Revenue Equivalence Without Differentiability Assumptions *

Jeffrey C. Ely†

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Abstract

1 Background

Two recent papers (Krishna and Maenner (2000), and Milgrom and Segal (2000)) provide versions of the envelope theorem and the revenue equivalence theorem that do not use the linearity of valuation functions used by Myerson in his original paper on optimal auctions. Krishna and Maenner (2000) assume that valuation functions are convex in type[1], while Milgrom and Segal (2000) assume that the valuation functions are differentiable and satisfy a kind of Lipschitz condition. In this note I explore the consequences for revenue equivalence of dropping convexity and differentiability assumptions altogether. I show by examples that the revenue equivalence theorem as usually stated fails. However, the examples suggest that a useful version can nevertheless be recovered.

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†ely@nwu.edu

[1] Another version of the result in Krishna and Maenner (2000) slightly weakens convexity at the cost of restrictions on the mechanism.

2 Examples

Consider an environment in which the set of public alternatives is $Q = [0, 1]$ and there is a single agent with type space $S = [0, 1]$, and valuation function $v : Q \times s \to \mathbf{R}$ given by $v(q, s) = -|q - s|$. We could think of Q as the set of possible locations for a library, and the agent lives at the point s and would pay a linear cost to travel to the library were it not located at s. Consider the efficient public decision rule $f : S \to Q$ given by $f(s) = s$. Clearly f is incentive compatible. What is unusual about this public decision rule is that the agent's valuation function is never differentiable at the point $(f(s), s)$. Thus, although a result of Milgrom and Segal (2000) tells us that for any incentive compatible transfer function $t : S \to \mathbf{R}$, the indirect utility function $U(s) = v(f(s), s) - t(s)$ is absolutely continuous, and hence equal to the integral of its derivative (which exists almost everywhere), the envelope theorem tells us nothing more than

$$1 = v_{s^-}(f(\bar{s}), \bar{s}) \geq U'(\bar{s}) \geq v_{s^+}(f(\bar{s}), \bar{s}) = -1$$

where v_{s^-} denotes the left-hand partial derivative with respect to s and v_{s^+} the right-hand partial derivative.[2] Hence, *a priori* we know nothing more than

$$U(s) = U(0) + \int_0^s \alpha(t)dt \tag{1}$$

for *some* measurable selection $\alpha(t)$ from the correspondence

$$\partial(t) = [v_{s^+}(f(\bar{s}), \bar{s}), v_{s^-}(f(\bar{s}), \bar{s})]$$

Notice the contrast with the standard revenue equivalence theorem which yields a unique $\alpha(t)$ which is *determined* by f given incentive compatibility. That theorem relies on the assumption that v is differentiable, or in the case of Krishna and Maenner (2000) convex.

The analysis to this point does not allow us to conclude that the indirect utility function is pinned down by f, and in particular leaves open the possibility that $\alpha(t)$ depends on the choice of incentive compatible transfer rule. I will now demonstrate that indeed *it does.*

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[2] Note that the envelope theorem being used here is a generalization of the one given in Milgrom and Segal (2000) to the case of valuation functions which have left and right-hand derivatives which do not necessarily coincide. The proof is a straightforward adaptation of the one in Milgrom and Segal (2000)

Because f selects the agent's optimal choice for each of his types, it is incentive compatible with the transfer rule which is identically zero, $t(\cdot) \equiv 0$. In this case, $U(s) \equiv 0$ and $U'(s) = 0$. Now consider the transfer rule $\tilde{t}(s) = s$. The mechanism $(f, \tilde{t})$ is incentive compatible because truthtelling yields $U(s) = -s$ while a report of $\tilde{s}$ gives

$$v(\tilde{s}, s) - \tilde{s} = \begin{cases} -s & \text{if } \tilde{s} < s \\ s - 2\tilde{s} & \text{if } \tilde{s} \geq s \end{cases}$$

which is no greater than $-s$.

Note in particular that $U(0) = 0$ so that type 0 obtains the same utility as in the original mechanism, yet the indirect utility function is not the same. In particular, $U'(s) = -1$, and equation (1) holds for $\alpha(t) = v_{s^+}(f(t), t)$.

Similarly, $\hat{t}(s) = -s$ also yields an incentive compatible mechanism which in this case gives $U(s) = s$, $U'(s) = 1$ and (1) satisfied for $\alpha(t) = v_{s^-}(f(t), t)$. In fact, for *any* measurable selection α from ∂, there exists a transfer rule which yields an incentive compatible mechanism whose indirect utility function satisfies (1).

Now consider a slightly more complicated example. Again, take $Q = S = [0, 1]$, but now suppose

$$v(q, s) = \begin{cases} sq & \text{if } q \leq s \\ 2s^2 - qs & \text{if } q > s \end{cases}$$

Here we can think of q as the quantity traded of some good, where the agent has positive marginal utility s for the first s units, and negative marginal utilty $-s$ for additional units. The efficient trade is again $f(s) = s$. Note that

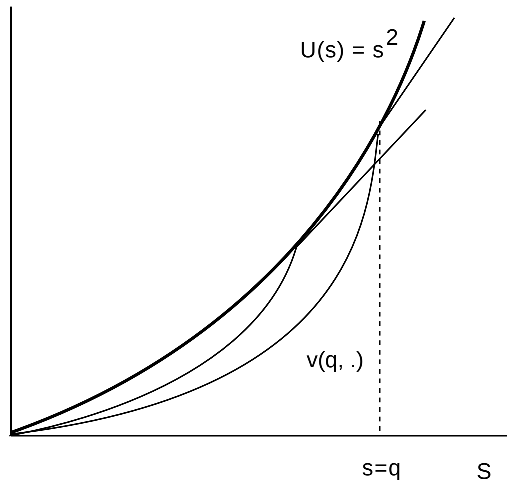
$$v_{s^+}(f(\bar{s}), \bar{s}) = \bar{s} \tag{2}$$

$$v_{s^-}(f(\bar{s}), \bar{s}) = 3\bar{s} \tag{3}$$

Consider the indirect utility function $U(s) = \frac{s^2}{2}$. Note that $U'(\bar{s}) = v_{s^+}(f(\bar{s}), \bar{s})$. This is the indirect utility function for the mechanism (f, t) where $t(s) = \frac{s^2}{2}$. To see that this is incentive compatible, observe that

$$v(f(\tilde{s}), s) - t(\tilde{s}) = \begin{cases} s\tilde{s} - \frac{\tilde{s}^2}{2} & \text{if } \tilde{s} < s \\ 2s^2 - \tilde{s}s - \frac{\tilde{s}^2}{2} & \text{if } \tilde{s} > s \end{cases}$$

which is increasing in $\tilde{s}$ for $\tilde{s} < s$ and decreasing for $\tilde{s} > s$. Below is an illustration of the situation. The figure shows the indirect utility function as the upper envelope of the family of functions $\{v(q, \cdot) : q \in Q\}$, two of which are represented. Notice that right-hand derivatives are tangent to the graph of the indirect utility function.



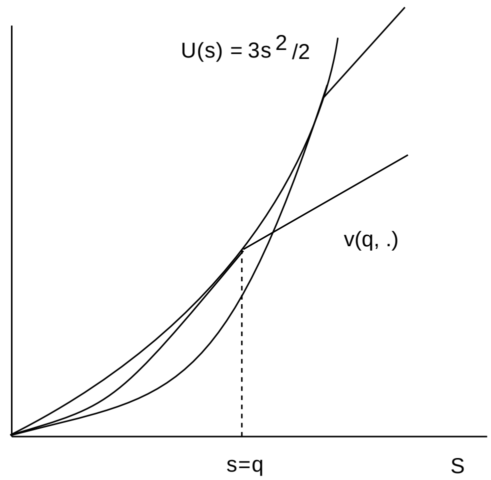
Similarly, $U(s) = \frac{3s^2}{2}$ is an incentive compatible utility assignment using transfers $\tilde{t}(s) = -\frac{s^2}{2}$. Incentive compatibility can be checked as above. Observe that $U'(\bar{s}) = v_{s^-}(f(\bar{s}), \bar{s})$. This mechanism is illustrated below. Here we have the left-hand derivatives tangent to the graph.

In fact, it appears that any measurable selection α from ∂ can be integrated to obtain an indirect utility function for some incentive compatible mechanism.

One last observation common to both examples: the revenue maximizing incentive compatible transfer rule is found by taking $\alpha(t)$ to be everywhere an extremal element of $[v_{s^+}(f(t), t), v_{s^-}(f(t), t)]$, in particular the mimimum.

3 Revenue Equivalence Under a Regularity Condition

In this section, I prove a version of the revenue equivalence theorem under a regularity condition on the valuation function. This condition is weaker



than differentiability and also weaker than the regularity condition used by Krishna and Maenner (2000). It is just enough to rule out the problems associated with the examples of Section 2

Begin with some notation.

$$\bar{D}_{-}v(q,t) := \liminf_{s\uparrow t} \frac{v(q,t)-v(q,s)}{t-s} \tag{4}$$

$$\underline{D}_{+}v(q,t) := \limsup_{s\downarrow t} \frac{v(q,s)-v(q,t)}{s-t} \tag{5}$$

I will use the following generalized envelope theorem.

Theorem 1 *Assume $U(s) = v(q,s)$. If $U'_{-}(s)$ exists then*

$$U'_{-}(s) \leq \bar{D}_{-}v(q,s)$$

and if $U'_{+}(s)$ exists then

$$U'_{+}(s) \geq \underline{D}_{+}v(q,s)$$

Proof: By incentive compatibility, $v(q,t) \leq U(t)$ for any t. Hence

$$U(s) - U(t) \leq v(q,s) - v(q,t)$$

For any $t < s$ we therefore have

$$\frac{U(s) - U(t)}{s - t} \leq \frac{v(q,s) - v(q,t)}{s - t}$$

Taking limits as t approaches s we have

$$U'_-(s) = \lim_{t \uparrow s} \frac{U(s) - U(t)}{s - t} \leq \liminf_{t \uparrow s} \frac{v(q,s) - v(q,t)}{s - t} = \bar{D}_- v(q,s)$$

The proof for the right-hand side derivative follows the same lines. ■

Note that the version of the envelope theorem proven above does not assume that the valuation function is differentiable, and hence generalizes the one in Milgrom and Segal (2000).

To prove the revenue equivalence theorem, I will make the following assumption on the valuation function.

Assumption 1 *For every*[3] *q and s,*

$$\bar{D}_- v(q,s) \leq \underline{D}_+ v(q,s)$$

This condition would be equivalent to the regularity condition in Krishna and Maenner (2000) if the valuation function was left and right handed differentiable at s.

Say that v is *weakly partially differentiable* in s at (q,s) if $\bar{D}_- v(q,s) = \underline{D}_+ v(q,s)$. In this case, the latter value will be called the weak partial derivative, denoted $Dv(q,s)$.

Theorem 2 *Suppose the valuation function satisfies assumption 1. If $U(s)$ is differentiable at at s, then v is weakly partially differentiable in s at $(f(s),s)$. Therefore, if U is absolutely continuous, then for any s,*

$$U(s) = U(0) + \int_0^s Dv(f(t),t)dt \tag{6}$$

Proof: Combine assumption 1 and the envelope theorem:

$$U'_-(s) \leq \bar{D}_- v(q,s) \leq \underline{D}_+ v(q,s) \leq U'_+(s)$$

[3] As will be clear below, it is enough to assume that the inequality holds at each $(f(s),s)$.

If U is differentiable at s, $U'_{-}(s) = U'_{+}(s)$, and thus v is weakly partially differentiable in s. In particular, $U'(s) = Dv(f(s), s)$.

Now if U is absolutely continuous, it is differentiable almost everywhere and equal to the integral of its derivative Thus (6) holds. ■

This theorem provides a generalization of the results of Milgrom and Segal (2000) and the one-dimensional version of Krishna and Maenner (2000). To see this note that the conditions given by Milgrom and Segal (2000) which guarantee the absolute continuity of U are weaker than the Lipschitz condition assumed in Krishna and Maenner (2000), and Assumption 1 is weaker than the regularity condition assumed in Krishna and Maenner (2000). Theorem 2 shows that Assumption 1 and the Milgrom and Segal (2000) conditions are sufficient for (6) and we have not made any assumption on the mechanism as in Krishna and Maenner (2000). Since we have not assumed differentiability, the theorem generalizes Milgrom and Segal (2000).

References

Krishna, V., and E. Maenner (2000): "Convex Potentials with an Application to Mechanism Design," forthcoming, *Econometrica.*

Milgrom, P., and I. Segal (2000): "Envelope Theorems for Arbitrary Choice Sets," mimeo.