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Growing Through Cycles in an Infinitely-lived Agent Economy

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Abstract

This paper develops an infinitely-lived representative agent economy, in which the relative contribution of the two engines of growth, investment and innovation, changes endogenously over time. The balanced growth path of the economy loses its stability when its endogenously determined growth rate is not sufficiently high, and the economy fluctuates, perpetually moving back and forth between two phases. In one phase, there is no innovation and the market structure is competitive, and the economy grows solely by capital accumulation, as in a neoclassical model. In the other phase, new goods are introduced and the market structure is monopolistic, as in a neo-Schumpetarian model. In the long run, both investment and innovation grow at the same rate, but the economy alternates between the periods of high investment and the periods of higher innovation.

Keywords: Endogenous Growth, Endogenous Fluctuations, Asynchronous Movements of Innovation and Investment

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1. Introduction.

The neoclassical growth models focus on factor accumulation as an engine of growth. As Solow (1956) showed, factor accumulation alone cannot sustain growth in the presence of diminishing returns in capital. Recently, neo-Schumpetarian models of endogenous growth, such as Romer (1987) and Rivera-Batiz & Romer (1991) among others, stressed the innovation of new products, motivated by monopoly profits, as a way of avoiding diminishing returns, and of sustaining growth indefinitely. They have shown that the equilibrium is characterized by a balanced growth path, in which new products are innovated at a constant rate, and the economy accumulates capital at the equal rate. One crucial feature of these models is that the monopoly power enjoyed by the innovators is assumed to last forever.

This paper develops an endogenous growth model, in which the relative contribution of these two engines of growth, investment and innovation, changes over time. Like other neo-Schumpertarian models, the model has a unique balanced growth path. Unlike others, however, the balanced growth path is unstable, when the growth rate is not sufficiently high. With the balanced growth path being unstable, the economy fluctuates perpetually moving back and forth between two phases. In one phase, there is no innovation and the market structure is competitive, and the economy grows solely by capital accumulation, as in a neoclassical model. In the other phase, new goods are introduced and the market structure is monopolistic, as in a neo-Schumpetarian model. In the long run, both investment and innovation grow at the same rate, but the economy moves back and forth between the periods of high investment and the periods of higher innovation. Both investment and innovation are essential in sustaining growth indefinitely, and yet the only one of them appears to play a dominant role in each phase.

The present model departs from the Rivera-Batiz & Romer (1991) model in that the monopoly power enjoyed by the innovators of new products is temporary. This assumption plays a dual role in generating fluctuations. First, the degree of monopoly prevailing in the economy can change over time. Second, a potential innovator wants to enjoy its temporary monopoly power when the degree of monopoly prevailing in the economy is higher. This is because the potential innovator needs to enter when the market for its product is large enough to recover the cost of innovation. The size of the market depends in part on how the products with which it competes with are priced. This leads to a synchronization of innovative activities. If the innovator chooses to introduce its product when others do, some of its competing products are monopolistically priced. On the other hand, if the innovator enters after others have innovated, the market for its product would be too small to recover the cost of innovation, because the competing products would become more competitively priced, as their innovators lose their monopoly power. As a result, the economy experiences the period of high innovation with a monopolistic market structure, followed by the period of no innovation with a competitive market structure. Once innovation stops, the output and investment growth go up, partially because the resources are now redirected from innovative activities to manufacturing activities and partially because the competitive market structure allocates the resources more efficiently among the existing products. And, as a result of high investment growth, the economy will eventually build up enough of a resource base to enter another period of innovative activities.

Matsuyama (1999) recently demonstrated essentially the same results obtained in this paper, under an additional departure from the Rivera-Batiz & Romer model; instead of deriving capital accumulation as a solution to the agent's intertemporal optimization, it was simply assumed that the economy maintains a constant capital/output ratio.[1] This assumption of a fixed saving rule greatly simplifies the dynamics of the model, to the extent that the global analysis of the system can be conducted. One might suspect that this assumption might not be innocuous; it might be responsible for endogenous fluctuations. One's intuition suggests that, if the economy is populated by the infinitely-lived agent, intertemporal substitution would eliminate fluctuations. What this paper shows is that introducing infinitely-lived agent in the model make little difference. In particular, the critical level of growth rate below which the balanced growth path loses its stability and the economy fluctuates endogenously is identical as in the case of a fixed saving rule. In this sense, what was crucial in Matsuyama (1999) was the assumption of a temporary monopoly power, not of a fixed saving rule.

This does not, however, mean that one's intuition is faulty. A sufficiently high rate of intertemporal substitution indeed restores the stability of the balanced growth path. This is not a

[1] A constant capital/output ratio can be justifed as the optimal choice of the agent, in a two-period-lived overlapping generations economy.

contradiction to the statement above. The reason is that the growth rate itself is endogenously determined in this model. A high rate of intertemporal substitution increases the growth rate of the economy, thereby pushing it above the critical level of the growth rate. Once the growth rate is controlled for, the condition for the instability of the balanced growth path and for endogenous fluctuations is the same, regardless of whether capital accumulation is determined by a fixed saving rule or by the optimizing infinitely-lived agent.

This paper is also closely related to Deneckere and Judd (1992), who demonstrated that the temporary monopoly power in dynamic monopolistic competition models leads to endogenous fluctuations. Their model does not have capital accumulation, so that it cannot capture the asynchronous movement of investment and innovation. In their model, there is no asset in the economy that the agents could carry over from one period to another. As a result, the equilibrium is independent of the rate of intertemporal substitution.

The rest of the paper is organized as follows. Section 2 develops the model and derives the equilibrium condition. Section 3 conducts the local stability analysis of the steady state. Section 4 considers the existence of period-2 cycles, and studies their properties.

2. The Model.

The time is discrete and extends from one to infinity: $t \in T = \{1,2,3,...\}$. There is a competitively supplied final good, which can either be consumed or invested. For the notational convenience, let K_t denote the capital stock at the end of period t, i.e., the amount of the final good left unconsumed in period t, and carried over to period t+1. Note that this means that the amount of capital stock available for use in period t is denoted by K_{t-1}. The economy inherits a positive capital stock, $K_0 > 0$, in the first period.

The economy is populated by the infinitely-lived representativeagent. In period t, the agent earns the capital income, $r_t K_{t-1}$, and the labor income, $w_t L$, and then consumes C_t and carries over K_t units of the final good to period t+1. The agent chooses a consumption path that maximizes the following discounted utility,

$$U = \sum_{t=1}^{\infty} \beta^t \frac{(C_t)^{1-1/\gamma} - 1}{1-1/\gamma} \qquad 0 < \gamma < \infty, \qquad \gamma \neq 1$$

or

$$U=\sum_{t=1}^{\infty}\beta^{t}\ln(C_{t}), \qquad \gamma=1,$$

($0<\beta<1$ is the discount rate and γ is the rate of intertemporal substitution), subject to the flow budget constraint,

(1) $$K_{t}=r_{t}K_{t-1}+w_{t}L-C_{t},$$

and the intertemporal solvency condition, which rules out a Ponzi-scheme,

(2) $$\lim_{t\to\infty}\frac{K_{t}}{\prod_{s=1}^{t}r_{s}}\geq 0.$$

As is well-known, the optimal consumption path is characterized by the Euler equation

(3) $$\frac{1}{C_{t}^{1/\gamma}}=\beta\frac{r_{t+1}}{C_{t-1}^{1/\gamma}}>0$$

and the binding intertemporal solvency condition, which can be written as

(4) $$\lim_{t\to\infty}\frac{K_{t}}{\prod_{s=1}^{t}r_{s}}=\lim_{t\to\infty}\beta^{t}\frac{K_{t}}{C_{t}^{1/\gamma}}=0.$$

There are two primary factors of production: capital (K) and labor (L). Labor goes directly into the production of the final good. Capital is first converted into a variety of differentiated intermediate products. These intermediates are aggregated into the composite by a symmetric CES, which in turn is combined with labor by a Cobb-Douglas technology. More specifically, the technology of the final goods producer is expressed as

(5) $$Y_{t}=\hat{A}(L)^{1/\sigma}\left\{\int_{0}^{N_{t}}\left[x_{t}(z)\right]^{1-1/\sigma}dz\right\}$$

where $x_t(z)$ is the amount of variety z employed in period t, $\sigma\in(1,\infty)$ is the direct partial elasticity of substitution between every pair of intermediate products, and $[0,N_t]$ represents the range of intermediates available in the marketplace in period t. Some features of this specification deserve comments. First, for a given availability of intermediate products, N_t, the technology of the final goods production satisfies the property of constant returns to scale, and hence it is consistent with the competitiveness of the final goods industry. Second, the final goods producer's demand for each intermediate product has a constant price elasticity equal to σ. Third, the labor share of the economy is equal to $1/\sigma$.

Prior to period t, the economy developed all the intermediate inputs in the range, $[0,N_{t-1}]$, with $N_0 > 0$. These "old" intermediates are manufactured by converting a units of capital into one unit of an intermediate, and sold competitively in period t. In addition, the intermediate inputs of variety $z \in [N_{t-1},N_t]$ may be introduced and sold exclusively by their innovators in period t. These "new" intermediates require F units of capital to innovate. The process of manufacturing new intermediates, just as old ones, requires a units of capital per output.

The marginal cost of all the intermediates in period t is thus equal to ar_t. The old ones are supplied competitively and hence at the marginal cost; $p_t(z) \equiv p_t^c = ar_t$ for $z \in [0,N_{t-1}]$. All the new products, if they exist, are sold at $p_t(z) \equiv p_t^m = a\sigma r_t/(\sigma-1)$, where $z \in [N_{t-1},N_t]$, because of the constant price elasticity, σ. Since all the intermediate products enter symmetrically in the production function of the final goods, we have $x_t(z) \equiv x_t^c$ for $z \in [0,N_{t-1}]$, and $x_t(z) \equiv x_t^m$ for $z \in [N_{t-1},N_t]$, and they satisfy

$$(6) \qquad \frac{x_t^c}{x_t^m} = \left[\frac{p_t^c}{p_t^m}\right]^{-\sigma} = \left[1-\frac{1}{\sigma}\right]^{-\sigma}.$$

The one-period monopoly enjoyed by the innovator provides an incentive for innovation, and there is no barrier to entry for innovative activities. The period t monopoly profit, net of the fixed cost, is $\pi_t = p_t^m x_t^m - r_t(ax_t^m+F)$; it is negative if and only if $x_t^m < (\sigma-1)F/a$. Thus, free entry ensures that, in equilibrium,

$$(7) \qquad ax_t^m \leq (\sigma-1)F\,,\ N_t \geq N_{t-1}\,,\ \left(ax_t^m - (\sigma-1)F\right)\left(N_t - N_{t-1}\right) = 0\,.$$

This is, when potential innovators do not expect the sale of a new product to reach the break-even point (i.e., $x_t^m < (\sigma-1)F/a$), there is no incentive for innovating new products (i.e., $N_t = N_{t-1}$). When innovation occurs and some new products are introduced (i.e., $N_t > N_{t-1}$), the innovator cannot earn any excess profit and must operate at the break even point (i.e., $x_t^m = (\sigma-1)F/a$).

The resource constraint on capital in period t is expressed as

$$K_{t-1} = N_{t-1}ax_t^c + \left(N_t - N_{t-1}\right)\left(ax_t^m + F\right).$$

From eqs. (6) and (7), the above resource constraint implies,

(8) $$ax_t^c = a\left[1-\frac{1}{\sigma}\right]^{-\sigma} x_t^m = \theta\sigma F \min\{k_{t-1}, 1\}.$$

and

(9) $$\frac{N_t}{N_{t-1}} = \max\{1, 1+\theta(k_{t-1}-1)\} \equiv \psi(k_{t-1}),$$

where

$$k_t \equiv \frac{K_t}{(\theta\sigma F)N_t}$$

and

$$\theta \equiv \left[1-\frac{1}{\sigma}\right]^{1-\sigma},$$

which depends positively on σ and its value can range from 1 to e = 2.71828.... as one varies σ from 1 to infinity. Since it is equal to $(p_t^c x_t^c)/(p_t^m x_t^m)$, θ can be interpreted as the extent to which each product expands its market size in goods in value when it becomes competitively priced.

From eq. (5), the total output is equal to

$$Y_t = \hat{A}(L)^{1/\sigma}\left[N_{t-1}\left(x_t^c\right)^{1-1/\sigma} + (N_t - N_{t-1})\left(x_t^m\right)^{1-1/\sigma}\right].$$

Using eqs. (7), (8), and (9), this can be rewritten as

(10) $$\frac{Y_t}{K_{t-1}} = A\max\{(k_{t-1})^{-1/\sigma}, 1\} \equiv A\phi(k_{t-1}),$$

where

$$A \equiv \frac{\hat{A}}{a}\left[\frac{aL}{\theta\sigma F}\right]^{1/\sigma}.$$

Eqs. (9) and (10) summarize what takes place on the production side of the economy in period t. If $k_{t-1} = K_{t-1}/\theta\sigma FN_{t-1} \leq k_c = 1$, the resource base of the economy, K, is too small relative to the number of the products, N, and there is no innovation. All the products are competitively produced, and the reduced form aggregate production function, given in eq. (10), has the standard neoclassical properties, including the law of diminishing returns in capital. In this case, we shall say that the economy is in *the Solow regime*. If $k_{t-1} = K_{t-1}/\theta\sigma FN_{t-1} > k_c = 1$, the resource base of the economy is sufficiently large relative to the number of the existing products, and some new products are introduced. Furthermore, the aggregate output is linear in capital, as in many endogenous growth models. In this case, we shall say that the economy is in *the Romer regime*. Note that k is normalized in such a way that k_c, the critical level of k that separates the two regimes, is one.

In equilibrium, $Y_t = w_tL + r_tK_{t-1}$ and $r_tK_{t-1} = (1-1/\sigma)Y_t$ hold. Hence, by using eqs. (1), (9) and (10), eq. (3) can be rewritten to

$$(11) \qquad \frac{\psi(k_{t-1})}{A\phi(k_{t-1})k_{t-1} - k_t\psi(k_{t-1})} = \frac{G[\phi(k_t)]^{\gamma}}{A\phi(k_t)k_t - k_{t+1}\psi(k_t)} > 0,$$

where

$$G \equiv \left[\beta\left(1-\frac{1}{\sigma}\right)A\right]^{\gamma}.$$

Eq.(11) can be viewed as a two-dimensional dynamical system, which maps (k_{t-1}, k_t) to (k_t, k_{t+1}). Therefore, it determines an entire trajectory, $\{k_t; t \in T\}$ for each (k_0, k_1), and yet only k_0 is given exogenously in the model. The restriction imposed by (4) is generally not strong enough to determine the unique equilibrium path. This is one significant difference from the model of Matsuyama (1999), which has a unique equilibrium path for any initial condition.

The Steady State: It is easy to verify that eq. (11) has a unique steady state. If $G \le 1$, the steady state is in the Solow regime, given by $k_t = k^* \equiv G^{\sigma/\gamma} \le k_c = 1$. Without innovation, all the goods are competitively supplied and the economy does not grow. The steady state is thus a *neoclassical stationary state*. Note also that in this steady state, eq. (4) is satisfied. Hence, if $k_0 = k^*$, $k_t = k^*$ for all $t > 0$ is an equilibrium path. If $G > 1$, the steady state is in the Romer regime, given by $k_t = k^{**} \equiv 1 + (G-1)/\theta > k_c = 1$. In this steady state, new products are introduced steadily, and K and N grow at the same rate. It is a *balanced growth path*. From (9) and (10),

$$\frac{Y_t}{Y_{t-1}} = \frac{K_t}{K_{t-1}} = \frac{N_t}{N_{t-1}} = \psi(k^{**}) = G > 1,$$

so that G is equal to the gross rate of growth. In order for this steady state to be an equilibrium path for $k_0 = k^{**}$, eq. (4) needs to be satisfied, which implies that, if $G > 1$, the following condition must hold:

$$(12) \qquad \beta G^{1-1/\gamma} < 1.$$

Otherwise, the discounted sum of the utility would be infinite, hence the consumer's maximization problem is ill-defined. In what follows, (12) is assumed to hold.[2]

Note that G is the key parameter determining the growth potential of the economy. When G > 1, it is equal to the (gross) growth rate of the economy along the balanced growth path, which goes up with a higher β, σ, A, and γ.

Before proceeding, it is worth pointing out the difference between the present model and the existing models in the literature. Unlike Rivera-Batiz and Romer (1991), the monopoly power of the innovators is temporary. Unlike Deneckere and Judd (1992), it has capital accumulation. Unlike Matsuyama (1999), capital accumulation is derived from intertemporal optimization of the infinitely-lived agent.

3. The Local Stability Analysis.

This section studies the local stability of the unique steady state along the equilibrium path. That is, when the initial condition, k_0, is sufficiently close to the steady state, is there an equilibrium path, along which the economy stays close to the steady state and converges to it? (The stability here should not be confused with the stability of the steady state in a two-dimensional dynamical system, eq.(11).)

Suppose G < 1, or $k^* = G^{\sigma/\gamma} < 1$, so that the neoclassical stationary state is in the interior of the Solow regime. Since $\phi(k) = k^{-1/\sigma}$ and $\psi(k) = 1$ in a neighborhood of k^*, eq. (11) becomes

$$k_{t+1} = A(k_t)^{1-1/\sigma} - G(k_t)^{-\gamma/\sigma}\left[A(k_{t-1})^{1-1/\sigma} - k_t\right].$$

Linearizing around $k^* = G^{\sigma/\gamma}$ yields the following second-order difference equation in $Dk_t \equiv k_t - k^*$:

$$Dk_{t+1} - \left(\frac{1}{\beta}\left(1+\frac{\gamma}{\sigma-1}\right)+1-\frac{\gamma}{\sigma}\right)Dk_t + \frac{1}{\beta}Dk_{t-1} = 0,$$

whose two characteristic roots are both positive: one of them is greater than one, and the other smaller than one. Thus, k^* is a saddle of the dynamical system, (11), and has a one-dimensional locally stable manifold. If the initial condition, k_0, is in a neighborhood, there exists a unique trajectory that remains in the neighborhood and converges to k^* monotonically. This trajectory is an

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[2] Since G > 1, eq. (12) does not imply any additional restriction, if $\gamma \leq 1$. If $\gamma > 1$, the balanced growth path exists if and

equilibrium trajectory, since eq. (4) is satisfied by any sequence converging to the stationary state.

Now, suppose $G > 1$, so that the balanced growth path, $k^{**} = 1 + (G-1)/\theta$, is in the interior of the Romer regime. Since $\phi(k) = 1$ and $\psi(k) = 1 + \theta(k-1)$ in a neighborhood of k^{**}, eq. (11) becomes

$$k_{t+1} = \left[(A+G)k_t - \frac{AGk_{t-1}}{1+\theta(k_{t-1}-1)} \right] \frac{1}{1+\theta(k_t - 1)}$$

Linearization around k^{**} yields the second-order difference equation in $Dk_t \equiv k_t - k^{**}$:

$$Dk_{t+1} - \frac{1-\theta+A}{G} Dk_t + \frac{(1-\theta)A}{G^2} Dk_{t-1} = 0$$

whose characteristic roots are

$$\frac{1-\theta}{G} < 0 \qquad \text{and} \qquad \frac{A}{G} = \left[\beta G^{1-1/\gamma} \left(1 - \frac{1}{\sigma} \right) \right]^{-1} > 1,$$

where the use has been made of (12). If $1 < G < \theta - 1$, the absolute values of the two roots are both greater than one, hence k^{**} is a source of the dynamical system, (11). Hence, the equilibrium trajectory, if it starts in a neighborhood, will not stay in the neighborhood. If $\theta - 1 < G$, then the negative root has the absolute value smaller than one, while the positive root is greater than one. In other words, k^{**} is a saddle, and has a one-dimensional locally stable manifold. If the initial condition, k_0, is in a neighborhood, there exists a unique trajectory that stays in the neighborhood and converges to k^{**} oscillatorily. This converging path is an equilibrium path, because it also satisfies eq. (4), under eq. (12).

By virtue of the Local Manifold Theorem, one can translate the above findings into the following form.

<u>Proposition 1.</u>

(i) If $G < 1$, the neoclassical stationary state, k^*, is locally stable in that there exists a neighborhood of k^*, U, such that, if $k_0 \in U$, there exists an equilibrium path, whose entire trajectory stays in U, and along which the economy converges monotonically to k^*; $\lim_{t\to\infty} k_t = k^*$.

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only if $1 < \beta(1-1/\sigma)A < \beta^{1/(1-\gamma)}$.

(ii) If $1 < G < \theta - 1$, the balanced growth path, k^{**}, is locally unstable in that there exists a neighborhood of k^{**}, U, such that, if $k_0 \in U$, there exists some t such that $k_t \notin U$, along any equilibrium path. That is, when the economy starts close to the balanced growth path, it will move away from it.

(iii) If $\theta - 1 < G$, the balanced growth path, k^{**}, is locally stable in that there exists a neighborhood of k^{**}, U, such that if $k_0 \in U$, there exists an equilibrium path, whose entire trajectory stays in U, and along which the economy converges oscillatorily to k^{**}: $\lim_{t\to\infty} k_t = k^{**}$.[3]

Compare this result with Proposition 1 of Matsuyama (1999). It shows that the local stability properties of the steady state do not change by introducing the infinitely-lived representative consumer. It should be pointed out, however, that the above result is weaker, and needs to be interpreted with great caution. First, the above proposition is concerned only with the existence of an equilibrium trajectory converging asymptotically to the steady state. Unlike in Matsuyama (1999), the equilibrium of this model may not be unique, and hence the above result does not say much about what cannot happen. In particular, the existence of a convergent equilibrium trajectory in the cases of $G < 1$ and of $G > \theta - 1$ does not necessarily rule out the existence of another equilibrium trajectory, which may not be convergent. This may be so even if the economy starts close to the steady state. Second, this proposition deals only with the local dynamics, so that it does not tell us whether the economy converges to the neoclassical stationary state (the balanced growth path) for an arbitrary initial condition, in the case of $G < 1$ ($G > \theta - 1$). For the case of $1 < G < \theta - 1$, however, the following statement about the global dynamics can be made as a corollary of the above Proposition.

<u>Corollary.</u> If $1 < G < \theta - 1$, the economy fluctuates forever, for $k_0 \in R_+ \backslash D$, where D is at most countable subset of R_+.

<u>Proof</u>. If $1 < G < \theta - 1$, its unique steady state, (k^{**}, k^{**}), is a source of the two-dimensional

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[3]For $\theta - 1 < G$, the convergent equilibrium trajectory is given by $k_t = Gk_{t-1}/(1+\theta(k_{t-1}-1))$.

dynamical system, (11). Hence, a trajectory cannot approach (k^{**}, k^{**}) asymptotically. Therefore, the economy converges to (k^{**}, k^{**}) along an equilibrium path, only if (k_0, k_1) is mapped to (k^{**}, k^{**}) in finite steps. The result thus follows from the fact that eq. (11), viewed as the two-dimensional dynamical system, is continuous, and has a finite number of pre-images. Q.E.D.

Therefore, the equilibrium dynamics of the economy exhibits endogenous fluctuations for almost all initial conditions, if $1 < G < \theta - 1$.[4]

The instability of the balanced growth path and the emergence of cyclical behavior are due to the complementarity in the timing of entry/innovation decisions. The timing matters in this model, because innovators could enjoy only a temporary monopoly power. Innovations take place only when the market for a new product is sufficiently large that the innovator can reach the break-even level of the output. The size of the market partially depends on how the products with which it competes are priced. If the innovator enters when other firms also enter, some of the products are monopolistically priced. If it enters in the following period, then these products become competitively priced, as their innovators lose the monopoly power. This consideration gives an incentive for firms to enter when other firms also enter. This effect is stronger when different products are highly substitutable, i.e., when θ is high. At the same time, a growing resource base gives an offsetting force of spreading innovative and entry activities, whose effect is stronger when G is high. When the former effect dominates the latter, there is a bunching of the entry activity, and the economy moves back and forth between the Romer regime (the period of innovation) and the Solow regime (the period of no innovation).

The main result of this section is that the unique steady state loses its stability and the economy fluctuates endogenously between the Romer and the Solow regimes, if $1 < G < \theta - 1$.

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[4] Because the system is discrete, the equilibrium dynamics may converge to the steady state even if it is a source for a countable set (hence with measure zero) of initial conditions. For example, simple algebra can verify that, not only for $k_0 = k^{**} > 1$, but also for $k_0 = (k^{**}/G)^{\sigma/(\sigma-1)} < 1$, $k_t = k^{**}$ for all $t \geq 1$ is an equilibrium path, even if $1 < G < \theta - 1$. One open question is the existence of a homoclinic orbit, which starts in the neighborhood of the steady state and wanders away from the neighborhood, and comes back to the steady state. The existence of such an orbit is of special interest as it suggests the existence of a chaotic equilibrium path. The recent discovery by Mitra (1999) that the model of Matsuyama (1999) has a chaotic equilibrium path under additional parameter restrictions strongly suggests that the existence of sucn a path in this model as well.

This condition is identical in the model of Matsuyama (1999), where it was simply assumed that the economy maintains a constant capital-output ratio. In other words, introducing infinitely-lived agent makes little difference. In particular, the discussion of the empirically plausibility of endogenous fluctuations, given in Matsuyama (1999, section 4), does not need to be changed. This may seem counterintuitive; one's intuition might suggest that intertemporal substitution should have some stabilizing effects. The intuition is actually consistent with the result here. The reason is that, in this model, the growth rate is determined endogenously. A higher γ increases G, whenever $G > 1$. As long as $\beta(1-1/\sigma)A > 1$, a sufficiently high γ makes $G > \theta - 1$, thereby restoring the stability of the balanced growth path. Once the growth rate is controlled for, the condition for the stability is the same whether capital accumulation is determined by a fixed saving rule or by the intertemporal optimization by the infinitely-lived agent.

4. Period-2 Cycles.

Characterizing the equilibrium dynamics for an arbitrary initial condition is beyond the scope of this paper. Nevertheless, one can obtain some ideas about the global equilibrium dynamics, by studying period-2 cycles.

Suppose that the economy alternates between the Solow regime and the Romer regime over the period-2 cycles, such that $k^L < 1$ and $k^H > 1$. Setting $k_{t-1} = k^H$, $k_t = k^L$, and $k_{t+1} = k^H$ in eq. (11) yields

$$(13) \qquad \frac{\psi(k^H)}{Ak^H - k^L\psi(k^H)} = \frac{G[\phi(k^L)]^\gamma}{A\phi(k^L)k^L - k^H} > 0$$

Likewise, setting $k_{t-1} = k^L$, $k_t = k^H$, and $k_{t+1} = k^L$ in eq. (11) yields

$$(14) \qquad \frac{1}{A\phi(k^L)k^L - k^H} = \frac{G}{Ak^H - k^L\psi(k^H)} > 0.$$

Multiplying each side of (13) and (14) yields

$$(15) \qquad \psi(k^H) = G^2[\phi(k^L)]^\gamma,$$

from which the average growth rate of the economy over the period-2 cycles is equal to $\sqrt{\psi(k^L)\psi(k^H)} = \sqrt{\psi(k^H)} = G[\phi(k^L)]^{\gamma/2} > G$. Therefore, the condition (4) is satisfied along the period-2 cycles if

(16) $$\beta G^{1-1/\gamma}\left(k^{L}\right)^{(1-\gamma)/2\sigma} < 1.$$

The period-2 cycles exist if there exist $k^H > 1 > k^L$ satisfying eqs. (14), (15), and (16).

In what follows, we focus on the case where $\gamma = 1$. Then, (16) becomes $\beta < 1$, and the other conditions can be rewritten to

(17) $$k^{H} = G\phi\left(k^{L}\right)k^{L}$$

and

(18) $$k^{L} = \frac{Gk^{H}}{\psi(k^{H})}.$$

These equations are illustrated by the two curves in Figure 1. Simple algebra verifies that they have a unique intersection in the range, $k^H > 1 > k^L$, if and only if $1 < G < \theta - 1$. The following proposition summarizes the main properties of the period-2 cycles.

<u>Proposition 2</u>. Let $\gamma = 1$. Then, a unique pair of the period-2 cycles, $k^H > 1 > k^L$, exists if and only if $1 < G < \theta - 1$. Furthermore,

(a) $g_N = g_K = g_Y = \sqrt{\psi\left(k^{H}\right)} = G\sqrt{\phi\left(k^{L}\right)} > G$ over the cycles,

(b) $g_N = 1 < G < G\phi(k^{L}) = g_K = g_Y$ in the Solow regime,

(c) $g_N = \psi(k^{H}) > G = g_K = g_Y$ in the Romer regime,

where g_X be the gross growth rate of variable X.

Proof: The existence has been established above. Let $k_{t-2} = k^H$, $k_{t-1} = k^L$, $k_t = k^H$. Then, from (9), $N_t/N_{t-1} = 1$ and $N_{t+1}/N_t = \psi(k^H)$. Hence, $K_t/K_{t-1} = (k^H/k^L)(N_t/N_{t-1}) = k^H/k^L$, and $K_{t+1}/K_t = (k^L/k^H)(N_{t+1}/N_t) = (k^L/k^H)\psi(k^H)$. From (10), $Y_t/Y_{t-1} = \phi(k^L)k^L\psi(k^H)/k^H$, and $Y_{t+1}/Y_t = k^H/\phi(k^L)k^L$. Inserting eqs. (17) and (18) into these expressions and taking into account (15) yields (a), (b), and (c). <u>Q.E.D.</u>

Period-2 cycles thus exist whenever the balanced growth path is unstable. These cycles are growth-enhancing: they allow the economy to grow even faster than along the balanced growth path.

Furthermore, both the output and the investment grow faster in the Solow regime, when there is no innovation, then in the Romer regime, when innovation takes place. Although the innovation of new goods is a crucial way of avoiding the diminishing returns and of sustaining growth indefinitely, the economy actually experiences a lower growth in the output and in the investment during the period of innovation. Only after the innovation stops, and when the market structure becomes competitive, the economy enjoys much of the benefits of the innovation. This proposition, by comparing Proposition 2 in Matsuyama (1999), shows that the fixed saving rule assumption made in that paper was not essential.

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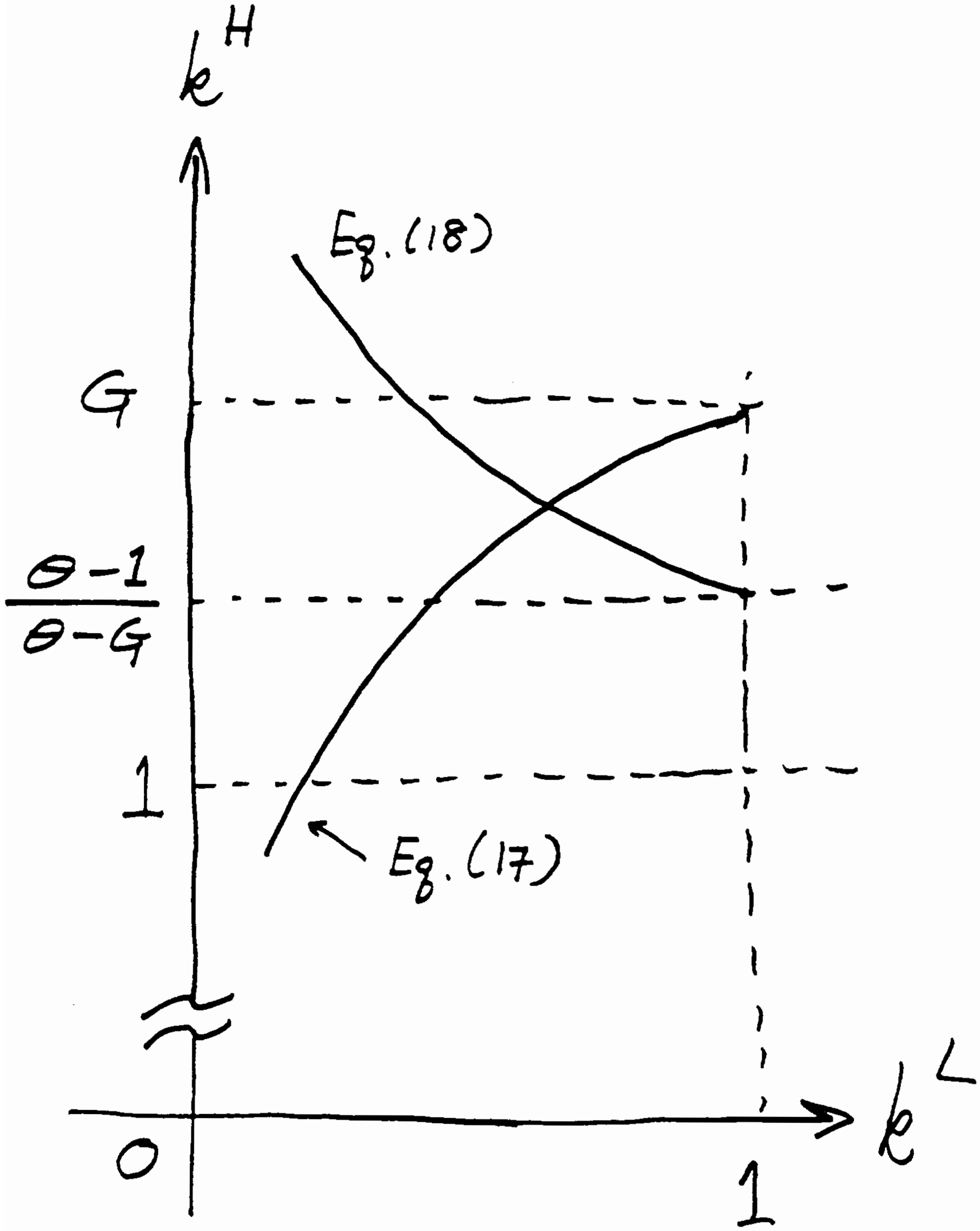

Figure 1