SUPPORTING INFORMATION
Appendix for “Varieties of Clientelism: Machine Politics During Elections”

Proofs of Propositions 1 - 3

We refer to opposing voters as $OV$; to supporting nonvoters as $SNV$; and to opposing nonvoters as $ONV$.

Also, for notational simplicity, let $h = g(c)f(x)dcdx$, $r = x - x^M$, and $s = -x - x^M$.

The proofs to Propositions 1 and 3 make use of the following lemma:

**Lemma 1:** For any allocation of budget $B$, a machine could buy more citizens if it had additional resources of any positive amount.

**Proof.** Let $A$ be an allocation of budget $B$. Define $M(A)$ to be the set of citizens who vote for a machine given this allocation: $M(A) \equiv \{(x_i, c_i) : b_i \geq \bar{b}_i\}$, where $b_i$ is the payment received by citizen $i$ under allocation $A$ and $\bar{b}_i$ is the payment required to buy this citizen. Limited resources means that for any allocation $A$, a machine cannot afford to buy all citizens: $\int \int b_ih > B$. It follows that there exists a set $Q \notin M(A)$ of positive measure such that $\bar{b}_i > b_i$ for all $(x_i, c_i) \in Q$. Let $(\hat{x}_i, \hat{c}_i)$ be any point on the interior of $Q$ and select $\eta$ sufficiently small such that $\Delta(\eta) \equiv [\hat{x}_i, \hat{x}_i + \eta] \times [\hat{c}_i, \hat{c}_i + \eta] \subset Q$. Let $\theta > 0$ represent some nonzero amount of resources. Then by the continuity of $f(x)$ and $g(c)$, there exists a $\eta_0 < \eta$ such that for any $\theta$, a machine can afford to buy all citizens in $\Delta(\eta_0)$: $\int_{\Delta(\eta_0)} \bar{b}_i h \leq \theta$.

**Proposition 1:** In an optimal allocation of resources, a machine sets $b^{*}_{TB} = 2b^{*}_{TB} = 2b^{*}_{DP} = 2b^{*}_{AB}$. 

**Proof.** We will show (i) $b^{*}_{TB} = b^{*}_{DP}$ and (ii) $b^{*}_{TB} = 2b^{*}_{TB}$. (The proof to $b^{*}_{TB} = b^{*}_{AB}$ follows identical logic).

(i) Let $b^{*}_{TB}$ and $b^{*}_{DP}$ be the upper bounds on a machine’s payments to $SNV$ and $ONV$, respectively. For contradiction, assume $A$ is an optimal allocation in which $b^{*}_{TB} \neq b^{*}_{DP}$. Without loss of generality, say $b^{*}_{TB} > b^{*}_{DP}$. We will show there exists an allocation $A'$ that is affordable and produces a strictly greater number of net votes. Thus, $A$ cannot be optimal.

Let $S$ be a set with positive measure of $SNV$ such that all citizens in set $S$ have a required payment $\bar{b}_i = b^{*}_{TB}$. Let $(\hat{x}, \hat{c})$ be any point on the interior of $S$ and take $\delta$ small enough such that $\Delta(\delta) \equiv [\hat{x}, \hat{x} + \delta] \times [\hat{c}, \hat{c} + \delta] \subset S$. Recall from Lemma 1 that $Q$ is a set of citizens who remain unbought under allocation $A$. Let $R \subset Q$ be a set with positive measure of $ONV$ such that all citizens in set $R$ have a required payment $b^{*}_{TB} > \bar{b}_i > b^{*}_{DP}$. Let $(\hat{x}, \hat{c})$ be any point on the interior of $R$. Take $\mu$ small enough such that $\Delta(\mu) \equiv [\hat{x}, \hat{x} + \mu] \times [\hat{c}, \hat{c} + \mu] \subset R$. By the continuity of $f(x)$ and $g(c)$, there exists a $\delta_0 < \delta$ and a $\mu_0 < \mu$ such that $\int_{\Delta(\delta_0)} h = \int_{\Delta(\mu_0)} h$ (call this Equation A1). Observe that $\Delta(\delta_0)$ and $\Delta(\mu_0)$ have the same number of citizens, so buying either set produces the same net votes. Let $\theta = \int_{\Delta(\delta_0)} \bar{b}_ih - \int_{\Delta(\mu_0)} \bar{b}_ih$ and note $\theta > 0$ because citizens on $\Delta(\delta_0)$ are more expensive than those on $\Delta(\mu_0)$. Finally, let $\Delta(\eta_0)$ be a set of citizens
who are mutually exclusive of set $\Delta(\mu_0)$ and who do not receive rewards under allocation $A$. Formally, $\Delta(\eta_0) \subset Q$ and $\Delta(\mu_0) \cap \Delta(\eta_0) = \emptyset$.

Consider an allocation $A'$ in which a machine buys all citizens in $\Delta(\mu_0)$, reduces payments to citizens on $\Delta(\delta_0)$ to zero, and redistributes the savings to citizens in $\Delta(\eta_0)$. Recall from Lemma 1 that citizens on $\Delta(\eta_0)$ can be bought with resources $\theta$. Formally, define $\Omega = [\underline{X}, \overline{X}] \times [0, \overline{C}] - (\Delta(\delta_0) \cup \Delta(\mu_0) \cup \Delta(\eta_0))$.

Let $A' = A$ for all $(x_i, c_i)$ on $\Omega$, $A' = 0$ for all $(x_i, c_i)$ on $\Delta(\delta_0)$, and $A' = \tilde{b}_i$ for all $(x_i, c_i)$ on $\Delta(\mu_0)$ and for all $(x_i, c_i)$ on $\Delta(\eta_0)$. The cost of $A'$ is $\leq$ the cost of allocation $A$, and $A'$ buys $\int_{\Delta(\eta_0)} h$ more citizens. Thus $A$ cannot be an optimal allocation.

(ii) To show $b^*_V = 2b^*_T$ (or, equivalently, $b^*_V = 2b^*_D$ or $b^*_V = 2b^*_{AB}$), we repeat the proof that $b^*_T = b^*_D$, replacing Equation (A1) with $\int_{\Delta(\delta_0)} h = 2 \int_{\Delta(\mu_0)} h$, where $\Delta(\delta_0)$ is a subset of $OV$ for whom $\tilde{b}_i = b^*_V > 2b^*_T$, and where $\Delta(\mu_0)$ is a subset of $SNV$ for whom $\frac{1}{2}b^*_V > \tilde{b}_i > b^*_T$. $\square$

Proposition 2: If a machine engages in electoral clientelism, then optimally it allocates resources across all three strategies of vote buying, turnout buying, and double persuasion.

Proof. Let $b^*_V = b^*$ and $b^*_T = b^*_D = b^*_{AB} = b^*$. In an optimal allocation, the number of vote-buying recipients is $VB = N \int_{0}^{\underline{X}} \int_{\underline{C}}^{\overline{C}} h$ (Equation A2), the number turnout-buying recipients is $TB = N \int_{0}^{\overline{X}} \int_{\underline{C}}^{\overline{C}} h$ (Equation A3), the number of double-persuasion recipients is $DP = N \int_{\underline{X}}^{\overline{X}} \int_{\underline{C}}^{\overline{C}} h$ (Equation A4), and the number of abstention buying recipients is $AB = N \int_{\underline{X}}^{\overline{X}} \int_{\underline{C}}^{\overline{C}} h + N \int_{\overline{X}}^{\overline{X}} \int_{\underline{C}}^{\overline{C}} h$ (Equation A5). By Proposition 1, $b^* = 2b^*$, so $b^* > 0 \iff b^* > 0$. It then follows from equations A2, A3, A4, and A5 that $VB > 0 \iff TB > 0 \iff DP > 0 \iff AB > 0$. $\square$

Proposition 3: If $\tilde{b}_i^V \leq b^*$ and $c_i \leq x^O$, a machine pays $\tilde{b}_i^V$ to a $OV$. If $\tilde{b}_i^{AB} \leq b^*$ and $c_i > x^O$, a machine pays $\tilde{b}_i^{AB}$ to a $OV$. If $\tilde{b}_i^T \leq b^*$, a machine pays $\tilde{b}_i^T$ to a $SNV$. If $\tilde{b}_i^D \leq b^*$, a machine pays $\tilde{b}_i^D$ to a $ONV$. All other citizens receive no payment.

Proof. We prove the TB case; identical logic holds for other strategies. We show (i) if $\tilde{b}_i^T \leq b^*$, a machine pays $\tilde{b}_i^T$ to a $SNV$; (ii) if $\tilde{b}_i^T > b^*$, a machine offers $b_i = 0$ to a $SNV$.

(i) Let $b^*$ be the upper bound on payments a machine makes to $SNV$. Define $M(A)$ to be the set of $SNV$ who vote for the machine given the payment allocation $A$. For contradiction, assume $A$ is an optimal allocation in which the machine does not buy all $SNV$ who are cheaper than $b^*$. Formally, there exists a set $Z$ with positive measure of $SNV$ receiving $b_i < \tilde{b}_i < b^*$. We will show there exists a $A'$ that is affordable and produces a strictly greater number of net votes. Thus, $A$ cannot be optimal.

Let $(\tilde{x}, \tilde{c})$ be any point on the interior of $M(A)$ and take $\delta$ small enough such that $\Delta(\delta) \equiv [\tilde{x}, \tilde{x} + \delta] \times [\tilde{c}, \tilde{c} + \delta] \subset M(A)$. Let $(\tilde{x}_i, \tilde{c}_i)$ be any point in $Z$ and select $\mu$ sufficiently small such that $\Delta(\mu) \equiv [\tilde{x}_i, \tilde{x}_i + \mu] \times [\tilde{c}_i, \tilde{c}_i + \mu] \subset Z$. By the continuity of $f(x)$ and $g(c)$ there exists a $\delta_0 < \delta$ and $\mu_0 < \mu$ such that $\int_{\Delta(\delta_0)} h = \int_{\Delta(\mu_0)} h$. Observe that $\Delta(\delta_0)$ and $\Delta(\mu_0)$ have the same number of $SNV$, so buying either set
produces the same net votes. Let \( \theta = \int_{\Delta(\delta_0)} b_i \cdot h - \int_{\Delta(\mu_0)} \tilde{b}_i \cdot h \) and note that \( \theta > 0 \) because citizens in \( \Delta(\delta_0) \) are cheaper than those in \( \Delta(\mu_0) \). Consider an allocation \( A' \) in which a machine buys all citizens in \( \Delta(\mu_0) \), reduces payments to citizens in \( \Delta(\delta_0) \) to zero, and redistributes the savings to citizens in \( \Delta(\eta_0) \). Recall from Lemma 1 that \( \Delta(\eta_0) \) is a set of citizens who remain unbought under allocation \( A \), and who could be bought with resources \( \theta \). Formally, define \( \Omega \equiv [x, \bar{x}] \times [0, \bar{c}] - (\Delta(\delta_0) \cup \Delta(\mu_0) \cup \Delta(\eta_0)) \). Let \( A' = A \) for all \( (x_i, c_i) \) on \( \Omega \), \( A' = 0 \) for all \( (x_i, c_i) \) on \( \Delta(\delta_0) \), and \( A' = \tilde{b}_i \) for all \( (x_i, c_i) \) on \( \Delta(\mu_0) \) and for all \( (x_i, c_i) \) on \( \Delta(\eta_0) \). The cost of \( A' \) is less than or equal to the cost of allocation \( A \) and \( A' \) buys \( \int_{\Delta(\eta_0)} h \) more citizens. Thus \( A \) cannot be an optimal allocation.

(ii) Recall that \( b^* \) is the upper bound on payments a machine makes to SNV. Offering \( b^* \) to a citizen for whom \( \tilde{b}_i^{\text{TB}} > b^* \) is insufficient to induce turnout (i.e., it is an underpayment). Formally, underpayment can be defined as a set of positive measure \( P \) of SNV receiving rewards \( b_i \) such that \( \tilde{b}_i > b_i > 0 \). For contradiction, assume \( A \) is an optimal allocation in which a machine underpays some SNV. We show there exists an affordable allocation \( A'' \) that produces strictly more net votes than \( A \). Thus, \( A \) cannot be optimal.

Define \( \theta = \int_P b_i h \) as the resources the machine devotes to citizens in set \( P \). In allocation \( A, \theta > 0 \). Observe that since the machine underpays these citizens, it receives 0 net votes in return. Recall from Lemma 1 that a machine can purchase all citizens on set \( \Delta(\eta_0) \) for resources \( \theta \), where \( \Delta(\eta_0) \) are citizens who remain unbought under allocation \( A \). Consider an allocation \( A'' \) in which a machine reduces payments to citizens on set \( P \) to 0 and uses the savings to purchase citizens on set \( \Delta(\eta_0) \). Formally, define \( \Omega \equiv [x, \bar{x}] \times [0, \bar{c}] - (P \cup \Delta(\eta_0)) \). Let \( A'' = A \) for all \( (x_i, c_i) \) on \( \Omega \), \( A'' = 0 \) for all \( (x_i, c_i) \) on \( P \), and \( A'' = \bar{b}_i \) for all \( (x_i, c_i) \) on \( \Delta(\eta_0) \). Then the costs of \( A'' \) are \( \leq \) the costs of \( A \), and \( A'' \) buys \( \int_{\Delta(\eta_0)} h \) more citizens. Thus \( A \) cannot be an optimal allocation.

**Comparative Statics**

For analysis of comparative statics, we assume \( f \) and \( g \) are distributed uniformly. The machine’s constrained optimization problem, where \( \lambda \) is the Lagrangian multiplier, is: \[ \max_{a_{TB}, b_{DP}, b_{VB}, b_{AB}} V^M - V^O - \lambda (E - B) \]

The machine maximizes the difference between its votes (\( V^M \)) and opposition votes (\( V^O \)), given that total expenditures (\( E \)) must be less than or equal to its budget \( B \). Note that \( V^O = \int_{x} f^{\text{VB}} \int_{c} f^{\text{O}} h \) and \( V^M = V_B + T_B + D_P + S \), where: Vote Buying (VB) = \( \int_{0}^{\bar{b}_V} \int_{\bar{c}}^{\bar{O}} f^{\text{O}} \), Turnout Buying (TB) = \( \int_{0}^{\bar{b}_T} \int_{\bar{c}}^{\bar{T}} h \), Double Persuasion (DP) = \( \int_{0}^{\bar{b}_D} \int_{\bar{c}}^{\bar{D}} h \), and Supporters (S) = \( \int_{0}^{X} \int_{0}^{h} h \). Total expenditures for the machine party are \( E = E_{VB} + E_{TB} + E_{DP} + E_{AB} \), where: VB Expenditures (\( E_{VB} \)) = \( \int_{0}^{\bar{b}_V} \int_{\bar{c}}^{\bar{O}} f^{\text{O}} \), Turnout Buying (TB) = \( \int_{0}^{\bar{b}_T} \int_{\bar{c}}^{\bar{T}} \), DP Expenditures (\( E_{DP} \)) = \( \int_{0}^{\bar{b}_D} \int_{\bar{c}}^{\bar{D}} h \), and AB Expenditures (\( E_{AB} \)) = \( \int_{0}^{X} \int_{\bar{c}}^{\bar{A}} B_{i}^{\text{AB}} h \). Solving the problem yields four first order conditions. Solving all first order conditions for \( \lambda \) yields the results from Proposition 1: \( b_{VB}^* = 2b_{TB}^* = \)
Compulsory Voting: Substitute $b^* = \frac{1}{2}b^{**}$ from the FOCs into the budget constraint. Implicit differentiation yields: 

$$\frac{\partial b^{**}}{\partial a} = \frac{\partial b^*}{\partial a} = \frac{1}{8(a+X-x^M-C)} - b^{**} < 0.$$ 

Substitute $b^{**} = 2b^*$ into the budget constraint.

Implicit differentiation yields: 

$$\frac{\partial b^*}{\partial a} = \frac{1}{4(4a+X-x^M-C)} - b^* < 0.$$ 

Comparative statics follow: (1) 

$$\frac{\partial V_B}{\partial a} = \frac{\Gamma}{4} \left[ 2b^* + (2a - x^M - C) + b^* \right] - b^* \frac{\partial b^*}{\partial a} - b^* \frac{\partial b^*}{\partial a} =$$

$$= \frac{\Gamma}{4} \left[ 2b^* - 2b^{**} \left( \frac{4(a-x^M-C)+2b^{**}}{8(a+X-x^M-C)} - b^* \right) - b^* \frac{\partial b^*}{\partial a} - b^* \frac{\partial b^*}{\partial a} > 0. \right.$$ 

$$\frac{\partial b^{**}}{\partial a} = \frac{\Gamma}{4} \left[ b^* \frac{\partial b^*}{\partial a} + (4X + b^*) \frac{\partial b^*}{\partial a} \right] = \frac{\Gamma}{4} \left[ \frac{\partial b^*}{\partial a} - \frac{\partial b^*}{\partial a} \right] < 0.$$

(4) Let $b_{TB} = \frac{1}{2}b^*$ and substitute 

$$\frac{\partial b^*}{\partial a} = \frac{\partial b^*}{\partial a} \text{ and implicit differentiation yields:}$$

$$\frac{\partial b^{**}}{\partial a} = \frac{\partial b^*}{\partial a} = \frac{1}{2}b^* \text{ and implicit differentiation yields:}$$

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$$\frac{\partial b^{**}}{\partial a} = \frac{\partial b^*}{\partial a} = \frac{1}{2}b^*, \text{ and } \frac{\partial b^{**}}{\partial a} = \frac{1}{2}b^*,$$

Ballot Secrecy: In the constrained optimization problem above, replace $E_{VB}$ with $\beta E_{VB}$ and $E_{DP}$ with $\beta E_{DP}$. The FOCs become $\beta b^*_B = 2\beta b^*_D = 2\beta b^*_TB = 2\beta b^*_AB$. Substitute $b^*_D = \frac{1}{2}b^*_B$ and $b^*_TB = b^*_AB = \frac{1}{2}b^*_B$ from the FOCs into the budget constraint. Implicit differentiation yields: 

$$\frac{\partial b^{**}}{\partial a} = \frac{b^*}{3(4a+X-x^M-C)} - b^{**} < 0.$$ 

Comparative statics follow: (1) 

$$\frac{\partial V_B}{\partial a} = \frac{\Gamma}{4} \left[ (b^*_B - 2b^*(M+C)) \frac{\partial b^*}{\partial a} - \frac{\partial b^*}{\partial a} - b^* \frac{\partial b^*}{\partial a} - b^* \frac{\partial b^*}{\partial a} \right] =$$

$$= \frac{\Gamma}{4} \left[ (b^*_B - 2b^*(M+C)) \frac{\partial b^*}{\partial a} - \frac{\partial b^*}{\partial a} - b^* \left( b^*_B + \beta b^*_B \right) \right] < 0.$$

(4) Let $b_{TB} = \beta b^*_B$ and substitute 

$$\frac{\partial b^{**}}{\partial a} = \frac{\partial b^*}{\partial a} = \frac{1}{2}b^* \text{ and implicit differentiation yields:}$$

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Salience of Political Preferences: Substituting FOCs into the budget constraint and implicitly 

$$\text{differentiating yields: } (1) \frac{\partial b^{**}}{\partial a} = \frac{b^*}{3(4a+X-x^M-C)} - b^{**} > 0 \text{ and } (2) \frac{\partial b^*}{\partial a} = \frac{b^*}{3(4a+X-x^M-C)} - b^* > 0.$$ 

Comparative statics follow: (1) 

$$\frac{\partial V_B}{\partial a} = -\frac{\Gamma}{8a} \left[ b^* (b^* - 2C) + 2\kappa (2a + \kappa x^M + C) + b^* - \kappa b^* \right] \frac{\partial b^*}{\partial a},$$

(2) 

$$\frac{\partial V_B}{\partial a} = \frac{\Gamma}{8a} \left[ b^* (b^* - 2C) + 2\kappa (2a + \kappa x^M + C) + b^* - \kappa b^* \right] \frac{\partial b^*}{\partial a} > 0.$$ 

Political Polarization: Note that by the assumption of symmetric party platforms, $x^M = x^O = 2x^M$.

Substitute $b^* = \frac{1}{2}b^{**}$ from the FOCs into the budget constraint. Implicit differentiation yields: 

$$\frac{\partial b^*}{\partial x^M} = \frac{1}{8a} \left[ \frac{b^*}{4a+X-x^M-C} - b^{**} > 0 \right. \text{ and } (2) \frac{\partial b^*}{\partial x^M} = \frac{1}{4a} \left[ \frac{b^*}{4a+X-x^M-C} - b^{**} > 0 \right.$$ 

$$\text{Comparative statics then follow: (1) }$$

$$\frac{\partial V_B}{\partial a} = \frac{\Gamma}{8} \left[ -\left( 2b^{**} + (2a + \kappa x^M + C) + b^* \right) \frac{\partial b^*}{\partial x^M} \right] - b^* \frac{\partial b^*}{\partial x^M} + b^* \frac{\partial b^*}{\partial x^M} = \frac{\Gamma}{8} \left[ -\left( 2b^{**} + (2a + \kappa x^M + C) + b^* \right) \frac{\partial b^*}{\partial x^M} \right] < 0.$$
\[ \frac{\partial TB}{\partial x} = \Gamma \left[ X \left( \frac{\partial b^*}{\partial x} \right) \right] > 0. \] (3) \[ \frac{\partial DP}{\partial x} = \frac{\Gamma}{2} \left[ b^* \frac{\partial b^*}{\partial x} \right] > 0. \] (4)

\[ \frac{\partial AB}{\partial x} = -\frac{\Gamma}{2} \left[ b^* \frac{\partial b^*}{\partial x} + (4X + b^*) \frac{\partial b^*}{\partial x} \right] = -\frac{\Gamma}{2} \left[ b^* \frac{\partial b^*}{\partial x} + 2X \frac{\partial b^*}{\partial x} \right] > 0 \] (recall that \( X < 0 \) and that under an optimal allocation of resources, \( b^* = \frac{1}{2} b^{**} \) and \( \frac{\partial b^*}{\partial x} = \frac{1}{2} \frac{\partial b^{**}}{\partial x} \)).

**Machine Support:** Substituting FOCs into the budget constraint and implicitly differentiating yields:

\[ \frac{\partial b^*}{\partial x} = \frac{\partial b^*}{\partial x} = 0. \] Comparative statics follow: (1) \[ \frac{\partial VB}{\partial x} = -\frac{\Gamma}{4} \left[ (2(x^M + C) - b^* + b^*) \frac{\partial b^*}{\partial x} + b^* \frac{\partial b^*}{\partial x} \right] = 0. \] (2)

\[ \frac{\partial TB}{\partial x} = \Gamma \left[ b^* + (X + x) \frac{\partial b^*}{\partial x} \right] = \Gamma b^* > 0. \] (3) \[ \frac{\partial DP}{\partial x} = \frac{\Gamma}{2} \left[ b^* \frac{\partial b^*}{\partial x} \right] = 0. \] (4)

\[ \frac{\partial AB}{\partial x} = -\frac{\Gamma}{4} \left[ b^* (4 + \frac{\partial b^*}{\partial x}) + (4(X + x) + b^*) \frac{\partial b^*}{\partial x} \right] = -\Gamma b^* < 0. \]