Beyond CES: Three Alternative Classes of Flexible Homothetic Demand Systems

Kiminori Matsuyama[1] Philip Ushchev[2]

October 2017

[1] Department of Economics, Northwestern University, Evanston, USA. Email: k-matsuyama@northwestern.edu

[2] National Research University Higher School of Economics, Russian Federation. Email: ph.ushchev@gmail.com

Homothetic preferences

- Common across many fields of applied general equilibrium, preferences are *homothetic* and technologies are *CRS*

- A preference $\gtrsim$ over $\mathbb{R}_+^n$ is called *homothetic* if any two indifference sets can be mapped one into the other by a *uniform rescaling*

- The direct utility function $u(\mathbf{x})$ is $linear\ homogeneous$

- The indirect utility $V(\mathbf{p}, h)$ can be represented as

$$V(\mathbf{p}, h) \equiv \max_{\mathbf{x}\in\mathbb{R}_+^n}\{u(\mathbf{x})|\mathbf{px} \le h\} = \frac{h}{P(\mathbf{p})}$$

  - h is consumer's income
  - $P(\mathbf{p})$ is an *ideal price index*

Why are homothetic preferences and CRS technologies important?

- Under identical homothetic preferences, aggregate consumption behavior is derived from utility maximization of a representative consumer, even though incomes may vary across households

- Perfect competition is valid only when the industry has CRS technologies

- Simple behavior of budget shares:
  - holding the prices constant, the budget share of each good (or factor) is independent of the household expenditure (or the scale of operation by industries)
  - this allows us to focus on the role of relative prices in the allocation of resources

- Ensure the existence of a balanced growth path in multi-sector growth models

CES and its restrictive features

In practice, most models assume that preferences/technologies also satisfy *constant-elasticity-of-substitution* (CES) property, which implies that

- the price elasticity of demand for each good/factor is *constant and identical* across goods/factors
- relative demand for any two goods/factors is *independent* of the prices of any other goods/factors
- the marginal rate of substitution between any two goods is *independent* of the consumption of any other goods
- in the case of *gross substitutes* (*complements*) all goods are *inessential* (*essential*)
- in a monopolistically competitive setting, each firm sells its product at a *markup independent of the market environment*

Our paper

- In this paper, we characterize three alternative classes of flexible homothetic demand systems

- In each of the three classes, the demand system only depends on one or two price aggregators for any number of goods

- Each of these classes contains CES as a special case

- Yet, they offer three *alternative* ways of departing from CES, because non-CES demand systems in these three classes do not overlap

- Each of these three classes is *flexible* in the sense that they are defined non-parametrically

Homothetic demand systems with a single aggregator (HSA)

HSA demand systems

- A *homothetic demand system with a single aggregator* (HSA) is given by:

$$x_i = \frac{h}{p_i} s_i\left(\frac{p_i}{A(\mathbf{p})}\right), \; i = 1, \dots, n$$

- The functions $s_i\colon \mathbb{R}_+ \to \mathbb{R}_+$ are normalized as follows: $\sum_{i=1}^{n} s_i(1) = 1$

- $A(\mathbf{p})$ is a *common price aggregator* defined as a solution to

$$\sum_{i=1}^{n} s_i\left(\frac{p_i}{A}\right) = 1$$

Example 1: Cobb-Douglas

- Set $s_i(z_i) = \alpha_i$, where $\alpha_1, \ldots, \alpha_n$ are positive constants such that

$$\sum_{i=1}^{n} \alpha_i = 1$$

- In this case, we obtain the *Cobb-Douglas* demand system

- $P(\mathbf{p}) = c \prod_{i=1}^{n} p_i^{\alpha_i}$, but $A(\mathbf{p})$ is *indeterminate*

Example 2: CES

- We obtain the CES demand system if we set $s_i(z_i) = \beta_i z_i^{1-\sigma}$
- Here $\sigma > 0$ is the constant elasticity of substitution
- The price aggregator $A(\mathbf{p})$ is proportional to the ideal price index:

$$A(\mathbf{p}) = \left(\sum_{i=1}^{n} \beta_i p_i^{1-\sigma}\right)^{\frac{1}{1-\sigma}} = cP(\mathbf{p}).$$

- NB: this need not be true in general!

Example 2: CES

The functions $s_i(z_i) = \beta_i z_i^{1-\sigma}$ are:

- increasing when $0 < \sigma < 1$ (the goods are gross complements)
- decreasing when $\sigma > 1$ (the goods are gross substitutes)
- constant when $\sigma = 1$ (the Cobb-Douglas case)

Example 2: CES and its restrictive nature

- Under $\sigma > 1$, the budget share $s_i(z_i) = \beta_i z_i^{1-\sigma}$ is not only decreasing but also goes to zero when $z_i \to \infty$. Each good is *inessential*.

- A good can be *essential* (or *indispensable*) only if $\sigma \leq 1$

- Thus, the CES cannot capture situations when only some goods are essential: *if one good is essential, all goods must be essential*

- Under the CES, the very concept of a good being essential or inessential is *redundant*: gross complements (respectively, substitutes) are always essential (inessential) goods

Question

- What are the restrictions to be imposed on the functions $s_i(\cdot)$ for a "candidate" HSA demand system to be compatible with *rational consumer behavior*?

- The answer is given by the following Proposition

A characterization of HSA

Proposition 1. *Consider a mapping* $\boldsymbol{s}(\boldsymbol{z}) = \left(s_1(z_1), \dots, s_n(z_n)\right)^{\mathrm{T}}$ *from* $\mathbb{R}_+^n$ *to* $\mathbb{R}_+^n$, *which is normalized by* $\sum_{k=1}^n s_k(1) = 1$ *and satisfies the conditions*:

$$z_i s_i'(z_i) < s_i(z_i), \quad s_i'(z_i) s_j'(z_j) \geq 0$$

Then:

(i) *there exists a unique monotone, convex, continuous and homothetic preference* $\gtrsim$ *over* $\mathbb{R}_+^n$, *such that the candidate HSA demand system associated with* $\boldsymbol{s}(\boldsymbol{z})$ *is generated by* $\gtrsim$

(ii) *the preference* $\gtrsim$ *is described by the following ideal price index*

$$\ln P(\mathbf{p}) = \ln A(\mathbf{p}) + \sum_{i=1}^{n} \int_{c_1}^{p_i/A(\mathbf{p})} \frac{s_i(\xi)}{\xi} \mathrm{d}\xi$$

(iii) *when* $n \geq 3$, $A(\mathbf{p}) = cP(\mathbf{p})$ *iff* $\gtrsim$ *is a CES preference*

Budget-share mapping as a primitive

- The budget-share mapping $\mathbf{s}(\mathbf{z})$ is the *primitive* of the HSA system

- $A(\mathbf{p})$ itself cannot serve as a primitive (see Example 5 below)

- $A(\mathbf{p})$ need *not* be proportional to $P(\mathbf{p})$

  - $A(\mathbf{p})$ captures the *cross-effects* in the demand system
  - $P(\mathbf{p})$ captures the *welfare consequences* of price changes

- The condition $n \geq 3$ is important, as under $n = 2$ *all* homothetic preferences are HSA

Self-duality of the HSA demand systems

- *The inverse HSA demand system* is given by

$$p_i = \frac{h}{x_i} s_i^*\left(\frac{x_i}{A^*(\boldsymbol{x})}\right), \; i = 1, \dots, n$$

- The functions $s_i^*: \mathbb{R}_+ \to \mathbb{R}_+$ are defined by $s_i^* = s_i(s_i^*/y_i)$

- $A^*(\boldsymbol{x})$ is a *common quantity aggregator* defined as a solution to

$$\sum_{i=1}^{n} s_i^*\left(\frac{x_i}{A^*}\right) = 1$$

- The two classes of HSA demand systems are *self-dual* to each other with a one-to-one correspondence between $\mathbf{s}(\mathbf{z})$ and $\mathbf{s}^*(\mathbf{y})$

Example 3a: Separable translog

- The translog ideal price index is given by

$$\ln P(\mathbf{p}) = \sum_{i=1}^{n} \delta_i \ln p_i - \frac{1}{2} \sum_{i,j=1}^{n} \gamma_{ij} \ln p_i \ln p_j - \ln c$$

- Here $\delta_i > 0$, while (γ_{ij}) is symmetric and positive semidefinite

- The following normalizations hold for all $i = 1, \dots, n$:

$$\sum_{j=1}^{n} \delta_j = 1, \quad \sum_{j=1}^{n} \gamma_{ij} = 0$$

Example 3a: Separable translog

- In general, the translog demand system is not HAS

- However, assume additionally the following separability:

$$\gamma_{ij} = \begin{cases} \gamma\beta_i(1-\beta_i), i = j \\ -\gamma\beta_i\beta_j, i \neq j \end{cases} \qquad \sum_{i=1}^{n} \beta_i = 1$$

- By setting $s_i(z_i) = \delta_i - \gamma\beta_i \ln z_i$, we get:

$$x_i = \frac{h}{p_i} s_i\left(\frac{p_i}{A(\boldsymbol{p})}\right) = \frac{h}{p_i}\left(\delta_i - \gamma\beta_i \ln\frac{p_i}{A(\boldsymbol{p})}\right)$$

Example 3a: Separable translog

- The price aggregator $A(\mathbf{p})$ is the weighted geometric mean of prices:

$$\ln A(\mathbf{p}) = \sum_{i=1}^{n} \beta_i \ln p_i$$

- The price index $P(\mathbf{p})$ *differs* from the price aggregator $A(\mathbf{p})$:

$$P(\mathbf{p}) = c \cdot \exp\left\{\sum_{i=1}^{n} \delta_i \ln p_i - \frac{\gamma}{2}\left[\sum_{i=1}^{n} \beta_i (\ln p_i)^2 - \left(\sum_{i=1}^{n} \beta_i \ln p_i\right)^2\right]\right\} \neq A(\mathbf{p})$$

Example 5: A Hybrid of Cobb-Douglas and CES

- Consider a convex combination of Cobb-Douglas budget shares and CES budget shares:

$$s_i(z) = \varepsilon\alpha_i + (1-\varepsilon)\beta_i z^{1-\sigma}$$

- Here $0 < \varepsilon < 1$, while α_i and β_i are such that

$$\alpha_i \geq 0, \ \ \beta_i > 0, \ \ \sum_{k=1}^{n} \alpha_k = \sum_{k=1}^{n} \beta_k = 1$$

Example 5: A Hybrid of Cobb-Douglas and CES

- The price aggregator $A(\mathbf{p})$ is independent of ε:

$$A(\mathbf{p}) = \left(\sum_{i=1}^{n} \beta_i p_i^{1-\sigma}\right)^{\frac{1}{1-\sigma}}$$

- The ideal price index is given by

$$P(\mathbf{p}) = c\left(\prod_{i=1}^{n} p_i^{\alpha_i}\right)^{\varepsilon}\left(\sum_{i=1}^{n} \beta_i p_i^{1-\sigma}\right)^{\frac{1-\varepsilon}{1-\sigma}}$$

Example 5: A Hybrid of Cobb-Douglas and CES

- When $\sigma > 1$, all goods are still gross substitutes, and yet, if $\alpha_i > 0$, good i is *essential*

- **Implication**: consider international trade between two countries, and suppose that some of the essential goods can be produced only in one country

- Trade elasticity is $\sigma > 1$. With a small ε, the demand system can be approximated by CES.

- Were the demand system CES ($\varepsilon = 0$), autarky would lead to a relatively small welfare loss

- But the welfare loss of autarky (measured by the price index change) is infinity for the country which cannot produce such essential goods

Implicitly additive homothetic preferences

HDIA preferences

- A preference $\succsim$ over $\mathbb{R}_+^n$ is said to be *homothetic with direct implicit additivity* (HDIA) if $u(\mathbf{x})$ is implicitly defined as a solution to

$$\sum_{i=1}^{n} \phi_i\left(\frac{x_i}{u}\right) = 1$$

- Here the sufficiently differentiable functions $\phi_i\colon \mathbb{R}_+ \to \mathbb{R}$ are
  - either strictly increasing and strictly concave (goods are gross substitutes)
  - or strictly decreasing and strictly convex (goods are gross complements)

- Moreover, $\phi_i(\cdot)$ are normalized as follows: $\sum_{i=1}^{n} \phi_i(1) = 1$

HDIA preferences

Proposition 2. *Assume* $\succsim$ *is a HDIA preference. Then*:

(i) *the Marshallian demands are given by*

$$x_i = \frac{h}{P(\mathbf{p})}({\phi_i}')^{-1}\left(\frac{p_i}{B(\mathbf{p})}\right),$$

where $P(p)$ *is the ideal price index, while* $B(p)$ *is another price aggregator*:

$$\sum_{k=1}^{n}\phi_k\left(({\phi_k}')^{-1}\left(\frac{p_k}{B}\right)\right) = 1, \quad P(p) = \sum_{k=1}^{n} p_k(\phi_k')^{-1}\left(\frac{p_k}{B(\mathbf{p})}\right);$$

(ii) *when* $n \geq 3$, *we have* $B(\mathbf{p}) = cP(\mathbf{p})$ *iff* $\succsim$ *is a CES preference.*

HIIA preferences

- A preference $\succsim$ over $\mathbb{R}^n_+$ is said to be homothetic with indirect implicit additivity (HIIA) if $P(\mathbf{p})$ is implicitly defined as a solution to

$$\sum_{i=1}^{n} \theta_i\left(\frac{p_i}{P}\right) = 1$$

- Here the sufficiently differentiable functions $\theta_i: \mathbb{R}_+ \to \mathbb{R}$ are
  - either strictly decreasing and strictly convex (goods are gross substitutes)
  - or strictly increasing and strictly concave (goods are gross complements)

- Moreover, $\theta_i(\cdot)$ are normalized as follows: $\sum_{i=1}^{n} \theta_i(1) = 1$

HIIA preferences

Proposition 3. *Assume a preference* $\succsim$ *is HIIA. Then*:

(i) the Marshallian demands are given by

$$x_i = \frac{h}{C(\boldsymbol{p})}\theta_i'\left(\frac{p_i}{P(\boldsymbol{p})}\right),$$

where $P(\boldsymbol{p})$ *is the ideal price index, while* $C(\boldsymbol{p})$ *is another price aggregator*:

$$C(\boldsymbol{p}) \equiv \sum_{k=1}^{n} p_k\theta_k'\left(\frac{p_k}{P(\boldsymbol{p})}\right);$$

(ii) when $n \geq 3$, *we have* $C(\boldsymbol{p}) = cP(\boldsymbol{p})$ *iff* $\succsim$ *is a CES preference.*

Comparing HSA, HDIA, and HIIA

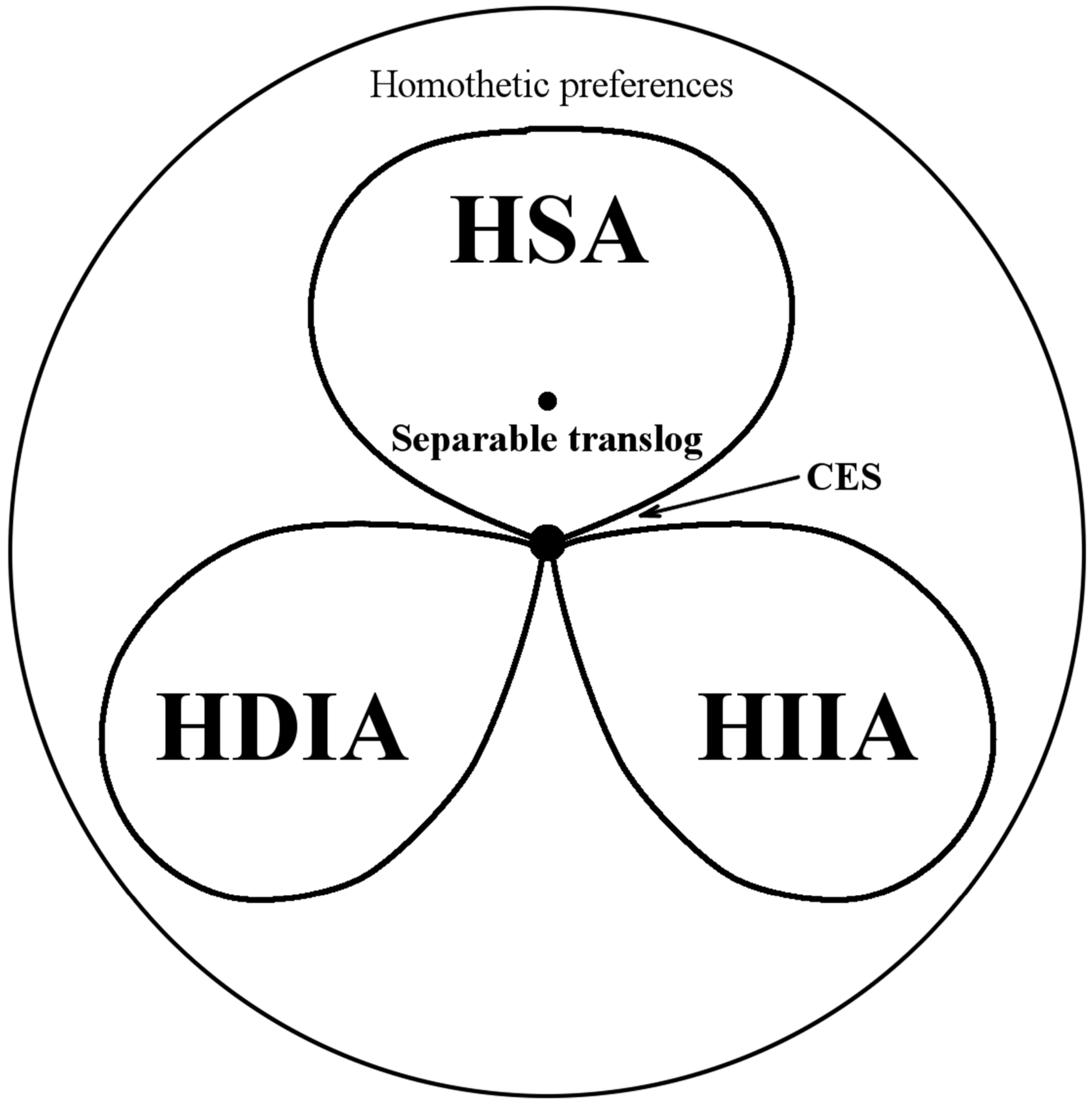
Three alternative ways of departure from CES

Proposition 4. *Assume that* $n \geq 3$. *Then*:

(i) HDIA ∩ HSA = CES;

(ii) HIIA ∩ HSA = CES;

(iii) HDIA ∩ HIIA = CES.


Homothetic preferences
HSA
Separable translog
CES
HDIA
HIIA

Thank you for your attention!

Homothetic demands and elasticities; A General Case

- The demand system associated with $P(\mathbf{p})$:

$$x_i = \frac{h}{p_i}\mathcal{E}_{p_i}(P)$$

- The inverse demand system associated with $u(\mathbf{x})$:

$$p_i = \frac{h}{x_i}\mathcal{E}_{x_i}(u)$$

- Here $\mathcal{E}_{p_i}(P)$ and $\mathcal{E}_{x_i}(u)$ are the elasticities defined by:

$$\mathcal{E}_{p_i}(P) \equiv \frac{\partial P}{\partial p_i}\frac{p_i}{P}, \quad \mathcal{E}_{x_i}(u) \equiv \frac{\partial u}{\partial x_i}\frac{x_i}{u}$$

HSA are GAS

- HSA demand systems are the homothetic restriction of what Pollak (1972) refers to as *generalized additively separable* (GAS) demand systems

- We prefer to call HSA instead of *homothetic generalized additively separable*, because it does not nest the demand systems generated by additively separable preferences.

- We provide sufficient conditions for the "candidate" HSA demand system to *actually* be a demand system generated by some *continuous and convex homothetic* preference

Example 3b: Modified translog

- Separable translog is incompatible with *gross complementarity*
- To overcome this, consider the following modification:

$$s_i(z_i) = \max\{\delta_i + \gamma\beta_i \ln z_i \,, \gamma\beta_i\}$$

- Here δ_i and β_i are all positive and such that

$$\sum_{k=1}^{n} \beta_k = \sum_{k=1}^{n} \delta_k = 1, \quad 0 < \gamma < \min_{k=1,\dots,n} \left\{\frac{\delta_k}{\beta_k}\right\}$$

Example 3b: Modified translog

- The price aggregator $A(\mathbf{p})$ has *the same form* as under the separable translog:

$$\ln A(\mathbf{p}) = \sum_{i=1}^{n} \beta_i \ln p_i$$

- The price index $P(\mathbf{p})$ is given by:

$$P(\mathbf{p}) = c \cdot \exp\left\{\sum_{i=1}^{n} \delta_i \ln p_i + \frac{\gamma}{2}\left[\sum_{i=1}^{n} \beta_i (\ln p_i)^2 - \left(\sum_{i=1}^{n} \beta_i \ln p_i\right)^2\right]\right\} \neq A(\mathbf{p})$$

Example 4: Linear expenditure shares

- Another natural extension of Cobb-Douglas is a demand system with *linear expenditure shares*:

$$s_i(z_i) = \max\{(1 - \delta)\alpha_i + \delta\beta_i z_i, 0\}$$

- Here $\delta < 1$, $\alpha_i > 0$, $\beta_i > 0$, and $\sum_{i=1}^{n} \alpha_i = \sum_{i=1}^{n} \beta_i = 1$

- The goods are

  - gross complements when $0 < \delta < 1$
  - gross substitutes when $\delta < 0$

Example 4: Linear expenditure shares

- The price aggregator $A(\mathbf{p})$ is the weighted arithmetic mean of prices:

$$A(\mathbf{p}) = \sum_{i=1}^{n} \beta_i p_i$$

- The ideal price index is given by

$$P(\mathbf{p}) = c[A(\mathbf{p})]^{\delta} \left(\prod_{i=1}^{n} p_i^{\alpha_i} \right)^{1-\delta} \neq A(\mathbf{p})$$