimal dominant strategy auction $\Gamma$ in Figure 4. First, according to $\mu^*$, conditional on bidder 1 having low valuation, the conditional probability that bidder 2 has high valuation is 1/2. This is different from the first-order belief of the low-valuation type of bidder 1, which is $b_1 = 2/5$. In other words, $\mu^*$ is not consistent with a common prior. This suggests the possibility of a mutually acceptable bet between the auctioneer and the low value type of bidder 1 about the realized value of bidder 2. One possible way to improve upon $\Gamma$ is to build this bet into the mechanism.

Second, since the two types of bidder 1 hold different beliefs, another potential way to improve upon $\Gamma$ is to introduce lotteries in the spirit of the surplus extraction mechanisms of Crémer and McLean (1985). Note that in the dominant strategy mechanism $\Gamma$, the object goes unsold when both bidders have low valuation resulting in a deadweight-loss of total surplus. If the auctioneer were to try to capture some of that surplus by selling the good to bidder 1, dominant strategy incentive compatibility would require that the high-valuation type of bidder 1 earn "information rents." On net, the auctioneer finds this unprofitable and this is why the auctioneer witholds the object when dominant strategy incentive compatibility is imposed.

So the auctioneer may try to improve upon dominant strategies by adding payments that depend on the reported valuation of bidder 2. Due to the differences in beliefs of the two types of bidder 1, such payments can be found that induce self-selection between these two types and thereby relax the constraint of incentive compatibility.

However, incentive compatibility prevents the auctioneer from profiting from either of these maneuvers, as we now show. First, consider a bet between the auctioneer and bidder 1 about the realized valuation of bidder 2. Let $x$ and $y$ be the amount bidder 1 pays the auctioneer in the event bidder 2 has low and high valuations respectively. This bet will be acceptable to both the auctioneer and the low-valuation type of bidder 1 only if

$$\begin{aligned} (1/2)x + (1/2)y &\geq 0, \quad \text{and} \\ (3/5)(-x) + (2/5)(-y) &\geq 0, \end{aligned}$$

with at least one inequality being strict unless $x = y = 0$. But then the high-valuation type of bidder 1 would find the bet acceptable as well because

$$(3/4)(-x) + (1/4)(-y) = (5/2)[(3/5)(-x) + (2/5)(-y)] + (3/2)[(1/2)x + (1/2)y],$$

is strictly bigger than the zero rent for the high-valuation type of bidder 1 under $\Gamma$. Thus, offering the bet to the low type but not the high type would violate (Bayesian) incentive compatibility. And when both types of bidder 1 accept, the bet turns sour for the auctioneer, as

$$(3/5)(-x) + (2/5)(-y) \leq 0.$$

This explains why introducing the first type of modification does not help.

Second, consider introducing a lottery in the style of Crémer and McLean (1985) to separate the high- and low-valuation types of bidder 1. By offering a bet $(x, y)$ depending on the realization of bidder 2's type, the seller may be able to relax the downward incentive compatibility constraint and sell to the low-valuation type of bidder 1 without leaving extra rent for the high-valuation type. If such a modification is successful then we must have

$$(3/5)(4-x) + (2/5)(-y) \geq 0, \quad \text{and}$$
$$(3/4)(9-x) + (1/4)(-y) \leq 0.$$

The first inequality would be the individual rationality constraint of the low type of bidder 1, and the second would be the incentive compatibility constraint of the high type. However, these together imply that any bet like this cannot be profitable for the auctioneer, as

$$(1/2)x + (1/2)y = (2/3)[(3/4)(-x) + (1/4)(-y)] - (5/3)[(3/5)(-x) + (2/5)(-y)] \leq -1.$$

This explains why introducing the second kind of bet does not help either.

More generally, these two types of modifications could be combined in various ways and there are conceivably a variety of other potential ways to improve upon the optimal dominant strategy mechanism $\Gamma$. However, in the formal proof of Theorem 1 we use a general technique to show that in fact that there is no modification of $\Gamma$ that could improve the seller's expected revenue.

# 4 Bayesian Foundations for Dominant Strategy Mechanisms

In this section, we shall investigate another possible foundation of dominant strategy mechanisms, namely the Bayesian foundation. Imagine the auctioneer as a Bayesian decision maker. When she needs to choose a mechanism under uncertainty of bidders' beliefs, she forms a subjective belief in $\mathcal{M}(\nu)$. She evaluates any mechanism according to, instead of its worst-case performance, its average performance with respect to this subjective belief. When we as outside observers observe that this auctioneer chooses a particular mechanism, say a dominant strategy mechanism, we can ask whether or not such a choice is consistent with Bayesian rationality; i.e., whether or not such a choice is optimal with respect to *some* subjective belief. If the answer is yes, then we say that such a choice is rationalizable. Given the predominant role of Bayesian rationality in the literature of mechanism design, it seems even more natural to pursue the Bayesian foundation.

To investigate the possibility of the Bayesian foundation, we only need minimal changes in our setting. We now interpret a probability measure over the universal type space as a belief held by the auctioneer over the belief-hierarchies held by the bidders. Such a belief is consistent with the given distribution of valuations $\nu$ if it belongs to $\mathcal{M}(\nu)$. A Bayesian foundation for dominant strategies then exists if there is a belief $\mu^* \in \mathcal{M}(\nu)$ such that

$$\Pi^D(\nu) = \sup_{\Gamma \in \Psi} R_{\mu^*}(\Gamma),$$

i.e. (3) holds.

It follows from the proof of Theorem 1 that there exists a Bayesian foundation for the use of dominant strategy mechanisms when the distribution of valuations is regular. However, we shall show by example below that beyond the regular case, a Bayesian foundation need not exist. As a negative result about the rationality of using dominant strategy mechanism, we view this as particularly strong: for some distributions of valuations, no Bayesian-rational auctioneer would optimally employ a dominant strategy mechanism, regardless of her subjective belief about bidders' beliefs.

In this example, there are two bidders and each has two possible valuations. The distribution of valuations $\nu$ is represented in Figure 6.[11]

| | $v_1 = 5$ | $v_1 = 10$ |
|---|---|---|
| $v_2 = 4$ | 1/6 | 0 |
| $v_2 = 2$ | 1/3 | 1/2 |

Figure 6: The distribution $\nu$.

The optimal dominant strategy mechanism is depicted in Figure 7, where we follow the convention from the previous example and use "$\alpha = i$" as the shorthand for "allocating the object to bidder $i$.".

| | $v_1 = 5$ | $v_1 = 10$ |
|---|---|---|
| $v_2 = 4$ | $\alpha = 2, t_1 = 0, t_2 = 2$ | $\alpha = 1, t_1 = 10, t_2 = 0$ |
| $v_2 = 2$ | $\alpha = 2, t_1 = 0, t_2 = 2$ | $\alpha = 1, t_1 = 10, t_2 = 0$ |

Figure 7: The optimal dominant strategy mechanism $\Gamma$.

It is helpful to pay attention to a few noteworthy aspects of this environment and the optimal dominant strategy mechanism. Notice that the valuation of bidder 1 is always higher than that of bidder 2. Nevertheless, the auctioneer chooses to sell to bidder 2 when bidder 1 has low valuation. This is optimal because conditional on bidder 2 having low valuation, the probability that bidder 1 has high valuation is greater than 1/2. This means that it is

[11]The distribution $\nu$ in this example does not have full support. This simplifies the exposition of the example, but the conclusion would be the same if the event $\{v_1 = 10, v_2 = 4\}$ had positive (but small) probability.

optimal to exclude the low-valuation type of bidder 1 to relax the incentive constraint and sell to the high-valuation type at his reservation price. Given this, the auctioneer may as well sell to bidder 2 when bidder 1 has a low valuation. If monotonicity were not a constraint, the auctioneer would choose to sell to bidder 1 when bidder 2 had high valuation. Thus, the monotonicity constraint binds here, and in order to satisfy it, the object is sold to bidder 2 in this case.

The following proposition says that, when bidders' valuations are distributed as in Figure 6, the dominant strategy mechanism in Figure 7 can never be optimal regardless of the auctioneer's belief. It should be obvious from the proof of the proposition that this example is robust.

**Proposition 2** *For the distribution $\nu$ depicted in Figure 6, the maximum revenue achievable by any mechanism is uniformly bounded away from the maximum revenue achievable by dominant strategy mechanisms regardless of the auctioneer's subjective belief; i.e.,*

$$\inf_{\mu \in \mathcal{M}(\nu)} \sup_{\Gamma' \in \Psi} R_\mu(\Gamma') > V^D(\nu).$$

The proof of Proposition 2 is in the appendix. Here we give a verbal sketch of the argument. There are a few different ways the auctioneer could conceivably improve on the dominant strategy mechanism and for any belief of the auctioneer at least one of them will indeed improve.

The outcome of the mechanism could be made to depend on the first-order belief of bidder 2. In particular, the mechanism could ask bidder 2 to report his belief in the probability that 1 has a low valuation. Suppose 2 were to report that his own value is low and that 1 is quite likely also to have a low valuation. In an incentive-compatible mechanism, 2's report can be assumed to be truthful. But this only means that 2 truthfully believes that 1 is likely to have a low valuation. What matters is the inference made by the auctioneer about 1's valuation conditional on learning that this is the 2's belief. There are two possibilities depending on the auctioneer's belief.

The auctioneer may disagree with bidder 2. But if this is the case, then a mechanism which involves a bet between the auctioneer and bidder 2 about 1's valuation would improve the seller's revenue. Alternatively, the auctioneer may agree with bidder 2. In that case, conditional on learning that both bidders have a low valuation, the object should be sold to bidder 1 (who is willing to pay more) contrary to the outcome of the dominant strategy mechanism.

Therefore, only if the seller believes that a low-valuation bidder 2 would never believe that bidder 1 has a low valuation could it be optimal to use a mechanism which, like the optimal dominant strategy mechanism does not depend on the beliefs of bidder 2. By a symmetric argument, only if the seller believes that a high-valuation bidder 2 would never believe that bidder 1 has a high valuation could a dominant strategy mechanism be optimal.

But this means that the auctioneer must believe that the two valuation-types of bidder 2 must have a strong difference in beliefs. In such a situation, the auctioneer could improve his mechanism by including Crémer-McLean separating bets to weaken incentive constraints.

# 5 Remarks on the Common Prior Assumption

The validity, in the regular case, of the maxmin and Bayesian foundations for the use of dominant strategy mechanisms was shown by construction of a particular assumption about bidders' beliefs. It is noteworthy that the assumption constructed in the proof of Theorem 1 is inconsistent with the widely-adopted common prior assumption (CPA).

Loosely speaking, the CPA says that there is a common probability measure (the common prior) from which each bidder derives his belief by computing the conditional probability of opponents' types conditional on his own "signal" or "information." In our current setting, where any assumption about bidders' types is already modeled as a probability distribution over bidders' types, we can relate any assumption $\mu$ to the CPA as follows. For any subset $A \in \Omega_i^*$, we shall write $\mu(A)$ as a short hand for $\mu(A \times \Omega_{-i}^*)$. In other words, we abuse notation and use the same notation for a probability measure as well as its marginal distributions.

**Definition 6** *We say that an assumption $\mu$ is a CPA-assumption if for any measurable subsets $A \subset \Omega_i^*$ and $B \subset \Omega_{-i}^*$,*

$$\int_A g_i^*(\omega_i)(B)\, \mu(d\omega_i) = \mu(A \times B).$$

It is apparent that the particular assumption $\mu^*$ we used in the proof of Theorem 1 is not an CPA-assumption. Is it possible to rationalize dominant strategy mechanisms using only CPA-assumptions? We investigate this possibility in the present section.

The following notation will be convenient. Let $\Omega$ be a type space which can be embedded via Lemma 1 by some mapping $m$ in $\Omega^*$, and let $\rho$ be any common prior over $\Omega$. The corresponding prior over $\Omega^*$ is defined by $\rho \circ m^{-1}$. We denote it by $m(\rho)$. If $\Omega$ is the naive type space $\Omega^\nu$, we abuse notation and use $\nu$ to denote also the common prior over $\Omega^\nu$, and use $m(\nu)$ to denote the corresponding distribution over $\Omega^*$. If $\mu$ takes the form of $m(\rho)$, where $\rho$ is the common prior of some type space $\Omega$ embeddable in the universal type space $\Omega^*$, then $\mu$ will be an CPA-assumption in the sense of Definition 6.

In Appendix C, we use an example to demonstrate that, when $\nu$ is far from being an independent distribution over $V$, the answer is negative. In this section, we shall present some positive results for the case when $\nu$ is close to an independent distribution.

We begin by noting that when $\nu$ is a product measure, i.e. the players' valuations are drawn independently, then the regular case reduces to the familiar monotone hazard rate condition. In this case, we can consider the naive type space $\Omega^\nu$ with the prior $\nu$, and it has been shown that the optimal BIC mechanism can be implemented in dominant strategies. When we embed $\Omega^\nu$ in the universal type space $\Omega^*$, the image of $\nu$ (i.e., $m(\nu)$) will be an CPA-assumption that rationalizes the use dominant strategy mechanisms. We record this observation as a lemma for ease of reference.

**Lemma 3** *Let $\nu$ be regular and independent, and let $m(\nu)$ be the corresponding distribution over the image of $\Omega^\nu$ in the universal type space. Then $m(\nu)$ is an CPA-assumption, and $\sup_{\Gamma\in\Psi} R_{m(\nu)}(\Gamma) = \Pi^D(\nu)$.*

When $\nu$ is close to an independent distribution, but not independent itself, Lemma 3 fails dramatically. Indeed, the Crémer and McLean (1985) mechanism extracts all buyers' surplus while the optimal dsIC mechanism must yield some information rent to high-value buyers. Thus, $m(\nu)$ itself cannot be used as a rationalizing CPA-assumption. However, we show that whenever $\nu$ is regular and close to an independent distribution, there exists an CPA-assumption $\mu$ which "almost" rationalizes the use of dominant strategy mechanisms. Precisely, the optimal dominant strategy mechanism achieves nearly the same revenue as the optimal detail-free mechanism under assumption $\mu$.

To state the general result, we introduce some necessary notation. We say that valuation distribution $\nu$ is $\varepsilon$-close to $\hat{\nu}$ if there exists some $\tilde{\nu}$ such that $\nu = (1-\varepsilon)\hat{\nu} + \varepsilon\tilde{\nu}$.

**Theorem 2** *For any regular and independent $\hat{\nu}$ and $\delta > 0$, there exists $\varepsilon > 0$ such that, if $\nu$ is $\varepsilon$-close to $\hat{\nu}$, then there exists an CPA-assumption $\mu$ such that*

$$\Pi^D(\nu) \geq \sup_{\Gamma\in\Psi} R_\mu(\Gamma) - \delta.$$

**Proof:** We begin by constructing a type space with a common prior. For each bidder $i$, and valuation $v_i$ there are two types $\hat{\omega}^{v_i}$ and $\tilde{\omega}^{v_i}$, with $f_i(\hat{\omega}^{v_i}) = f_i(\tilde{\omega}^{v_i}) = v_i$. There is a common prior $\rho$ over the set $\Omega$ of type profiles defined as follows.

$$\rho(\omega) = \begin{cases} (1-\varepsilon)\hat{\nu}(v_1,\ldots,v_n) & \text{if } \omega_i = \hat{\omega}^{v_i} \text{ for each } i, \\ \varepsilon\tilde{\nu}(v_1,\ldots,v_n) & \text{if } \omega_i = \tilde{\omega}^{v_i} \text{ for each } i, \\ 0 & \text{otherwise.} \end{cases}$$

That is, with probability $1-\varepsilon$, the valuations will be drawn from the independent distribution $\hat{\nu}$, and with the remaining probability from the distribution $\tilde{\nu}$. This type space thus has two belief-closed subspaces corresponding to the value distributions $\hat{\nu}$ and $\tilde{\nu}$. We denote these subspaces $\hat{\Omega}$ and $\tilde{\Omega}$. We define the belief mappings $g_i$ by Bayesian updating from $\rho$. By lemma 1, $(\Omega, f, g)$ can be embedded by some mapping $m$ in the universal type space because each type of bidder $i$ with the same valuation has a distinct hierarchy of beliefs.[12] We thus

[12]Consider the types belonging to $\tilde{\Omega}$. For these types it is common-knowledge that values are drawn from $\tilde{\nu}$. A type in $\Omega$ can have such a hierarchy if and only if the valuation is indeed drawn from $\tilde{\nu}$. Thus, the remaining types (i.e., types in $\hat{\Omega}$) have different hierarchies.

take $\mu$ to be $m(\rho)$, the corresponding distribution over the image of $\Omega$. It is an immediate consequence of the construction and Definition 6 that $\mu^*$ is an CPA-assumption and also that $\mu \in \mathcal{M}(\nu)$. Similarly, we define $\hat{\mu} = m(\hat{\nu})$ and $\tilde{\mu} = m(\tilde{\nu})$. Note that $\mu = (1-\varepsilon)\hat{\mu} + \varepsilon\tilde{\mu}$.

Let $\Gamma$ be any mechanism in $\Psi$. We have

$$R_\mu(\Gamma) = (1-\varepsilon)R_{\hat{\mu}}(\Gamma) + \varepsilon R_{\tilde{\mu}}(\Gamma).$$

Hence,

$$\begin{aligned}\sup_{\Gamma\in\Psi} R_\mu(\Gamma) &\leq (1-\varepsilon)\sup_{\hat{\Gamma}\in\Psi} R_{\hat{\mu}}(\hat{\Gamma}) + \varepsilon \sup_{\tilde{\Gamma}\in\Psi} R_{\tilde{\mu}}(\tilde{\Gamma}) \\ &= (1-\varepsilon)\Pi^D(\hat{\nu}) + \varepsilon \sup_{\tilde{\Gamma}\in\Psi} R_{\tilde{\mu}}(\tilde{\Gamma}),\end{aligned}$$

where the equality follows from Lemma 3.

From the maximum theorem, for any $\kappa > 0$ we can choose $\varepsilon$ small enough so that $\Pi^D(\nu) \geq \Pi^D(\hat{\nu}) - \kappa$. Thus,

$$\Pi^D(\nu) \geq \sup_{\Gamma\in\Psi} R_{\mu^*}(\Gamma) - \varepsilon\left[\sup R_{\tilde{\mu}}(\tilde{\Gamma}) - \Pi^D(\hat{\nu})\right] - \kappa.$$

Because $\tilde{\mu}$ is an CPA-assumption, we have the standard accounting identity: total surplus equals buyers' expected utility plus $R_{\tilde{\mu}}(\tilde{\Gamma})$. Because $\tilde{\Gamma} \in \Psi$ satisfies individual rationality, buyers' surplus is non-negative, so $R_{\tilde{\mu}}(\tilde{\Gamma})$ is bounded by the total surplus which is itself uniformly bounded by $v^M = \max V < \infty$. We thus have

$$\Pi^D(\hat{\nu}) \geq \sup_{\Gamma\in\Psi} R_{\mu^*}(\Gamma) - \varepsilon v^M - \kappa,$$

which yields the statement of the theorem when we take $\kappa = \delta/2$ and $\varepsilon < \delta/(2v^M)$. ■

Theorem 2 shows that when the auctioneer believes that valuations are distributed nearly-independently, then the use of dominant strategy mechanisms has an approximate maxmin foundation *even if we limit ourselves to CPA-assumptions*. In this case, any slight loss in revenue might be compensated for by the other virtues of dominant strategy mechanisms, e.g. simplicity and transparency of equilibrium play for the bidders.

# 6 Conclusion

We have identified a sufficient condition, a direct generalization of the regular case in Myerson (1981), under which dominant strategy mechanisms can be rationalized as optimal mechanisms, either by appeal to maxmin or Bayesian optimality criteria. Let us conclude by pointing out one additional implication of this result. Suppose that in addition to the regularity assumption, the distribution of valuations $\nu$ is symmetric, a natural assumption for a seller who does not know the identities or characteristics of the bidders. Then the English auction with a suitably chosen reserve price is an optimal dominant strategy auction.[13] We have thus shown that in symmetric, regular environments, the widespread use of the English auction as a selling mechanism can be justified as an optimal response to uncertainty about the bidders' beliefs.

[13] Lopomo (2000) proved that the English auction is the optimal dominant strategy mechanism in (almost) this setting.

## Appendix A: Universal Type Space

In this appendix, we review the Mertens and Zamir (1985) (hereafter MZ) construction of the universal type space and show how to apply it in our setting.[14] In general, the set of possible first-order beliefs for bidder $i$ is

$$\mathcal{T}_i^1 := \Delta V_{-i},$$

and the set of all possible $k$th-order beliefs is

$$\mathcal{T}_i^k := \Delta(V_{-i} \times \mathcal{T}_{-i}^{k-1}).$$

An infinite hierarchy of beliefs for bidder $i$ is a sequence $h_i = (h_i^1, h_i^2, \ldots)$ satisfying $h_i^k \in \mathcal{T}_i^k$

The projections $\phi_i^k : \mathcal{T}_i^k \to \mathcal{T}_i^{k-1}$, defined inductively by $\phi_i^2(h_i^2)(v_{-i}) = h_i^2(\{v_{-i}\} \times \mathcal{T}_{-i}^1)$, and for each measurable subset $\{v_{-i}\} \times B \subset V_{-i} \times \mathcal{T}_{-i}^{k-2}$,

$$\phi_i^k(h_i^k)(\{v_{-i}\} \times B) = h_i^k(\{v_{-i}\} \times \left[\phi_{-i}^{k-1}\right]^{-1}(B)),$$

demonstrate that each $k$th-order belief for bidder $i$ implicitly defines beliefs at lower orders as well. A hierarchy is said to be *coherent* if these implicitly defined beliefs are consistent with those explicitly defined at lower orders; i.e., $\phi_i^k(h_i^k) = h_i^k$.

Recall (see footnote 7) that for any type $\omega_i$ in any type space, it is possible to identify the hierarchy of beliefs of $\omega_i$. Let $h_i(\omega_i)$ represent this hierarchy.

**Lemma 4** *There exists a type space $\Omega^*$ such that for each player $i$, each value $v_i \in V_i$ and each coherent infinite hierarchy of beliefs $\hat{h}_i$ over $V_{-i}$, there is a type $\omega_i \in \Omega_i^*$ such that $f_i(\omega_i) = v_i$ and $h_i(\omega_i) = \hat{h}_i$. Moreover, each $\Omega_i^*$ is a compact topological space.*

This lemma is a straightforward application of the results in MZ which we briefly sketch. We take the space of basic uncertainty (what MZ call the parameter space) to be $V$. The main theorem in MZ (Theorem 2.9) shows the existence of the "universal belief-space" $\mathcal{Y}$ generated by $V$. Because all possible beliefs are included in $\mathcal{Y}$, it allows for the possibility that player $i$ is not certain of which element of $v_i$ has been realized. Thus $\mathcal{Y}$ is too large for our purposes. Instead, MZ's remark 2.17 derives a compact belief-subspace $C$ in which it is common-knowledge that each player $i$ knows his own value.

[14] It has been recently discovered that the MZ belief-hierarchies are limited in a certain sense: some rationalizable and equilibrium behaviors can arise when modeled using a small type space but cannot be captured in the universal type space. See for example Ely and Pęski (2006) and Dekel, Fudenberg, and Morris (2006). To capture all possible assumptions that are relevant for Bayesian equilibrium behavior, the universal type space would have to be enlarged. This issues can arise only in models with a common-value element. They would not alter any of the results in our private-value setting.

Formally, $C$ is a compact space such that there exist for each $i$, spaces[15] $\Omega_i$ where $\Omega_i \subset \Delta(V_{-i} \times \Omega_{-i})$ such that

$$C \cong V \times \Omega_1 \times \ldots \times \Omega_N$$

where $\cong$ denotes homeomorphism.[16] Each $\Omega_i$ is a set of possible beliefs for $i$ about the valuations and beliefs of the others. The hierarchies derived from $C$ will have the property that it is common-knowledge that each player knows his own valuation. Moreover, the MZ construction of $C$ (see remark 2.17 and property 6 in MZ) ensures that every profile of hierarchies of beliefs satisfying this restriction is represented in $C$.

To obtain our setup, we take $\Omega_i^* = V_i \times \Omega_i$, and let $f_i^* : \Omega_i^* \to V_i$ and $g_i^* : \Omega_i^* \to \Omega_i$ be the projection mappings. Because $V_i$ is finite and $\Omega_i$ is compact, we have that $\Omega_i^*$ is compact. Our universal type space (corresponding to the "private values" universal type space of Heifetz and Neeman (2006)) is then $\Omega^* = (\Omega_i^*, f_i^*, g_i^*)_{i \in N}$.

Also, Lemma 1 is an immediate consequence of Property 5 from MZ when we note that the non-redundancy condition of MZ is satisfied if the mapping $\omega_i \to (f_i(\omega_i), h_i(\omega_i))$ is one-to-one.

## Appendix B: Proof of Theorem 1

In this section, we shall first review the properties of the optimal dominant strategy mechanism design problem that will be used in the proof of Theorem 1. We use a version of a standard argument to show that the dominant strategy incentive compatibility constraints can be replaced by a monotonicity constraint on the allocation rule. We then show that regularity implies that the monotonicity constraint is not binding in the optimal dominant strategy mechanism design problem. This sets the stage for the proof of Theorem 1. The latter proceeds by constructing an assumption against which the optimal BIC mechanism design problem reduces to the same objective function but without the monotonicity constraint. It follows that the optimal values in the two problems are the same.

We can formulate the optimal dominant strategy mechanism design problem as follows:

$$\max_{p(\cdot),t(\cdot)} \qquad \sum_{v_i \in V} \nu(v) \sum_{i=1}^{N} t_i(v) \tag{4}$$

$$\text{subject to:} \quad \forall\ i = 1, \ldots, N,\ \forall\ m, l = 1, \ldots, M,\ \forall\ v_{-i} \in V_{-i},$$

$$p_i(v^m, v_{-i})v^m - t_i(v^m, v_{-i}) \geq 0, \qquad \langle DIR_i^m \rangle$$

$$p_i(v^m, v_{-i})v^m - t_i(v^m, v_{-i}) \geq p_i(v^l, v_{-i})v^m - t_i(v^l, v_{-i}). \qquad \langle DIC_i^{m \to l} \rangle$$

We omit the proof of the following standard lemma which establishes that the constraints in (4) can be replaced by a single monotonicity constraint on the allocation rule.

[15] MZ call these type spaces. Our use of that terminology is therefore slightly different, but follows standard usage in mechanism design.

[16] The product structure of $C$ is not explicitly noted in MZ, but it follows from the construction, see remark 2.17 and property 6 in MZ.

**Lemma 5** *Say that an allocation rule $p$ is dsIC if there exists a transfer rule $t$ such that the auction mechanism $(p,t)$ satisfies the constraints in (4). A necessary and sufficient condition for $p$ to be dsIC is the following monotonicity condition: $\forall i = 1, \ldots, N$,*

$$p_i(v^m, v_{-i}) \geq p_i(v^{m-1}, v_{-i}), \qquad \forall\ m = 2, \ldots, M, \qquad \forall\ v_{-i} \in V_{-i}. \qquad \langle M_i \rangle$$

It follows from standard arguments that in an optimal dominant strategy mechanism, the constraints $\langle DIR_i^1 \rangle$ and $\langle DIC_i^{m \to m-1} \rangle$ are binding and (given that $p$ is monotonic) all other constraints can be ignored. Combining the resulting equalities, we see that when the other bidders report valuation profile $v_{-i}$, bidder $i$'s net utility ("rent") will be

$$U_i(v^1, v_{-i}) = 0$$

for type $v^1$ and

$$U_i(v^m, v_{-i}) = p_i(v^{m-1}, v_{-i})(v^m - v^{m-1}) + U_i(v^{m-1}, v_{-i}) = \gamma \sum_{m'=1}^{m-1} p_i(v^{m'}, v_{-i})$$

for type $v^m$, $m > 1$. By definition, the total transfer received by the auctioneer is the total surplus generated by any sale of the object less the rent received by the bidders. Thus, an equivalent formulation of the problem is to choose a dsIC (i.e., monotonic) allocation rule to maximize the expected value of this difference.

$$\max_{p(\cdot)} \qquad \sum_{i=1}^{N} \sum_{m=1}^{M} \sum_{v_{-i} \in V_{-i}} \nu(v^m, v_{-i}) \left[ p_i(v^m, v_{-i}) v^m - \gamma \sum_{m'=1}^{m-1} p_i(v^{m'}, v_{-i}) \right] \qquad (5)$$
$$\text{subject to } \langle M_i \rangle,\ i = 1, \ldots, N.$$

A typical approach to solving a problem in this form is to first consider the unconstrained maximization of (5), and check whether the solution satisfies the monotonicity constraint. In the following proposition we show that this is guaranteed to be the case under the regularity condition. The second part is Proposition 1 presented in the text.

**Proposition 3**

1. *If $\nu$ is regular, then any solution to the unconstrained problem (5) also satisfies the constraints $\langle M_i \rangle$.*
2. *If $\nu$ satisfies both the monotone hazard rate condition and affiliation, then $\nu$ is regular.*

**Proof:** First, consider the relaxed problem ignoring the monotonicity constraint in (5). Fix a valuation profile $v$ and notice that the derivative of the maximand with respect to $p_i(v)$

is bidder $i$'s virtual valuation: $v_i - \gamma \sum_{\hat{v}_i > v_i} \nu(\hat{v}_i, v_{-i})/\nu(v)$. It will be optimal at valuation profile $v$ to award the object for sure to the bidder with the greatest non-negative virtual valuation, with the object going unsold if all virtual valuations are negative.[17] (In the event that two or more bidders tie for the greatest non-negative virtual valuation, the tie can be broken arbitrarily.)

For part 1, suppose that the virtual valuations satisfy the single-crossing condition, and let $p$ be an allocation rule that solves the unconstrained maximization of (5). Then $p_i(v) > 0$ only if $\gamma_i(v) \geq \max_j \gamma_j(v)$, and $p_i(v) = 1$ if $\gamma_i(v) > \max_{j \neq i} \gamma_j(v)$. Fix $v$ such that $p_i(v) > 0$, (so that $\gamma_i(v) \geq \max_j \gamma_j(v)$) and consider an increase in the valuation of bidder $i$ to $\hat{v}_i > v_i$. By the single-crossing condition, $\gamma_i(v) > \max_{j \neq i} \gamma_j(v)$ and hence $p_i(v) = 1$. This shows that $\langle M_i \rangle$ is satisfied.

For part 2, suppose that both affiliation and the monotone hazard rate condition are satisfied and let $v$ be a valuation profile at which $\gamma_i(v) \geq \gamma_j(v)$. Consider an increase in the valuation of bidder $i$ to $\hat{v}_i > v_i$. Write $\hat{v} = (\hat{v}_i, v_{-i})$. It is well-known that affiliation implies that this "increases" the conditional distribution of other bidders' valuations in the sense of the monotone likelihood ratio ordering. That is, for any pair of valuations $v'_j > v_j$, $\frac{\nu(v'_j, \hat{v}_{-j})}{\nu(\hat{v})} \geq \frac{\nu(v'_j, v_{-j})}{\nu(v)}$.

The new virtual valuation for any bidder $k$ is

$$\gamma_k(\hat{v}_i, v_{-i}) = \hat{v}_k - \gamma \frac{1 - F_k(\hat{v})}{\nu(\hat{v})}$$

By the monotone hazard rate condition $\gamma_i(\hat{v}) > \gamma_i(v)$. By affiliation, for each bidder $j \neq i$,

$$\begin{aligned} \frac{1 - F_j(\hat{v})}{\nu(\hat{v})} &= \sum_{v'_j > v_j} \frac{\nu(v'_j, \hat{v}_{-j})}{\nu(\hat{v})} \\ &\geq \sum_{v'_j > v_j} \frac{\nu(v'_j, v_{-j})}{\nu(v)} \\ &= \frac{1 - F_j(v)}{\nu(v)} \end{aligned}$$

and this implies $\gamma_j(\hat{v}) \leq \gamma_j(v)$. And for the seller ($j = 0$), the latter inequality holds by definition.

Combining these results we have $\gamma_i(\hat{v}) > \gamma_j(\hat{v})$. Since $j$ was arbitrary, this proves that the single crossing condition holds. ■

We are now in a position to prove Theorem 1. The structure of the proof is as follows. We begin by supposing that $\nu$ is regular and satisfies an additional condition, called non-singularity. We show that the maxmin foundation exists for dominant strategy mechanisms

[17] For related derivations, see Lopomo (2000) and Segal (2003).

in this case. Next we show that we can find a sequence of such distributions to approach any $\nu$ satisfying the hypotheses of the theorem. We then apply a limiting argument to show that the maxmin foundation for dominant strategy mechanisms exists in this case as well.

Given $\nu$, write $\nu_i^m$ for the marginal probability of valuation $v_i = v^m$, and write $G_i(m) = \sum_{m'=m}^M \nu_i^{m'}$ for the associated de-cumulative distribution function. Let $\sigma_i^m = \nu(\cdot|v^m)$ be the conditional distribution over the valuations of bidders $j \neq i$ conditional on bidder $i$ having valuation $v^m$. Say that $\nu$ is *non-singular* if the collection of vectors $\{\sigma_i^m\}_{m=1}^M$ is linearly independent.

Say that a type space is *simple* if for each player $i$ and valuation $v_i$ there is a unique type for $i$ with valuation $v_i$; i.e., the mapping $f$ is one-to-one.[18] By Lemma 1, a simple type space can be embedded via a mapping $m$ into the universal type space. Say that an assumption $\mu$ is *simple* if it concentrates on the image in $\Omega^*$ of a simple type space. In this case, for any mechanism $(p, t)$ defined over $\Omega^*$ we can consider the reduced mechanism $(\bar{p}, \bar{t})$ defined over $V$, where

$$\bar{p}_i(v) = p_i(m(f^{-1}(v))), \qquad \bar{t}_i(v) = t_i(m(f^{-1}(v))).$$

A further notational simplification will be convenient. Let $\bar{p}_i^m$ and $\bar{t}_i^m$ denote respectively the vectors $\langle \bar{p}_i(v_i^m, \cdot)\rangle_{v_{-i} \in V_{-i}}$ and $\langle \bar{t}_i(v_i^m, \cdot)\rangle_{v_{-i} \in V_{-i}}$ in $\mathbf{R}^{M^{N-1}}$.

Suppose $\nu$ is non-singular and regular. We begin by constructing a simple type space which will then be embedded in the universal type space using Lemma 1. Let the set of types for player $i$ be equal to the set of possible valuations, i.e. $\hat{\Omega}_i = V_i$. We take $f_i$ to be the identity, and for notational ease we will write $\tau_i^m = g_i(v^m)$ for the beliefs of type $v^m$ of bidder $i$ about the types of the other bidders.

These beliefs are defined as follows:

$$\forall i, \forall m, \quad \tau_i^m = \frac{1}{G_i(m)} \sum_{m'=m}^{M} \nu_i^{m'} \sigma_i^{m'}.$$

Thus, conditional on having valuation $v^m$, bidder $i$'s belief over opponents' valuations is the average of the auctioneer's beliefs conditional on $i$ having valuation *at least* $v^m$.[19] Note that the collection $\{\tau_i^m\}_{m=1}^M$ is linearly independent by the non-singularity of $\nu$. The following equivalent recursive definition of $\tau_i^m$ is useful:

$$\begin{aligned} \tau_i^M &= \sigma_i^M, \\ \tau_i^m &= \frac{1}{G_i(m)} \left(\nu_i^m \sigma_i^m + G_i(m+1)\tau_i^{m+1}\right), \quad \forall m < M. \end{aligned} \tag{6}$$

Finally, because it is simple, the type space $\hat{\Omega} = (\hat{\Omega}_i, f_i, g_i)_{i \in N}$ can be embedded in the universal type space by Lemma 1. Let $\Omega_i \subset \Omega_i^*$ be the image in the universal type space,

[18] The naive type space $\Omega^\nu$ is one example of simple type spaces.

[19] Thus, each bidder type has beliefs which are a distortion of those that would be derived from $\nu$, except for the highest valuation type, where there is "no distortion at the top."

and write $\omega_i^m = m_i(v_i^m)$. Type $\omega_i^m$ has valuation $f_i^*(\omega_i^m) = v_i^m$ and belief $g_i^*(\omega_i^m) = \tau_i^m$ (up to the relabeling). We can now define the assumption $\mu^*$ by setting $\mu^*(m(f^{-1}(v))) = \nu(v)$. Clearly $\mu^* \in \mathcal{M}(\nu)$. We will show that $\mu^*$ is a rationalizing assumption.

Under the simple assumption $\mu^*$, the optimal BIC auction design problem can be expressed as follows:

$$\begin{aligned} &\max_{p(\cdot),t(\cdot)} \quad \sum_{i=1}^{N} \sum_{v \in V} \nu(v) \bar{t}_i(v) && (7) \\ &\text{subject to:} \quad \forall\, i = 1, \ldots, N,\ \forall\, m = 1, \ldots, M,\ \forall\, l = 1, \ldots, M, && \\ &\qquad \tau_i^m \cdot (\bar{p}_i^m v^m - \bar{t}_i^m) \geq 0, && \langle IR_i^m \rangle \\ &\qquad \tau_i^m \cdot (\bar{p}_i^m v^m - \bar{t}_i^m) \geq \tau_i^m \cdot \left(\bar{p}_i^l v^m - \bar{t}_i^l\right). && \langle IC_i^{m \to l} \rangle \end{aligned}$$

We have used the inner product notation such as $\tau_i^m \cdot \bar{t}_i^m$ for expectations with respect to the belief $\tau_i^m$. Note that the *IR* and *IC* constraints for all types outside of the support of $\mu^*$ have been omitted.[20]

Say that an allocation rule $p$ is BIC if there exists a transfer rule $t$ such that the auction mechanism $(p, t)$ satisfies the constraints in (7). Because the beliefs of those types of each bidder that appear in (7) are linearly independent, every allocation rule is BIC. Indeed, by exploiting the differences in beliefs, the incentive compatibility and individual rationality constraints can be satisfied by building into the transfer rule lotteries which have positive expected value to the intended type and arbitrarily large negative expected values to the other types. This kind of construction is due to Crémer and McLean (1985), and we shall omit the details.

While the above argument shows that any allocation rule is implementable by some appropriate choice of transfer rule, we can further sharpen the conclusion and argue that certain constraints in (7) can be manipulated or even ignored without cost to the auctioneer. To begin with, each "upward" incentive constraint (i.e., $\langle IC_i^{m \to l} \rangle$ for $m < l$) can be ignored. Indeed, because bidder $i$'s beliefs are linearly independent, there exists a lottery $\lambda \in \mathbf{R}^{M^{N-1}}$ such that $\tau_i^m \cdot \lambda = 0$ for all $m \geq l$ and $\tau_i^m \cdot \lambda < 0$ for all $m < l$. Since by (6) $\sigma_i^l$ is a linear combination of $\tau_i^l$ and $\tau_i^{l+1}$, we also have $\sigma_i^l \cdot \lambda = 0$. By adding (some sufficiently large scale of) $\lambda$ to $\bar{t}_i^l$, each $\langle IC_i^{m \to l} \rangle$ for $m < l$ can be relaxed. No other constraints are affected and the resulting change in the auctioneer's revenue is $\sigma_i^l \cdot \lambda = 0$.

We next show that for any auction mechanism $(p, t)$ that satisfies the remaining constraints, there exists an auction mechanism $(p', t')$ which satisfies the constraints $\langle IR_i^m \rangle$, for $m = 1, \ldots, M$, and $\langle IC_i^{m \to m-1} \rangle$, for $m = 2, \ldots, M$, with equality, and achieves at least as high an $\mu^*$-expected revenue as $(p, t)$ does.

[20] More precisely, we are looking at the relaxed problem. The solution to the relaxed problem 7 provides an upper bound for the auctioneer's revenue that can be achieved by any mechanism under the assumption $\mu^*$.

To prove this, fix any auction mechanism $(p, t)$ that satisfies the remaining constraints. Suppose $\langle IC_i^{m\to m-1}\rangle$ holds with strict inequality. Let $\boldsymbol{\tau}$ denote the matrix whose $M$ rows are the vectors $\{\tau_i^m\}_{m=1}^M$, and let $(\boldsymbol{\tau}^{-m}, \sigma_i^{m-1})$ be the matrix obtained by replacing the $m$th row of $\boldsymbol{\tau}$ with the vector $\sigma_i^{m-1}$. Note that the matrix $(\boldsymbol{\tau}^{-m}, \sigma_i^{m-1})$ has rank $M$. We can thus solve the following equation for $\lambda$:

$$(\boldsymbol{\tau}^{-m}, \sigma_i^{m-1}) \cdot \lambda = x^m,$$

where $x^m$ denotes the $m$th elementary basis vector in $\mathbf{R}^M$. Note that because $\tau_i^{m-1} \cdot \lambda = 0 < \sigma_i^{m-1} \cdot \lambda$, and because $\tau_i^{m-1}$ is a convex combination of $\sigma_i^{m-1}$ and $\tau_i^m$ according to (6), we have $\tau_i^m \cdot \lambda < 0$.

We will add the vector $\varepsilon\lambda$ to $\bar{t}_i^{m-1}$ for some scalar $\varepsilon > 0$. Because $\tau_i^{m'} \cdot \lambda = 0$ for $m' \neq m$, no constraints for types $\omega_i^{m'}$ are affected. As for type $\omega_i^m$, the constraint $\langle IR_i^m\rangle$ is unaffected. The only incentive constraint of type $\omega_i^m$ that is affected is $\langle IC_i^{m\to m-1}\rangle$, and this constraint was slack by assumption. Let $S_i^m > 0$ be the slack in $\langle IC_i^{m\to m-1}\rangle$, and choose $\varepsilon = -S_i^m/(\tau_i^m \cdot \lambda) > 0$. Then, with the resulting transfer rule, $\langle IC_i^{m\to m-1}\rangle$ holds with equality. Finally, because $\varepsilon\sigma_i^{m-1} \cdot \lambda > 0$, the auctioneer profits from this modification.

We next show that each $\langle IR_i^m\rangle$ can be treated as an equality without loss of generality. Define $S_i^m = \tau_i^m \cdot (\bar{p}_i^m v^m - \bar{t}_i^m) \geq 0$ to be the slack in $\langle IR_i^m\rangle$. Construct a lottery $\lambda$ that satisfies

$$\tau_i^m \cdot \lambda = S_i^m, \quad m = 1, \ldots, M.$$

By the full-rank arguments such a lottery $\lambda$ can be found. We will add $\lambda$ to each $\bar{t}_i^m$. No constraint of the form $\langle IC_i^{m\to l}\rangle$ will be affected, but now each constraint of the form $\langle IR_i^m\rangle$ holds with equality. Finally, we check that the auctioneer profits from this modification. Indeed, the auctioneer nets

$$\begin{aligned}
\sum_{m=1}^{M} \nu_i^m \left(\sigma_i^m \cdot \lambda\right) &= \sum_{m=1}^{M-1} \left(G_i(m)\tau_i^m - G_i(m+1)\tau_i^{m+1}\right) \cdot \lambda + \nu_i^M \tau_i^M \cdot \lambda \\
&= G_i(1)\tau_i^1 \cdot \lambda \\
&= G_i(1)S_i^1 \\
&\geq 0.
\end{aligned}$$

The proof for the non-singular case is now concluded as follows. Based on the preceding arguments, we consider the modified program in which the constraints $\langle IR_i^m\rangle$ and $\langle IC_i^{m\to m-1}\rangle$ are satisfied with equality. We will use these constraints to substitute out for the transfers in the objective function and reduce the problem to an *unconstrained* optimization with the only choice variable being the allocation rule (recall that any allocation rule is BIC). The resulting objective function will be identical to the objective function (4) for the dsIC problem. Thus the only difference between the two problems is the absence of any monotonicity constraint in the BIC case. It then follows that $(i)$ the modified problem and hence the original problem (7) will have a solution, and $(ii)$ this solution will be the same as the solution to the optimal dominant strategy mechanism design problem by Part 1 of Proposition 3. In particular, equation (3) holds and $\mu^*$ rationalizes the use of dominant strategy mechanisms.

We rewrite the objective function in (7) as below, and impose the constraints as equalities:

$$\begin{aligned}
\max_{p(\cdot),t(\cdot)} \quad & \sum_{i=1}^{N}\sum_{m=1}^{M} \nu_i^m \sigma_i^m \cdot \bar{t}_i^m && (8)\\
\text{subject to:} \quad & \forall\, i = 1,\ldots,N,\ \forall\, m = 1,\ldots,M, \\
& \tau_i^m \cdot (\bar{p}_i^m v^m - \bar{t}_i^m) = 0, && \langle \overline{IR}_i^m \rangle \\
& \tau_i^m \cdot (\bar{p}_i^m v^m - \bar{t}_i^m) = \tau_i^m \cdot \left(\bar{p}_i^{m-1} v^m - \bar{t}_i^{m-1}\right). && \langle \overline{IC}_i^{m\to m-1} \rangle
\end{aligned}$$

By definition, $\sigma_i^M = \tau_i^M$, so $\langle \overline{IR}_i^M \rangle$ becomes $\sigma_i^M \cdot \bar{t}_i^M = v^M \sigma_i^M \cdot \bar{p}_i^M$. Now, for arbitrary $m < M$,

$$\begin{aligned}
\sigma_i^m \cdot \bar{t}_i^m &= \frac{1}{\nu_i^m}\left[G_i(m)\tau_i^m - G_i(m+1)\tau_i^{m+1}\right] \cdot \bar{t}_i^m \\
&= \frac{1}{\nu_i^m}\left\{G_i(m) v^m \tau_i^m \cdot \bar{p}_i^m - G_i(m+1)\left[\tau_i^{m+1} \cdot (\bar{p}_i^m - \bar{p}_i^{m+1}) v^{m+1} + \tau_i^{m+1} \cdot \bar{t}_i^{m+1}\right]\right\} \\
&= \frac{1}{\nu_i^m}\left[G_i(m) v^m \tau_i^m \cdot \bar{p}_i^m - G_i(m+1) v^{m+1} \tau_i^{m+1} \cdot \bar{p}_i^m\right].
\end{aligned}$$

In the first line we used the recursive definition in (6), in the second line we used $\langle \overline{IR}_i^m \rangle$ and $\langle \overline{IC}_i^{m+1\to m} \rangle$, and in the third line we used $\langle \overline{IR}_i^{m+1} \rangle$.

Substituting the constraints into the objective function, it becomes:

$$\begin{aligned}
& \sum_{i=1}^{N}\left\{ v^M \nu_i^M \sigma_i^M \cdot \bar{p}_i^M + \sum_{m=1}^{M-1}\left[v^m G_i(m)\tau_i^m \cdot \bar{p}_i^m - v^{m+1} G_i(m+1)\tau_i^{m+1} \cdot \bar{p}_i^m\right]\right\} \\
= \ & \sum_{i=1}^{N}\left\{ v^M \nu_i^M \sigma_i^M \cdot \bar{p}_i^M + \sum_{m=1}^{M-1}\left[v^m \left(\nu_i^m \sigma_i^m + G_i(m+1)\tau_i^{m+1}\right) \cdot \bar{p}_i^m - v^{m+1} G_i(m+1)\tau_i^{m+1} \cdot \bar{p}_i^m\right]\right\} \\
= \ & \sum_{i=1}^{N}\left[\sum_{m=1}^{M} v^m \nu_i^m \sigma_i^m \cdot \bar{p}_i^m - \sum_{m=2}^{M}(v^m - v^{m-1}) G_i(m)\tau_i^m \cdot \bar{p}_i^{m-1}\right].
\end{aligned}$$

Applying the definition of $\tau_i^m$, the objective function becomes:

$$\begin{aligned}
& \sum_{i=1}^{N}\left[\sum_{m=1}^{M} v^m \nu_i^m \sigma_i^m \cdot \bar{p}_i^m - \sum_{m=2}^{M} \gamma \left(\sum_{m'=m}^{M} \nu_i^{m'} \sigma_i^{m'}\right) \cdot \bar{p}_i^{m-1}\right] \\
= & \sum_{i=1}^{N}\left[\sum_{m=1}^{M} v^m \nu_i^m \sigma_i^m \cdot \bar{p}_i^m - \gamma \sum_{m=2}^{M} \sum_{m'=2}^{m} \nu_i^m \sigma_i^m \cdot \bar{p}_i^{m'-1}\right] \\
= & \sum_{i=1}^{N} \sum_{m=1}^{M} \nu_i^m \sigma_i^m \cdot \left[v^m \bar{p}_i^m - \gamma \sum_{m'=2}^{m} \bar{p}_i^{m'-1}\right] \\
= & \sum_{i=1}^{N} \sum_{m=1}^{M} \sum_{v_{-i} \in V_{-i}} \nu_i(v^m, v_{-i}) \cdot \left[v^m \bar{p}_i(v^m, v_{-i}) - \gamma \sum_{m'=1}^{m-1} \bar{p}_i(v^{m'}, v_{-i})\right].
\end{aligned}$$

This is identical to the objective function in (5). We have thus shown that $\mu^*$ is a rationalizing assumption and that equation (3) is satisfied for any regular, non-singular $\nu$.

Now consider an arbitrary regular $\nu$, not necessarily non-singular. There exists a sequence $\nu_n$ converging to $\nu$ such that each $\nu_n$ is non-singular. Moreover, for $\nu_n$ close enough to $\nu$, the strict inequalities in the definition of single-crossing will be preserved, and hence $\nu_n$ will all be regular once $n$ is large enough. For each such $\nu_n$, construct the simple type space $\hat{\Omega}^n$ exactly as in the first half of the proof. Let $\tau_i^m(n)$ denote the belief of type $v^m$ of bidder $i$ in the type space $\hat{\Omega}_i^n$. Passing to a subsequence if necessary, take $\tau_i^m(n) \to \tau_i^m$ for each $i$ and $m$. Let $\hat{\Omega}$ be the "limit" type space where $\hat{\Omega}_i = V_i$, $f_i$ is the identity, and $g_i(v^m) = \tau_i^m$. By Lemma 1, each of these type spaces can be embedded in the universal type space. Let $\Omega^n$ and $\Omega$ be the corresponding images in $\Omega^*$, and define the respective simple assumptions $\mu_n^*$ and $\mu^*$ constructed as in the first part of the proof. In particular, each $\mu_n^*$ rationalizes the use of dominant strategy mechanisms.

Let $\Gamma = (p, t)$ be any mechanism in $\Psi$. As before, we denote by $\bar{p}_i^m$ and $\bar{t}_i^m$ the vectors $\bar{p}_i(\omega_i^m, \cdot)$ and $\bar{t}_i(\omega_i^m, \cdot)$ respectively, where each $\omega$ is an element of $\Omega$, the image in $\Omega^*$ of the "limit" type space $\hat{\Omega}$. Since $\Gamma \in \Psi$, these vectors satisfy the constraints in (7). From the vectors $(\bar{p}, \bar{t})$, we will construct a sequence of vectors $(\bar{p}, \bar{t}(n))$, each satisfying the constraints in the $\nu_n$-version of (7), as follows.

We will say that $\bar{t}(n)$ satisfies $BIC(n)$ if the following incentive constraints hold:

$$\tau_i^m(n) \cdot (\bar{p}_i^m v^m - \bar{t}_i^m(n)) \geq 0, \qquad \langle IR_i^m(n) \rangle$$

$$\tau_i^m(n) \cdot (\bar{p}_i^m v^m - \bar{t}_i^m(n)) \geq \tau_i^m(n) \cdot \left(\bar{p}_i^l v^m - \bar{t}_i^l(n)\right). \qquad \langle IC_i^{m \to l}(n) \rangle$$

For each $i$, $m$, and $n$, let

$$S_i^m(n) = \max\{0, \tau_i^m(n) \cdot (\bar{t}_i^m - \bar{p}_i^m \cdot v^m)\}$$

be the amount by which the $\langle IR_i^m(n) \rangle$ constraint is violated by the transfers $\bar{t}_i^m$ for type $\omega_i^m(n)$. Because $(\bar{p}, \bar{t})$ satisfies the constraints in (7), and $\tau_i^m(n) \to \tau_i^m$, we have $S_i^m(n) \to 0$ for each $i$ and $m$.

Let $\tilde{t}_i^m(n)$ be the sum of the vector $\bar{t}_i^m$ and the constant vector $-S_i^m(n)$. Next, for each $i$, $m$, $l$, and $n$, let

$$L_i^{m\to l}(n) = \max\{0, \tau_i^l(n)\cdot(\bar{p}_i^l v^m - \tilde{t}_i^l(n)) - \tau_i^m(n)\cdot(\bar{p}_i^m v^m - \tilde{t}_i^m(n))\}$$

be the amount by which $\langle IC_i^{m\to l}(n)\rangle$ is violated by the transfers $\tilde{t}_i(n)$. Note that $L_i^{m\to m}(n) = 0$. Again, because $(\bar{p},\bar{t})$ satisfies the constraints in (7), and because $\tilde{t}(n) \to \bar{t}$, we have $L_i^{m\to l}(n) \to 0$ for each $i$, $m$, and $l$. For each $n$, we construct $\lambda_i^l(n)$ to solve the system

$$\tau_i^m(n)\cdot\lambda_i^l(n) = L_i^{m\to l}(n), \qquad \forall\, i, m, l.$$

We can now define $\bar{t}(n)$ by setting

$$\bar{t}_i^m(n) = \bar{t}_i^m - S_i^m(n) + \lambda_i^m(n).$$

By construction, $\bar{t}(n)$ satisfies $BIC(n)$, and together with $\bar{p}$ satisfies the constraints in the $\nu_n$-version of (7). Because each $\nu_n$ is regular and non-singular, and each $\mu_n^*$ is an assumption that rationalizes the use of dominant strategy mechanisms, the first part of the proof implies that

$$\sum_i \sum_v \nu_n(v)\bar{t}_i(n)(v) \leq \Pi^D(\nu_n)$$

for each $n$.

Because the constraint set in the optimal dominant strategy mechanism design problem (5) is compact, the maximum theorem implies

$$\Pi^D(\nu_n) \to \Pi^D(\nu).$$

Hence,

$$\begin{aligned}
\Pi^D(\nu) - R_{\mu^*}(\Gamma) &= \lim_{n\to\infty}\left[\Pi^D(\nu_n) - R_{\mu^*}(\Gamma)\right] \\
&\geq \sum_{i=1}^N \lim_{n\to\infty}\left[\mathbf{E}_{\nu_n}\bar{t}_i(n) - \mathbf{E}_{\nu}\bar{t}_i(n)\right] \\
&= \sum_{i=1}^N \lim_{n\to\infty}\left[\mathbf{E}_{\nu_n}(\bar{t}_i(n) - \bar{t}_i) + \mathbf{E}_{\nu_n}\bar{t}_i - \mathbf{E}_{\nu}\bar{t}_i\right] \\
&= \sum_{i=1}^N \lim_{n\to\infty}\left[\mathbf{E}_{\nu_n}(\bar{t}_i(n) - \bar{t}_i)\right] \\
&= \sum_{i=1}^N \lim_{n\to\infty}\sum_{m=1}^M \sigma_i^m(n)\cdot(S_i^m(n) - \lambda_i^m(n)) \\
&= 0.
\end{aligned}$$

The last equality follows because $S_i^m(n) \to 0$ for every $m$ and $\tau_i^m(n) \cdot \lambda_i^l(n) \to 0$ for each $m$ and $l$ implies by (6) that $\sigma_i^m(n) \cdot \lambda_i^m(n) \to 0$ for each $m$.

This establishes that $\mu^*$ rationalizes the use of dominant strategy mechanisms for the distribution $\nu$ and thus concludes the proof. ■

## Appendix C: An Example for Section 5

In Section 5, we claim that there exists a distribution $\nu$ that satisfies the regularity condition, and such that there is no CPA-assumption $\mu$ under which equation (3) holds. We shall provide an example of such a distribution here.

Consider the same example as in Section 3, where there are two bidders, and each bidder has two possible valuations. The distribution of valuations is as depicted in Figure 3, and the corresponding optimal dominant strategy mechanism is as depicted in Figure 4.

Suppose there exists an CPA-assumption $\mu \in \mathcal{M}(\nu)$ for which equation (3) holds. We shall prove that there exists a detail-free mechanism that generates higher $\mu$-expected revenue than $\Gamma$ does. This would contradict the supposition that equation (3) holds.

It suffices to work only with bidder 2's first-order beliefs in order to complete this proof. So, following the convention in Section 3, we shall continue to use $a$ ($b$) to denote the first-order belief of a high-valuation (low-valuation) type of bidder 2 that bidder 1 has high valuation. Let $\underline{b} = \sup\{x \in [0,1] : \mu(b < x) = 0\}$.

First, observe that $\underline{b} \geq 4/9$. Suppose, on the contrary, $\underline{b} < 4/9$. Then pick any number $z$ between $\underline{b}$ and $4/9$, and consider the mechanism $\Gamma(z)$ as depicted in Figure 8.

| | $v_1 = 4$ | $v_1 = 9$ |
|---|---|---|
| $a \in [0,1]$ | $\alpha = 2, t_1 = 0, t_2 = 11$ | $\alpha = 2, t_1 = 0, t_2 = 11$ |
| $b \geq z$ | $\alpha = 0, t_1 = 0, t_2 = 0$ | $\alpha = 1, t_1 = 9, t_2 = 0$ |
| $b < z$ | $\alpha = 1, t_1 = 4, t_2 = 0$ | $\alpha = 1, t_1 = 4, t_2 = 0$ |

Figure 8: The mechanism $\Gamma(z)$.

It is obvious that $\Gamma(z)$ is BIC for the universal type space. The only difference between $\Gamma(z)$ and $\Gamma$ is in the ($\mu$-non-null) event of $b < z$, in which case $\Gamma(z)$ generates $\mu$-expected revenue of 4, whereas $\Gamma$ only generates $\mu$-expected revenue of $9\mu(v_1 = 9 | b < z) < 9z < 9(4/9) = 4$, where the first inequality comes from the fact that $\mu$ is an CPA-assumption. Since this would have contradicted the supposition that equation (3) holds, we must have $\underline{b} \geq 4/9$.

Then, consider the mechanism $\Gamma''$ as depicted in Figure 9.

| | $v_1 = 4$ | $v_1 = 9$ |
|---|---|---|
| $a \in [0,1]$ | $\alpha = 2, t_1 = 0, t_2 = 11$ | $\alpha = 1, t_1 = 9, t_2 = -15/2$ |
| $b \geq 4/9$ | $\alpha = 2, t_1 = 0, t_2 = 11$ | $\alpha = 1, t_1 = 9, t_2 = -15/2$ |
| $b < 4/9$ | $\alpha = 0, t_1 = 0, t_2 = 0$ | $\alpha = 0, t_1 = 0, t_2 = 0$ |

Figure 9: The mechanism $\Gamma''$.

To see that $\Gamma''$ is BIC for the universal type space, it suffices to observe that, for low-valuation types of bidder 2 with $b \geq 4/9$, truth-telling gives them a non-negative rent of $(5 - 11)(1 - b) + (15/2)b \geq (-6)(5/9) + (15/2)(4/9) = 0$.

Since $b < 4/9$ is a $\mu$-null event, $\Gamma''$ generates $\mu$-expected revenue of $9(4/10) + 11(6/10) - (15/2)(4/10) = 72/10$, whereas $\Gamma$ only generates $\mu$-expected revenue of $9(3/10) + 11(4/10) = 71/10$. This proves that equation (3) does not hold, a contradiction.

## Appendix D: Proof of Proposition 2

We shall first prove a weaker version of Proposition 2.

**Proposition 4** *For the distribution $\nu$ depicted in Figure 6, the optimal dominant strategy mechanism $\Gamma$ depicted in Figure 7 cannot be rationalized by any element in $\mathcal{M}(\nu)$; i.e., $\forall \mu \in \mathcal{M}(\nu)$,*

$$\sup_{\Gamma' \in \Psi} R_\mu(\Gamma') > V^D(\nu).$$

Proposition 2 further strengthens Proposition 4 by asserting that $\sup_{\Gamma' \in \Psi} R_\mu(\Gamma')$ is uniformly bounded away from $V^D(\nu)$ for all $\mu \in \mathcal{M}(\nu)$. This second result will be proved after we have proved Proposition 4

We prove Proposition 4 by way of contradiction. Fix any element $\mu$ in $\mathcal{M}(\nu)$ that rationalizes the optimal dominant strategy mechanism $\Gamma$, we shall prove that there exists a mechanism in $\Psi$ that generates higher $\mu$-expected revenue than $\Gamma$ does. This would contradict the assumption that $\mu$ rationalizes $\Gamma$ and complete the proof.

The proof proceeds by a sequence of lemmas. In each we derive conditions that must be satisfied by $\mu$. Finally we show that no $\mu$ can satisfy them all.

For the purpose of this proof, it suffices to work only with bidder 2's first-order beliefs in order to arrive at a contradiction. So we shall maintain the notational convention used in the example of Section 3 and summarize bidder 2's belief by his first-order belief that bidder 1 has high valuation. The belief of a type with high (resp. low) valuation is denoted $a$ ($b$.) Now because there may be many types in the support of $\mu^*$ with the same valuation, we need some notation to refer to different sets of types. For any (measurable) subset $A \subset [0,1]$, we shall use "$a \in A$" to denote the event that 2 has high valuation and believes with some probability in $A$ that 1 has a high valuation. Likewise $b \in B$ is the event that 2 has low valuation and believes with some probability in $B$ that 1 has high valuation.[21]

The first lemma says that, conditional on any $\mu$-non-null subset of low-valuation types of bidder 2, the $\mu$-conditional-probability that bidder 1 has high valuation cannot be too low, otherwise the auctioneer can improve upon $\Gamma$ by selling to some low-valuation types of bidder 1.[22]

**Lemma 6** *For any $x \in (0,1]$ such that $\mu(b = x) = 0$, if $\mu(b < x) > 0$, then $\mu(v_1 = 10|b < x) \geq 3/8$.*

**Proof:** Suppose there exists $x \in (0,1]$ such that $\mu(b < x) = \mu(b \leq x) > 0$, and yet $\mu(v_1 = 10|b < x) < 3/8$. Consider the mechanism $\Gamma(x)$ as depicted in Figure 10.

| | $v_1 = 5$ | $v_1 = 10$ |
|---|---|---|
| $a \in [0,1]$ | $\alpha = 2, t_1 = 0, t_2 = 2$ | $\alpha = 1, t_1 = 10, t_2 = 0$ |
| $b \geq x$ | $\alpha = 2, t_1 = 0, t_2 = 2$ | $\alpha = 1, t_1 = 10, t_2 = 0$ |
| $b < x$ | $\alpha = 1, t_1 = 5, t_2 = 0$ | $\alpha = 1, t_1 = 5, t_2 = 0$ |

Figure 10: The mechanism $\Gamma(x)$.

To see that $\Gamma(x)$ is BIC for the universal type space, note that (*i*) truth-telling continues to be a dominant strategy of bidder 1, (*ii*) low-valuation types of bidder 2 always have zero rent regardless of what they announce, and (*iii*) high-valuation types of bidder 2 would not announce the (newly added) message "$b < x$" as that gives them zero rent.

The only difference between $\Gamma(x)$ and $\Gamma$ is in the ($\mu$-non-null) event of $b < x$, in which case $\Gamma(x)$ generates $\mu$-expected revenue of $5\mu(v_1 = 5|b < x) + 5\mu(v_1 = 10|b < x) = 5$, whereas $\Gamma$ only generates $\mu$-expected revenue of $2\mu(v_1 = 5|b < x) + 10\mu(v_1 = 10|b < x) < 2(5/8) + 10(3/8) = 5$, contradicting the assumption that $\mu$ rationalizes $\Gamma$. ■

The second lemma says that for any low-valuation type of bidder 2 that is possible under $\mu^*$, the first-order belief $b$ also cannot be too low, otherwise his belief would be too different from the auctioneer's belief, so much so that the auctioneer can improve upon $\Gamma$ by betting against him.

**Lemma 7** $\mu(b < 3/13) = 0$.

[21] Formally, for any type $\omega_2$ of bidder 2, if $f_2^*(\omega_2) = 4$ (i.e., if $v_2 = 4$), $a$ denotes $g_2^*(\omega_2)[(f_1^*)^{-1}(10)]$ and $a \in A$ denotes the event $\{\omega : f_2^*(\omega_2) = 4, g_2^*(\omega_2)[(f_1^*)^{-1}(10)] \in A\}$.

[22] In Lemma 6 (and similarly in Lemmas 7-9), the seemingly redundant requirement of $\mu(b = x) = 0$ is a null-boundary property used only in the proof of Proposition 2.

**Proof:** Suppose not. Then pick $x < 3/13$ such that $\mu(b < x) > 0$ and $\mu(b = x) = 0$,[23] and consider the mechanism $\Gamma'(x)$ as depicted in Figure 11.

| | $v_1 = 5$ | $v_1 = 10$ |
|---|---|---|
| $a \in [0,1]$ | $\alpha = 2, t_1 = 0, t_2 = 2$ | $\alpha = 1, t_1 = 10, t_2 = 0$ |
| $b \geq x$ | $\alpha = 2, t_1 = 0, t_2 = 2$ | $\alpha = 1, t_1 = 10, t_2 = 0$ |
| $b < x$ | $\alpha = 0, t_1 = 0, t_2 = -2$ | $\alpha = 1, t_1 = 10, t_2 = 2(1-x)/x$ |

Figure 11: The mechanism $\Gamma'(x)$.

To see that $\Gamma'(x)$ is BIC for the universal type space, note that $(i)$ truth-telling continues to be a dominant strategy of bidder 1, $(ii)$ low-valuation types of bidder 2 would have strict incentive to announce the (newly added) message "$b < x$" if and only if the resulting rent of $2(1-b) - [2(1-x)/x]b = 2(1-b/x)$ is positive, or equivalently if and only if $b < x$, and $(iii)$ high-valuation types of bidder 2 would not announce the (newly added) message "$b < x$" as that gives them rent of $2(1-a) - [2(1-x)/x]a = 2(1-a/x)$, which is lower than the rent of $2(1-a)$ if they tell the truth.

The only difference between $\Gamma'(x)$ and $\Gamma$ is in the ($\mu$-non-null) event of $b < x$, in which case $\Gamma'(x)$ collects from bidder 2 an $\mu$-expected amount of

$$
\begin{aligned}
& (-2)\mu(v_1 = 5|b < x) + [2(1-x)/x]\mu(v_1 = 10|b < x) \\
\geq \; & (-2)(5/8) + [2(1-x)/x](3/8) \\
= \; & 3/(4x) - 2 \\
> \; & [3/4(3/13)] - 2 \\
= \; & 5/4
\end{aligned}
$$

(where the first inequality follows from Lemma 6), whereas $\Gamma$ only collects from bidders 2 an $\mu$-expected amount of $2\mu(v_1 = 5|b < x) \leq 2(5/8) = 5/4$, contradicting the assumption that $\mu$ rationalizes $\Gamma$. ■

The third lemma says that the first-order belief $a$ of high-valuation types of bidder 2 cannot be too low. Otherwise beliefs held by high- and low-valuation types of bidder 2 would be too different, and this would enable the auctioneer to improve upon $\Gamma$ by introducing Crémer and McLean (1985) bets to separate these types and relax incentive compatibility constraints.

**Lemma 8** $\mu(a < 1/11) = 0$.

[23] It is always possible to pick such an $x$, as any distribution over $[0, 1]$ can have at most countably many mass points.

**Proof:** If not then let $y < 1/11$ such that $\mu(a = y) = 0$ and $\mu(a < y) > 0$. Notice that $y < 1/11$ implies $y < 3y/(2y+1) < 3/13$, and hence we can also choose $x$ between $3y/(2y+1)$ and $3/13$ such that $\mu(b = x) = 0$. Consider the mechanism $\Gamma(x, y)$ as depicted in Figure 12.

| | $v_1 = 5$ | $v_1 = 10$ |
|---|---|---|
| $a < y$ | $\alpha = 1, t_1 = 5, t_2 = -2x(1-y)/(x-y)$ | $\alpha = 1, t_1 = 5, t_2 = 2(1-x)(1-y)/(x-y)$ |
| $a \geq y$ | $\alpha = 2, t_1 = 0, t_2 = 2$ | $\alpha = 1, t_1 = 10, t_2 = 0$ |
| $b < x$ | $\alpha = 1, t_1 = 5, t_2 = -2x(1-y)/(x-y)$ | $\alpha = 1, t_1 = 5, t_2 = 2(1-x)(1-y)/(x-y)$ |
| $b \geq x$ | $\alpha = 2, t_1 = 0, t_2 = 2$ | $\alpha = 1, t_1 = 10, t_2 = 0$ |

Figure 12: The mechanism $\Gamma(x, y)$.

To see that $\Gamma(x, y)$ is BIC for the universal type space, note that ($i$) truth-telling continues to be a dominant strategy of bidder 1, ($ii$) low-valuation types of bidder 2 would have strict incentive to announce the (newly added) message "$b < x$" if and only if the resulting rent of $[2x(1-y)/(x-y)](1-b) - [2(1-x)(1-y)/(x-y)]b = 2(1-y)(x-b)/(x-y)$ is positive, or equivalently if and only if $b < x$, and ($iii$) high-valuation types of bidder 2 would have strict incentive to announce the (newly added) message "$a < y$" if and only if the resulting rent of $[2x(1-y)/(x-y)](1-a) - [2(1-x)(1-y)/(x-y)]a = 2(1-y)(x-a)/(x-y)$ is strictly higher than the truth-telling rent of $2(1-a)$, or equivalently if and only if $a < y$.

Since the event of $b < x$ is a $\mu$-null event by Lemma 7, the only real difference between $\Gamma(x, y)$ and $\Gamma$ is in the ($\mu$-non-null) event of $a < y$, in which case $\Gamma(x, y)$ generates $\mu$-expected revenue of

$$\begin{aligned}
& 5 - 2x(1-y)/(x-y) \\
= \; & 5 - 2(x - y + y)(1-y)/(x-y) \\
= \; & 5 - 2(1-y) - 2y(1-y)/(x-y) \\
> \; & 5 - 2(1-y) - 2y(1-y)(2y+1)/[3y - y(2y+1)] \\
= \; & 5 - 2(1-y) - 2y(1-y)(2y+1)/[2y(1-y)] \\
= \; & 2,
\end{aligned}$$

whereas $\Gamma$ only generates $\mu$-expected revenue of 2, contradicting the assumption that $\mu$ rationalizes $\Gamma$. ■

Finally, the fourth lemma says that the first-order belief $a$ of high-valuation types of bidder 2 cannot be too high. Otherwise the beliefs of such types would be too different from the auctioneer's subjective belief, and this would enable the auctioneer to profit by offering an incentive compatible and individually rational bet. Obviously lemmas 8 and 9 deliver the contradiction and thus prove Proposition 4.

**Lemma 9** $\mu(a < 1/11) > 0$.

**Proof:** Suppose $\mu(a < 1/11) = 0$. Consider the mechanism $\Gamma'$ as depicted in Figure 13.

| | $v_1 = 5$ | $v_1 = 10$ |
|---|---|---|
| $a \geq 1/12$ | $\alpha = 2, t_1 = 0, t_2 = 123/61$ | $\alpha = 2, t_1 = 0, t_2 = 233/61$ |
| $a < 1/12$ | $\alpha = 2, t_1 = 0, t_2 = 2$ | $\alpha = 1, t_1 = 10, t_2 = 0$ |
| $b \in [0, 1]$ | $\alpha = 2, t_1 = 0, t_2 = 2$ | $\alpha = 1, t_1 = 10, t_2 = 0$ |

Figure 13: The mechanism $\Gamma'$.

To see that $\Gamma'$ is BIC for the universal type space, note that $(i)$ truth-telling continues to be a dominant strategy of bidder 1, $(ii)$ low-valuation types of bidder 2 would not announce the (newly added) message "$a \geq 1/12$" as that gives them strictly negative rent regardless of what bidder 1 announces, and $(iii)$ high-valuation types of bidder 2 would have weak incentive to announce the (newly added) message "$a \geq 1/12$" if and only if the resulting rent of $(4 - 123/61)(1 - a) + (4 - 233/61)a$ is weakly higher than their original rent of $2(1 - a)$, or equivalently if and only if $a \geq 1/12$.

Since the event $a < 1/12 < 1/11$ is a $\mu$-null event by assumption, the only real difference between $\Gamma'$ and $\Gamma$ is in the ($\mu$-non-null) event of $a \geq 1/12$, in which case $\Gamma'$ generates $\mu$-expected revenue of $123/61 > 2$, whereas $\Gamma$ only generates $\mu$-expected revenue of 2. This proves that $\mu$ does not rationalize $\Gamma$. ■

This completes the proof of Proposition 4. We now prove the remaining part Proposition 2 through two lemmas.

**Lemma 10** *Suppose $K$ is a compact topological space and that $\mathcal{F}$ is a family of real-valued functions on $K$ such that, for each $x \in K$, there is some $f_x \in \mathcal{F}$ which is continuous at $x$ and satisfies $f_x(x) > 0$. Then we have* $\inf_{x \in K} \sup_{f \in \mathcal{F}} f(z) > 0$.

**Proof:** For each $x \in K$, there exists an open neighborhood $U_x$ such that, for each $y \in U_x$, we have $f_x(y) > f_x(x)/2$. The collection $\{U_x : x \in K\}$ forms an open covering of the compact space $K$, and hence there exists a finite sub-covering. Let $\{U_{x_1}, \ldots, U_{x_n}\}$ be a finite sub-covering and let $\varepsilon = \min\{f_{x_1}(x_1), \ldots, f_{x_k}(x_n)\} > 0$. For each $x \in K$, we have $x \in U_{x_l}$ for some $l = 1, \ldots, n$ so that $\sup_{f \in \mathcal{F}} f(x) \geq f_{x_l}(x) > f_{x_l}(x_l)/2 \geq \varepsilon/2 > 0$. ■

**Lemma 11** *Suppose $\mathcal{O}_1, \ldots, \mathcal{O}_n$ are disjoint open subsets of $\Omega^*$ such that $\mu(\cup \mathcal{O}_l) = 1$, and $t : \Omega^* \to \mathbf{R}$ is a bounded real function that is constant on each $\mathcal{O}_l$. Then the mapping*

$$\mu' \to \int_{\Omega^*} t \, \mu'(d\omega)$$

*is continuous at the point $\mu$.*

**Proof:** Fix any $\varepsilon > 0$. Let $\bar{t} > 0$ be an upper bound for $|t|$. The function $\mu' \to \mu'(\mathcal{O}_i)$ is lower semi-continuous (see Aliprantis and Border (1999)), hence we can set

$$\delta = \frac{\varepsilon}{\bar{t}n^2}$$

and find a neighborhood $U$ of $\mu$ such that, for all $\mu' \in U$, $\mu'(\mathcal{O}_l) > \mu(\mathcal{O}_l) - \delta$ for $l = 1, \ldots, n$. Since $\mu(\cup \mathcal{O}_l) = 1$, it follows that $\mu'(\mathcal{O}_l) < \mu(\mathcal{O}_l) + (n-1)\delta$ and $\mu'(\Omega^* \setminus \cup \mathcal{O}_l) < \mu(\Omega^* \setminus \cup \mathcal{O}_l) + n\delta = n\delta$.

We can write

$$\int_{\Omega^*} t \, d\mu' = \sum_{l=1}^{n} \mu'(\mathcal{O}_l) t(\mathcal{O}_l) + \int_{\Omega^* \setminus \cup \mathcal{O}_l} t(\omega) \, d\mu',$$

so that

$$\begin{aligned}
& \sum_{l=1}^{n} \mu'(\mathcal{O}_l) t(\mathcal{O}_l) - \mu(\Omega^* \setminus \cup \mathcal{O}_l)\bar{t} \le \int_{\Omega^*} t \, \mu'(d\omega) \le \sum_{l=1}^{n} \mu'(\mathcal{O}_l) t(\mathcal{O}_l) + \mu'(\Omega^* \setminus \cup \mathcal{O}_l)\bar{t} \\
\Longrightarrow \quad & \sum_{l=1}^{n} [\mu(\mathcal{O}_l) - \delta] t(\mathcal{O}_l) - n\delta\bar{t} < \int_{\Omega^*} t \, \mu'(d\omega) < \sum_{l=1}^{n} [\mu(\mathcal{O}_l) + (n-1)\delta] t(\mathcal{O}_l) + n\delta\bar{t} \\
\Longrightarrow \quad & -\delta \sum_{l=1}^{n} t(\mathcal{O}_l) - n\delta\bar{t} < \int_{\Omega^*} t \, \mu'(d\omega) - \int_{\Omega^*} t \, \mu(d\omega) < (n-1)\delta \sum_{l=1}^{n} t(\mathcal{O}_l) + n\delta\bar{t} \\
\Longrightarrow \quad & -2n\delta\bar{t} < \int_{\Omega^*} t \, \mu'(d\omega) - \int_{\Omega^*} t \, \mu(d\omega) < n^2\delta\bar{t}.
\end{aligned}$$

This proves that $\left| \int_{\Omega^*} t \, \mu'(d\omega) - \int_{\Omega^*} t \, \mu(d\omega) \right| < \max\{2n\delta\bar{t}, n^2\delta\bar{t}\} = \varepsilon$. ■

**Proof of Proposition 2** Notice that, for each of the mechanisms used in the proof of Proposition 4, the total transfer $(t_1 + t_2)(\omega)$ satisfies the conditions of Lemma 11. For example, consider the mechanism $\Gamma(x)$ in Lemma 6. For any $(v_1, v_2)$, the set of universal type profiles in which the valuation pair is $(v_1, v_2)$ is open in the product topology with $\mu$-null boundary. Moreover, since $\mu(b = x) = 0$, the event $b < x$ is also open in the product topology with $\mu$-null boundary. Therefore, we can take $\mathcal{O}_1, \ldots, \mathcal{O}_6$ to be the interiors of the sets represented by the cells of the table in Figure 10. These open sets are disjoint, have $\mu$-null boundaries, and have total $\mu$-measure equal to 1 as required.

Thus, for any $\mu \in \mathcal{M}(\nu)$, there exists a mechanism $\Gamma(\mu) \in \Psi$ such that $R_\mu(\Gamma(\mu)) - \Pi^D(\nu) > 0$, and the mapping $\mu' \to R_{\mu'}(\Gamma(\mu)) - \Pi^D(\nu)$ is continuous at the point $\mu' = \mu$. We can hence apply Lemma 10, taking $K = \mathcal{M}(\nu)$ and $\mathcal{F} = \{R_{(\cdot)}(\Gamma) - \Pi^D(\nu) : \Gamma \in \Psi\}$. ■

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