Beyond CES: Three Alternative Classes of Flexible Homothetic Demand Systems

Kiminori Matsuyama\textsuperscript{1} \hspace{1cm} Philip Ushchev\textsuperscript{2}

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\textsuperscript{1} Department of Economics, Northwestern University, Evanston, USA. Email: \texttt{k-matsuyama@northwestern.edu}

\textsuperscript{2} National Research University Higher School of Economics, Russian Federation. Email: \texttt{ph.ushchev@gmail.com}
Homothetic preferences: A general case

• Common across many fields of applied general equilibrium, preferences are homothetic and technologies are CRS

• A preference $\succeq$ over $\mathbb{R}^n_+$ is called homothetic if any two indifference sets can be mapped one into the other by a uniform rescaling

• The direct utility function $u(x)$ is linear homogeneous

• The indirect utility $V(p, h)$ can be represented as
  \[ V(p, h) \equiv \max_{x \in \mathbb{R}^n_+} \{ u(x) \mid px \leq h \} = \frac{h}{P(p)} \]
  
  o $h$ is consumer’s income
  o $P(p)$ is an ideal price index
Homothetic demands and elasticities; A general case

• The demand system associated with $P(p)$:

$$x_i = \frac{h}{p_i} \mathcal{E}_{p_i}(P)$$

• The inverse demand system associated with $u(x)$:

$$p_i = \frac{h}{x_i} \mathcal{E}_{x_i}(u)$$

• $\mathcal{E}_{p_i}(P)$ and $\mathcal{E}_{x_i}(u)$ are the elasticities defined by:

$$\mathcal{E}_{p_i}(P) \equiv \frac{\partial P}{\partial p_i} \frac{p_i}{P}, \quad \mathcal{E}_{x_i}(u) \equiv \frac{\partial u}{\partial x_i} \frac{x_i}{u}$$
Why are homothetic preferences and CRS technologies important?

- Under identical homothetic preferences, aggregate consumption behavior is derived from utility maximization of a representative consumer, even though incomes may vary across households.

- Perfect competition is valid only when the industry has CRS technologies.

- Simple behavior of budget shares:
  - Holding the prices constant, the budget share of each good (or factor) is independent of the household expenditure (or the scale of operation by industries).
  - This allows us to focus on the role of relative prices in the allocation of resources.

- Ensure the existence of a balanced growth path in multi-sector growth models.
CES and its restrictive features

In practice, most models assume that preferences/technologies also satisfy constant-elasticity-of-substitution (CES) property, which implies that

- the price elasticity of demand for each good/factor is constant and identical across goods/factors

- relative demand for any two goods/factors is independent of the prices of any other goods/factors

- the marginal rate of substitution between any two goods is independent of the consumption of any other goods

- in the case of gross substitutes (complements) all goods are inessential (essential)

- in a monopolistically competitive setting, each firm sells its product at a markup independent of the market environment
Our paper

• In this paper, we characterize three alternative classes of flexible homothetic demand systems

• In each of the three classes, the demand system only depends on one or two price aggregators for any number of goods

• Each of these classes contains CES as a special case

• Yet, they offer three *alternative* ways of departing from CES, because non-CES demand systems in these three classes do not overlap

• Each of these three classes is *flexible* in the sense that they are defined non-parametrically
Homothetic demand systems
with a single aggregator (HSA)
HSA demand systems

• Consider a mapping $s(z) = (s_1(z_1), \ldots, s_n(z_n))^T$ from $\mathbb{R}_+^n$ to $\mathbb{R}_+^n$,

• A homothetic demand system with a single aggregator (HSA) is given by:

$$x_i = \frac{h}{p_i} s_i \left( \frac{p_i}{A(p)} \right), \; i = 1, \ldots, n$$

where $A(p)$ is a common price aggregator defined as a solution to

$$\sum_{i=1}^{n} s_i \left( \frac{p_i}{A} \right) = 1$$
Example 1: Cobb-Douglas

- Set $s_i(z_i) = \alpha_i$, where $\alpha_1, \ldots, \alpha_n$ are positive constants such that

$$
\sum_{i=1}^{n} \alpha_i = 1
$$

- In this case, we obtain the *Cobb-Douglas* demand system

- $P(p) = c \prod_{i=1}^{n} p_i^{\alpha_i}$, but $A(p)$ is indeterminate
Example 2: CES

- We obtain the CES demand system if we set $s_i(z_i) = \beta_i z_i^{1-\sigma}$

- Here $\sigma > 0$ is the constant elasticity of substitution

- The price aggregator $A(p)$ is proportional to the ideal price index:

$$A(p) = \left( \sum_{i=1}^{n} \beta_i p_i^{1-\sigma} \right)^{1\over 1-\sigma} = cP(p).$$

- NB: this need not be true in general!
Example 2: CES

The functions $s_i(z_i) = \beta_i z_i^{1-\sigma}$ are:

- increasing when $0 < \sigma < 1$ (the goods are *gross complements*)
- decreasing when $\sigma > 1$ (the goods are *gross substitutes*)
- constant when $\sigma = 1$ (the Cobb-Douglas case)
Example 2: CES and its restrictive nature

Definition:
• Good $i$ is *essential* (or *indispensable*) if $x_i = 0$ implies $u(x) = 0$ (or equivalently, if $p_i \to \infty$ implies $P(p) \to \infty$).
• Good $i$ is *inessential* (or *dispensable*), otherwise.

Under CES
• Each good is inessential if $\sigma > 1$.
• A good is essential only if $\sigma \leq 1$
• CES cannot capture situations when only some goods are essential: *if one good is essential, all goods must be essential*
• The very distinction of a good being essential or inessential is *redundant*: gross complements (respectively, substitutes) are always essential (inessential) goods
Integrability Question

• What are the restrictions to be imposed on the functions $s_i(\cdot)$ for a “candidate” HSA demand system to be compatible with rational consumer behavior?

• The answer is given by the following Proposition
A characterization of HSA

Proposition 1. Consider a mapping $s(z) = (s_1(z_1), \ldots, s_n(z_n))^T$ from $\mathbb{R}_+^n$ to $\mathbb{R}_+^n$, which is normalized by $\sum_{k=1}^n s_k(1) = 1$ and satisfies the conditions:

$$z_i s_i'(z_i) < s_i(z_i), \quad s_i'(z_i) s_j'(z_j) \geq 0$$

Then:

(i) there exists a unique monotone, convex, continuous and homothetic preference $\succeq$ over $\mathbb{R}_+^n$, such that the candidate HSA demand system associated with $s(z)$ is generated by $\succeq$.

(ii) the preference $\succeq$ is described by the following ideal price index

$$\ln P(p) = \ln A(p) + \sum_{i=1}^n \int_{c_1}^{p_i/A(p)} \frac{s_i'(\xi)}{\xi} d\xi$$

(iii) when $n \geq 3$, $A(p) = cP(p)$ iff $\succeq$ is a CES preference.
Budget-share mapping as a primitive

- The budget-share mapping $s(z)$ is the *primitive* of the HSA system.

- $A(p)$ itself cannot serve as a primitive (see Example 5 below).

- $A(p)$ need *not* be proportional to $P(p)$
  - $A(p)$ captures the *cross-price effects* in the demand system.
  - $P(p)$ captures the *welfare consequences* of price changes.

- The condition $n \geq 3$ is important, as under $n = 2$ *all* homothetic preferences are HSA.
Self-duality of the HSA demand systems

- Consider a mapping \( s^*(y) = (s_1^*(y_1), \ldots, s_n^*(y_n))^T \) from \( \mathbb{R}_+^n \) to \( \mathbb{R}_+^n \).

- The inverse HSA demand system is given by
  \[
p_i = \frac{h}{x_i} s_i^* \left( \frac{x_i}{A^*(x)} \right), \quad i = 1, \ldots, n
\]

  where \( A^*(x) \) is a common quantity aggregator defined as a solution to
  \[
  \sum_{i=1}^{n} s_i^* \left( \frac{x_i}{A^*} \right) = 1
  \]

- The two classes of HSA demand systems are self-dual to each other with a one-to-one correspondence between \( s(z) \) and \( s^*(y) \), defined by \( s_i^* = s_i(s_i^*/y_i) \).
Example 3a: Separable translog

• The translog ideal price index is given by

\[
\ln P(p) = \sum_{i=1}^{n} \delta_i \ln p_i - \frac{1}{2} \sum_{i,j=1}^{n} \gamma_{ij} \ln p_i \ln p_j - \ln c
\]

• Here \( \delta_i > 0 \), while \( (\gamma_{ij}) \) is symmetric and positive semidefinite

• The following normalizations hold for all \( i = 1, \ldots, n \):

\[
\sum_{j=1}^{n} \delta_j = 1, \quad \sum_{j=1}^{n} \gamma_{ij} = 0
\]
Example 3a: Separable translog

• In general, the translog demand system is not HSA

• However, assume additionally the following separability:

\[ \gamma_{ij} = \begin{cases} \gamma \beta_i (1 - \beta_i), & i = j \\ -\gamma \beta_i \beta_j, & i \neq j \end{cases} \]

\[ \sum_{i=1}^{n} \beta_i = 1 \]

• By setting \( s_i(z_i) = \delta_i - \gamma \beta_i \ln z_i \), we get:

\[ x_i = \frac{h}{p_i} s_i \left( \frac{p_i}{A(p)} \right) = \frac{h}{p_i} \left( \delta_i - \gamma \beta_i \ln \frac{p_i}{A(p)} \right) \]
Example 3a: Separable translog

- The price aggregator $A(p)$ is the weighted geometric mean of prices:

$$\ln A(p) = \sum_{i=1}^{n} \beta_i \ln p_i$$

- The price index $P(p)$ differs from the price aggregator $A(p)$:

$$P(p) = c \cdot \exp \left\{ \sum_{i=1}^{n} \delta_i \ln p_i - \frac{\gamma}{2} \left[ \sum_{i=1}^{n} \beta_i (\ln p_i)^2 - \left( \sum_{i=1}^{n} \beta_i \ln p_i \right)^2 \right] \right\} \neq A(p)$$
Example 5: A Hybrid of Cobb-Douglas and CES

• Consider a convex combination of Cobb-Douglas budget shares and CES budget shares:

\[ s_i(z) = \varepsilon \alpha_i + (1 - \varepsilon) \beta_i z^{1-\sigma} \]

• Here \( 0 < \varepsilon < 1 \), while \( \alpha_i \) and \( \beta_i \) are such that

\[ \alpha_i \geq 0, \beta_i > 0, \sum_{k=1}^{n} \alpha_k = \sum_{k=1}^{n} \beta_k = 1 \]
Example 5: A Hybrid of Cobb-Douglas and CES

- The price aggregator $A(p)$ is independent of $\varepsilon$:

$$A(p) = \left( \sum_{i=1}^{n} \beta_i p_i^{1-\sigma} \right)^{\frac{1}{1-\sigma}}$$

- The ideal price index is given by

$$P(p) = c \left( \prod_{i=1}^{n} p_i^{\alpha_i} \right)^{\varepsilon} \left( \sum_{i=1}^{n} \beta_i p_i^{1-\sigma} \right)^{\frac{1-\varepsilon}{1-\sigma}}$$
Example 5: A Hybrid of Cobb-Douglas and CES

• When \( \sigma > 1 \), all goods are still gross substitutes, and yet, if \( \alpha_i > 0 \), good \( i \) is essential

• **Implication:** consider international trade between two countries, and suppose that some of the essential goods can be produced only in one country

• Trade elasticity is \( \sigma > 1 \). With a small \( \varepsilon \), the demand system can be approximated by CES.

• Were the demand system CES (\( \varepsilon = 0 \)), autarky would lead to a relatively small welfare loss

• But the welfare loss of autarky (measured by the price index change) is infinity for the country which cannot produce such essential goods
Implicitly additive homothetic preferences
**HDIA preferences**

- A preference \( \succeq \) over \( \mathbb{R}_+^n \) is said to be *homothetic with direct implicit additivity* (HDIA) if \( u(x) \) is implicitly defined as a solution to
  \[
  \sum_{i=1}^{n} \phi_i \left( \frac{x_i}{u} \right) = 1
  \]

- Here the sufficiently differentiable functions \( \phi_i : \mathbb{R}_+ \to \mathbb{R} \) are
  - either strictly increasing and strictly concave (goods are gross substitutes)
  - or strictly decreasing and strictly convex (goods are gross complements)

- Moreover, \( \phi_i(\cdot) \) are normalized as follows: \( \sum_{i=1}^{n} \phi_i(1) = 1 \)
HDIA preferences

Proposition 2. Assume $\succeq$ is a HDIA preference. Then:

(i) the Marshallian demands are given by

$$x_i = \frac{h}{P(p)} (\phi_i')^{-1} \left( \frac{p_i}{B(p)} \right),$$

where $P(p)$ is the ideal price index, while $B(p)$ is another price aggregator:

$$\sum_{k=1}^{n} \phi_k \left( (\phi_k')^{-1} \left( \frac{p_k}{B} \right) \right) = 1, \quad P(p) = \sum_{k=1}^{n} p_k (\phi_k')^{-1} \left( \frac{p_k}{B(p)} \right);$$

(ii) when $n \geq 3$, we have $B(p) = cP(p)$ iff $\succeq$ is a CES preference.
HIIA preferences

- A preference $\succsim$ over $\mathbb{R}_+^n$ is said to be homothetic with indirect implicit additivity (HIIA) if $P(p)$ is implicitly defined as a solution to

$$\sum_{i=1}^{n} \theta_i \left( \frac{p_i}{P} \right) = 1$$

- Here the sufficiently differentiable functions $\theta_i: \mathbb{R}_+ \rightarrow \mathbb{R}$ are

  - either strictly decreasing and strictly convex (goods are gross substitutes)
  - or strictly increasing and strictly concave (goods are gross complements)

- Moreover, $\theta_i(\cdot)$ are normalized as follows: $\sum_{i=1}^{n} \theta_i(1) = 1$
HIIA preferences

**Proposition 3.** Assume a preference \( \succeq \) is HIIA. Then:

(i) the Marshallian demands are given by

\[
x_i = \frac{h}{C(p)} \theta_i' \left( \frac{p_i}{P(p)} \right),
\]

where \( P(p) \) is the ideal price index, while \( C(p) \) is another price aggregator:

\[
C(p) \equiv \sum_{k=1}^{n} p_k \theta_k' \left( \frac{p_k}{P(p)} \right);
\]

(ii) when \( n \geq 3 \), we have \( C(p) = cP(p) \) iff \( \succeq \) is a CES preference.
Comparing HSA, HDIA, and HIIA
Three alternative ways of departure from CES

Proposition 4. Assume that $n \geq 3$. Then:

(i) $\text{HDIA} \cap \text{HSA} = \text{CES}$;

(ii) $\text{HIIA} \cap \text{HSA} = \text{CES}$;

(iii) $\text{HDIA} \cap \text{HIIA} = \text{CES}$. 
Thank you for your attention!
HSA are GAS

- HSA demand systems are the homothetic restriction of what Pollak (1972) refers to as generalized additively separable (GAS) demand systems

- We prefer to call HSA instead of homothetic generalized additively separable, because it does not nest the demand systems generated by additively separable preferences.

- We provide sufficient conditions for the “candidate” HSA demand system to actually be a demand system generated by some continuous and convex homothetic preference
Example 3b: Modified translog

- Separable translog is incompatible with *gross complementarity*

- To overcome this, consider the following modification:

\[ s_i(z_i) = \max \{ \delta_i + \gamma \beta_i \ln z_i , \gamma \beta_i \} \]

- Here \( \delta_i \) and \( \beta_i \) are all positive and such that

\[ \sum_{k=1}^{n} \beta_k = \sum_{k=1}^{n} \delta_k = 1, \quad 0 < \gamma < \min_{k=1,\ldots,n} \left\{ \frac{\delta_k}{\beta_k} \right\} \]
Example 3b: Modified translog

- The price aggregator $A(p)$ has the same form as under the separable translog:

\[
\ln A(p) = \sum_{i=1}^{n} \beta_i \ln p_i
\]

- The price index $P(p)$ is given by:

\[
P(p) = c \cdot \exp \left\{ \sum_{i=1}^{n} \delta_i \ln p_i + \frac{\gamma}{2} \left[ \sum_{i=1}^{n} \beta_i (\ln p_i)^2 - \left( \sum_{i=1}^{n} \beta_i \ln p_i \right)^2 \right] \right\} \neq A(p)
\]
Example 4: Linear expenditure shares

- Another natural extension of Cobb-Douglas is a demand system with *linear expenditure shares*:

\[
s_i(z_i) = \max\{(1 - \delta)\alpha_i + \delta \beta_i z_i, 0\}
\]

- Here \(\delta < 1\), \(\alpha_i > 0\), \(\beta_i > 0\), and \(\sum_{i=1}^{n} \alpha_i = \sum_{i=1}^{n} \beta_i = 1\)

- The goods are
  - gross complements when \(0 < \delta < 1\)
  - gross substitutes when \(\delta < 0\)
Example 4: Linear expenditure shares

- The price aggregator $A(p)$ is the weighted arithmetic mean of prices:

$$A(p) = \sum_{i=1}^{n} \beta_i p_i$$

- The ideal price index is given by

$$P(p) = c[A(p)]^\delta \left( \prod_{i=1}^{n} p_i^{\alpha_i} \right)^{1-\delta} \neq A(p)$$