Endogenous Ranking and Equilibrium Lorenz Curve Across (ex-ante) Identical Countries

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1. Introduction:

- Rich countries tend to have higher TFPs & *K/L* than the poor, typically interpreted as the ***causality*** from TFPs and/or *K/L* to *Y/L*, often under the maintained hypotheses
  - These countries offer independent observations
  - Cross-country variations would disappear without any exogenous variations.

- A complementary approach in ***trade*** (and ***economic geography***): even if countries are ex-ante identical, interaction through trade (and factor mobility) could lead to:
  - Ex-post heterogeneity of countries (***symmetry-breaking***) with joint dispersions in *Y/L*, TFPs, & *K/L* emerge as (only) stable patterns. ***Two-way causality***
  - An explanation for ***Great Divergence***, ***Growth Miracle***

- Most existing studies (Krugman etc.) show this insight in 2-country/2-tradeables. Absent analytical results, the message is unclear with many countries/many tradeables.
  - Does symmetry-breaking split the world into the rich and poor clusters (a polarization)? Or
  - keep splitting into finer clusters until they become more dispersed & fully ranked?
  - What determines the shape of the distribution generated by this mechanism?

- This paper offers an ***analytically solvable*** symmetry-breaking model of trade and inequality among many (ex-ante) identical countries to answer these questions.

- **Main Ingredients** of the model

  - A finite number (*J*) of (ex-ante) identical countries (or regions)
  - A unit interval [0,1] of tradeable consumption goods with Cobb-Douglas preferences (indices are normalized so that the expenditure share is uniform, WLOG) *à la* Dornbusch-Fischer-Samuelson
  - Endogenous productivity due to the variety of nontradeable differentiated intermediates, "local producer services," *à la* Dixit-Stiglitz
  - Tradeables produced with Cobb-Douglas tech. with the share of local producer services $\gamma(s)$ increasing (ordered so that the higher indexed are more dependent, WLOG)

- **Symmetry-Breaking: Two-way causality** between patterns of trade and productivity

  - More variety of local services gives a country CA in tradeables that depend more on such services.
  - Having CA in tradeables that depend more on the local services means a larger market for such services and hence more variety.

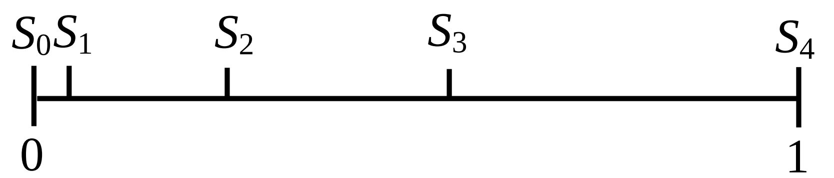
- **What makes the model tractable:** Countries are vastly ***outnumbered*** by tradeables

A Preview of the Main Results

- **Endogenous comparative advantage**: For a finite *J*, countries sort themselves into different tradeable goods in any stable equilibrium;

  - A unit interval [0,1] is partitioned into *J* subintervals.

Illustrated for *J* = 4

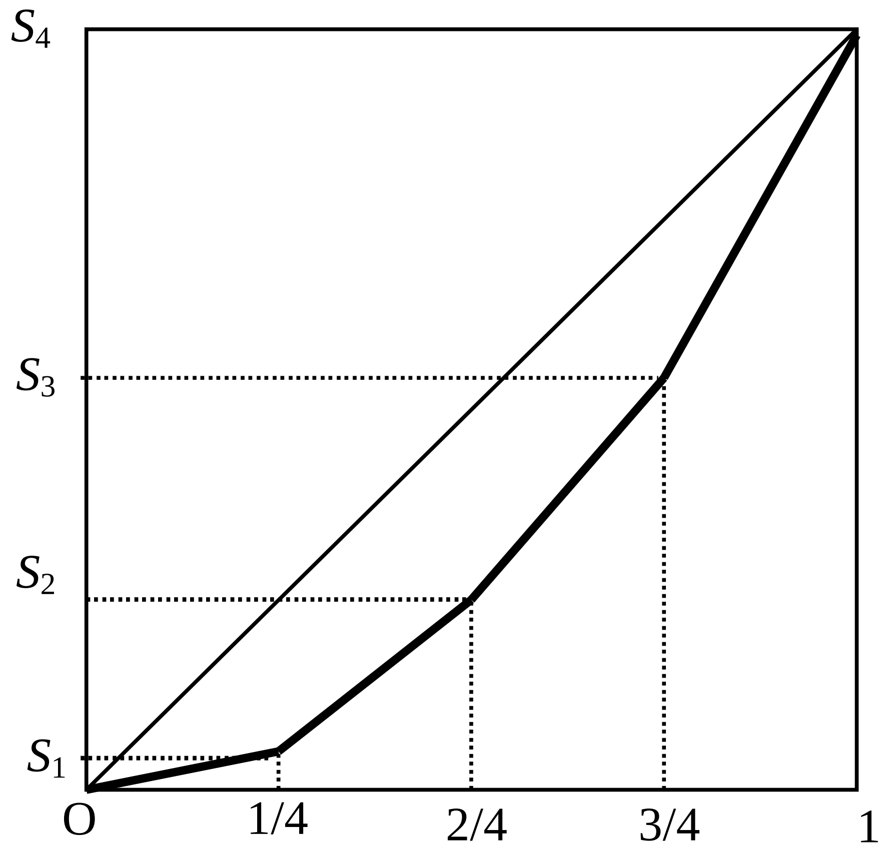


S_j: (Cumulative) share of the *j* poorest countries, characterized by **2nd order difference equation with the 2 terminal conditions**

NB: The subintervals are monotone increasing in length

  - **Strict ranking** of countries in *Y/L*, TFP, and *K/L*, which are (perfectly) correlated.

Equilibrium Lorenz curve, Φ^J: Illustrated for J = 4



- As $J \to \infty$, the limit Lorenz curve converges to the unique solution of the **2nd order differential equation with the 2 terminal conditions.** Furthermore, it is analytically solvable.

  - **Shape of Lorenz Curve** is determined by how the tradeables vary in their dependency on local producer services.

  - **Comparative Statics:** Many key parameters entering in log-submodular way, easy to show their changes cause a Lorenz-dominant shift.

- **Welfare effects of trade;** We can also answer questions like;

  - When is trade Pareto-improving?
  - If it is not Pareto-improving, "what fractions of countries would lose from trade?
  - The answers depend on the diversity of tradeables in their dependence of the services, measured by the Theil index (or entropy).

Organization of this slides (not the paper):

1. Introduction

2. Basic Model (Fixed Factor Supply; Without Nontradeable Consumption Goods)
   - Single-country (Autarky) equilibrium ($J = 1$)
   - Two-country equilibrium ($J = 2$)
   - Multi-country equilibrium ($2 < J < \infty$)
   - Limit case ($J \to \infty$); Power-law (truncated Pareto) examples, comparative statics
   - Welfare Effects of Trade

3. An Extension with Nontradeable Consumption Goods; Effects of Globalization
   - Multi-country equilibrium ($2 \leq J < \infty$)
   - Limit case ($J \to \infty$)

4. An Extension with Variable Factor Supply
   - Multi-country equilibrium ($2 \leq J < \infty$)
   - Limit case ($J \to \infty$)

5. Concluding Remarks

2. Basic Model: All Factors in Fixed Supply, All Consumer Goods Tradeable

J (inherently) identical countries in the World Economy

Representative Consumers:

- Endowed with V units of the (nontradeable) primary factor of production, which may be a composite of capital, labor, etc., as $V = F(K, L, \ldots)$.

- Cobb-Douglas preferences over **Tradeable Consumer Goods**, $s \in [0,1]$

$$\log U = \int_0^1 \log(X(s))dB(s) = \int_0^1 \log(X(s))ds$$

WLOG, we can index the goods by the cumulative expenditure share, B(s) = s.

Tradeable Consumer Goods Sectors $s \in [0,1]$**:** ***Competitive***

Cobb-Douglas unit cost function: $C(s) = \zeta(s)(\omega)^{1-\gamma(s)}(P_N)^{\gamma(s)}$

ω: the price of the primary factor of production (TFP in equilibrium).

P_N: the Dixit-Stiglitz price index of nontradeable producer services, defined by

$$P_N = \left\{\int_0^n [p(z)]^{-\frac{1}{\theta}} dz\right\}^{-\theta} \qquad (\theta = \frac{1}{\sigma - 1} > 0)$$

n: Equilibrium variety of producer services

θ: the degree of differentiation

$\gamma(s)$: the share of services in sector-s, increasing in $s \in [0,1]$

Nontradeable Producer Services Sector: *Monopolistically Competitive*

- Primary factor required to supply q units of each variety: $T(q) = f + mq$
- **Constant Mark-Up Pricing:** $p(z) = (1+\nu)\omega m$ $(0 < \nu \leq \theta)$
  - ➢ Unconstrained (Dixit-Stiglitz) monopoly pricing: $\nu = \theta$
  - ➢ Limit pricing: $\nu < \theta$
- **Free Entry-Zero Profit:** $\nu mq = f$

Unit Cost in Sector-s:

$$C(s) = \zeta(s)(\omega)^{1-\gamma(s)}\left\{\int_0^n [p(z)]^{-\frac{1}{\theta}} dz\right\}^{-\theta\gamma(s)} = \zeta(s)\{(1+\nu)m\}^{\gamma(s)}(n)^{-\theta\gamma(s)}\omega$$

- ➢ Decreasing in n; productivity gains from variety *à la* Ethier-Romer
- ➢ High-indexed sectors gain more from greater variety
- ➢ This effect is stronger for a larger θ.

In stable equilibrium, ω and n will end up being different across countries.

Single-country* (*J* = 1) *or Autarky Case: The economy produces all $s \in [0,1]$.

Let $\Gamma^A \equiv \int_0^1 \gamma(s)ds$.

- **Producer Services Market:** $npq = n(1+v)m\omega q = \Gamma^A Y$,
- **Primary Factor Market:** $\omega V = (1-\Gamma^A)Y + n\omega(f + mq)$,
- With **the Zero Profit** condition: $vmq = f$,

→

$$n^A \propto \Gamma^A;$$

Equilibrium variety depends on the market size for services, which are proportional to the average share of services over all consumer goods sector.

$$Y^A = \omega^A V = \omega^A F(K,L,\ldots).$$

ω = the price of the primary factor composite = TFP

Two-Country* (*J* = 2) *Case: Home & Foreign (*). **Suppose *n* < *n**.** Then,

- $\dfrac{C(s)}{C^*(s)} = \left(\dfrac{n}{n^*}\right)^{-\theta\gamma(s)}\left(\dfrac{\omega}{\omega^*}\right)$, increasing in *s*.

A country with a higher *n* has comparative advantage in higher-indexed sectors.

- H exports $s \in [0, S)$ & F exports $s \in (S, 1]$, where $\dfrac{C(S)}{C^*(S)} = \left(\dfrac{n}{n^*}\right)^{-\theta\gamma(S)}\left(\dfrac{\omega}{\omega^*}\right) = 1$

Each country must be the least cost producer for a positive measure of tradeables.

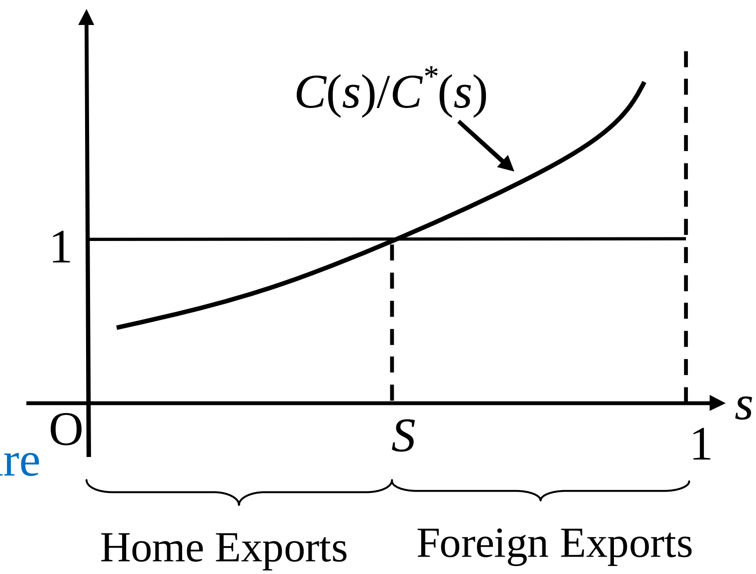
- $\dfrac{\omega}{\omega^*} = \left(\dfrac{n}{n^*}\right)^{\theta\gamma(S)} < 1$. productivity gains from variety
- $S(Y+Y^*) = Y = \omega V$ & $(1-S)(Y+Y^*) = Y^* = \omega^* V$

A country's share = the world's expenditure share of the consumer goods it produces.

- $n \propto \Gamma^-(S) \equiv \dfrac{1}{S}\displaystyle\int_0^S \gamma(s)ds < n^* \propto \Gamma^+(S) \equiv \dfrac{1}{1-S}\int_S^1 \gamma(s)ds$

In each country, variety is proportional to the average share of services among its (active) tradeable sectors.

$$\Rightarrow \frac{S}{1-S} = \frac{Y}{Y^*} = \frac{\omega}{\omega^*} = \left(\frac{\Gamma^-(S)}{\Gamma^+(S)}\right)^{\theta\gamma(S)} < 1.$$



A Symmetric Pair of Stable Asymmetric Equilibria

- Home produces $s \in [0, S]$ and Foreign produces $s \in [S, 1]$,

$$\frac{S}{1-S} = \frac{Y}{Y^*} = \frac{\omega}{\omega^*} = \left(\frac{\Gamma^-(S)}{\Gamma^+(S)}\right)^{\theta\gamma(S)} < 1;$$

- Foreign produces $s \in [0, S]$ and Home produces $s \in [S, 1]$,

$$\frac{1-S}{S} = \frac{Y}{Y^*} = \frac{\omega}{\omega^*} = \left(\frac{\Gamma^+(S)}{\Gamma^-(S)}\right)^{\theta\gamma(S)} > 1.$$

Instability of Symmetric Equilibrium: $n = n^*$ $(= n^A)$

Stable Equilibrium Patterns in the *J*-Country World:

Index the countries so $\{n_j\}_{j=1}^{J}$ is monotone increasing. Then,

- $\dfrac{C_j(s)}{C_{j+1}(s)} = \left(\dfrac{n_j}{n_{j+1}}\right)^{-\theta\gamma(s)} \left(\dfrac{\omega_j}{\omega_{j+1}}\right)$, is strictly increasing in *s:*

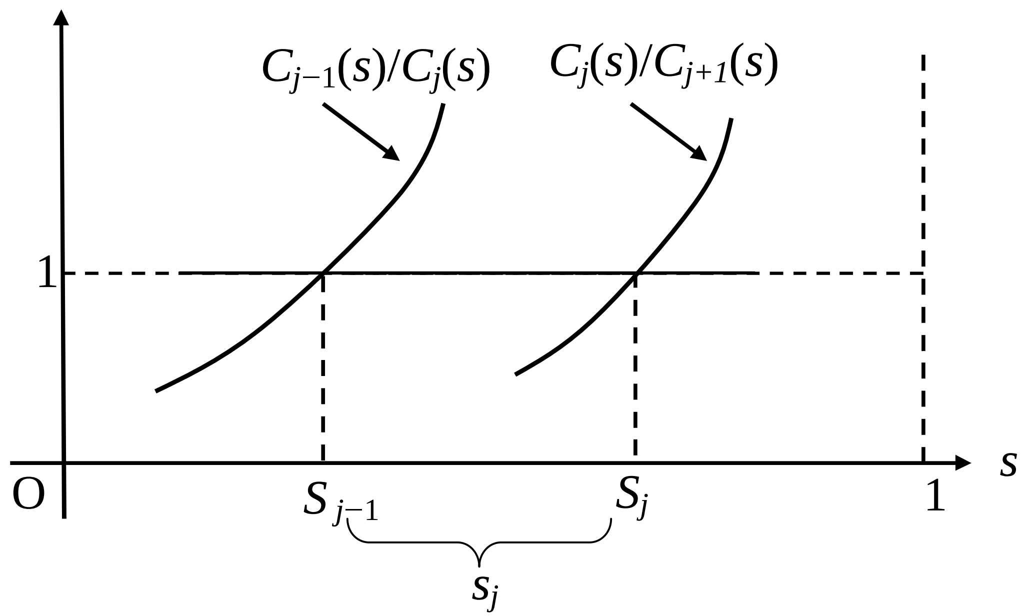
- The unit interval is partitioned into *J*-subintervals: the *j*-th exports $s \in (S_j, S_{j+1})$, where $\{S_j\}_{j=1}^{J}$ is given by $S_0 = 0$, $S_J = 1$ and $\dfrac{C_j(S_j)}{C_{j+1}(S_j)} = \left(\dfrac{n_j}{n_{j+1}}\right)^{-\theta\gamma(S_j)} \left(\dfrac{\omega_j}{\omega_{j+1}}\right) = 1.$

- $\{\omega_j\}_{j=1}^{J}$ is monotone increasing.

- $Y_j = \omega_j F(K, L, ...) = (S_j - S_{j-1})Y^W$

- $n_j \propto \Gamma_j \equiv \dfrac{1}{S_j - S_{j-1}} \displaystyle\int_{S_{j-1}}^{S_j} \gamma(s)ds,$

hence monotone increasing, as assumed.



This can be summarized as:

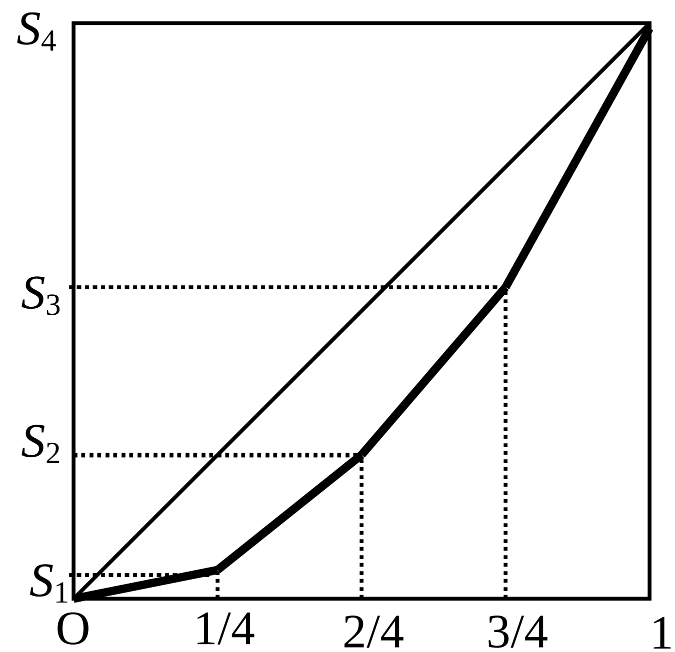
Proposition 1 (the *J*-country case):
$\{S_j\}_{j=0}^{J}$ solves the nonlinear 2nd-order difference equation with the 2 terminal conditions:

$$\frac{S_{j+1}-S_j}{S_j - S_{j-1}} = \left(\frac{\Gamma(S_j, S_{j+1})}{\Gamma(S_{j-1}, S_j)}\right)^{\theta\gamma(S_j)} > 1 \text{ with } S_0 = 0 \ \& \ S_J = 1,$$

where $$\Gamma(S_{j-1}, S_j) \equiv \frac{1}{S_j - S_{j-1}} \int_{S_{j-1}}^{S_j} \gamma(s)ds.$$

The Lorenz curve, $\Phi^J : [0,1] \to [0,1]$, is the piece-wise linear function, $\Phi^J(j/J) = S_j$. Clearly,

- Φ^J is strictly increasing & convex;
- $\Phi^J(0) = 0$ & $\Phi^J(1) = 1$.



But, it is not analytically solvable.

- Uniqueness?
- Comparative statics?
- Welfare evaluations?

These problems disappear by $J \to \infty$.

Calculating the limit Lorenz Curve: $\Phi^J \to \Phi$, as $J \to \infty$

$$\frac{S_{j+1}-S_j}{S_j - S_{j-1}} = \left(\frac{\Gamma(S_j, S_{j+1})}{\Gamma(S_{j-1}, S_j)}\right)^{\theta\gamma(S_j)} \quad \text{with } \Gamma(S_{j-1}, S_j) \equiv \frac{1}{S_j - S_{j-1}} \int_{S_{j-1}}^{S_j} \gamma(s)ds$$

By setting $x = j/J$ and $\Delta x = 1/J$,

$$S_{j+1} - S_j = \Phi(x+\Delta x) - \Phi(x) = \Phi'(x)\Delta x + \Phi''(x)\tfrac{|\Delta x|^2}{2} + o\left(|\Delta x|^2\right),$$

$$S_j - S_{j-1} = \Phi(x) - \Phi(x-\Delta x) = \Phi'(x)\Delta x - \Phi''(x)\tfrac{|\Delta x|^2}{2} + o\left(|\Delta x|^2\right),$$

from which

$$\text{LHS} = \frac{S_{j+1}-S_j}{S_j - S_{j-1}} = 1 + \frac{\Phi''(x)}{\Phi'(x)}\Delta x + o(|\Delta x|).$$

Likewise,

$$\Gamma(S_j, S_{j+1}) = \frac{\int_{\Phi(x)}^{\Phi(x+\Delta x)} \gamma(s)ds}{\Phi(x+\Delta x) - \Phi(x)} = \gamma(\Phi(x)) + \frac{1}{2}\gamma'(\Phi(x))\Phi'(x)\Delta x + o(|\Delta x|)$$

$$\Gamma(S_j, S_{j-1}) = \frac{\int_{\Phi(x-\Delta x)}^{\Phi(x)} \gamma(s)ds}{\Phi(x) - \Phi(x-\Delta x)} = \gamma(\Phi(x)) - \frac{1}{2}\gamma'(\Phi(x))\Phi'(x)\Delta x + o(|\Delta x|)$$

from which

$$\text{RHS} = \left(\frac{\Gamma(S_j, S_{j+1})}{\Gamma(S_{j-1}, S_j)}\right)^{\theta\gamma(S_j)} = \left(1 + \frac{\gamma'(\Phi(x))}{\gamma(\Phi(x))}\Phi'(x)\Delta x + o(|\Delta x|)\right)^{\theta\gamma(\Phi(x))}$$
$$= 1 + \theta\gamma'(\Phi(x))\Phi'(x)\Delta x + o(|\Delta x|)$$

Combining these yields

$$1 + \frac{\Phi''(x)}{\Phi'(x)}\Delta x + o(|\Delta x|) = 1 + \theta\gamma'(\Phi(x))\Phi'(x)\Delta x + o(|\Delta x|).$$

Hence, as $J \to \infty$, $\Delta x = 1/J \to 0$,

$$\frac{\Phi''(x)}{\Phi'(x)} = \theta\gamma'(\Phi(x))\Phi'(x).$$

By integrating once,

$$\log(\Phi'(x)) - \theta\gamma(\Phi(x)) = c_0 \qquad \Leftrightarrow \qquad \exp(-\theta\gamma(\Phi(x)))\Phi'(x) = e^{c_0}$$

By integrating once again,

$$\int_0^{\Phi(x)} e^{-\theta\gamma(s)} ds = c_1 + e^{c_0} x.$$

From $\Phi(0) = 0$ & $\Phi(1) = 1$, $\Phi : [0,1] \to [0,1]$, is determined ***uniquely*** by

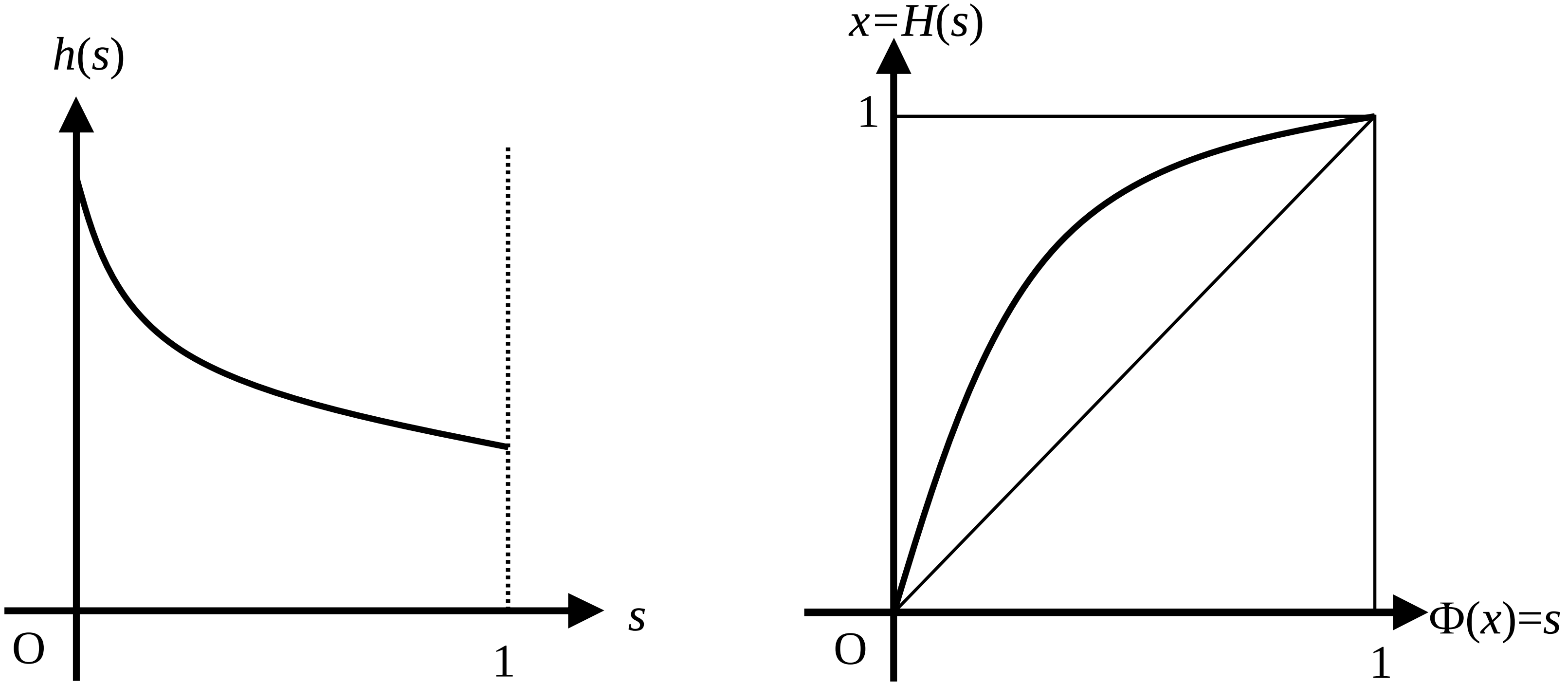
$$\int_0^{\Phi(x)} e^{-\theta\gamma(s)} ds = \left[\int_0^1 e^{-\theta\gamma(u)} du\right] x.$$

Proposition 2 ($J \rightarrow \infty$) The limit equilibrium Lorenz curve, $\lim_{J\to\infty} \Phi^J = \Phi$, solves:

$$\frac{\Phi''(x)}{\Phi'(x)} = \theta\gamma'(\Phi(x))\Phi'(x) \text{ with } \Phi(0)=0 \ \& \ \Phi(1)=1$$

Its unique solution is

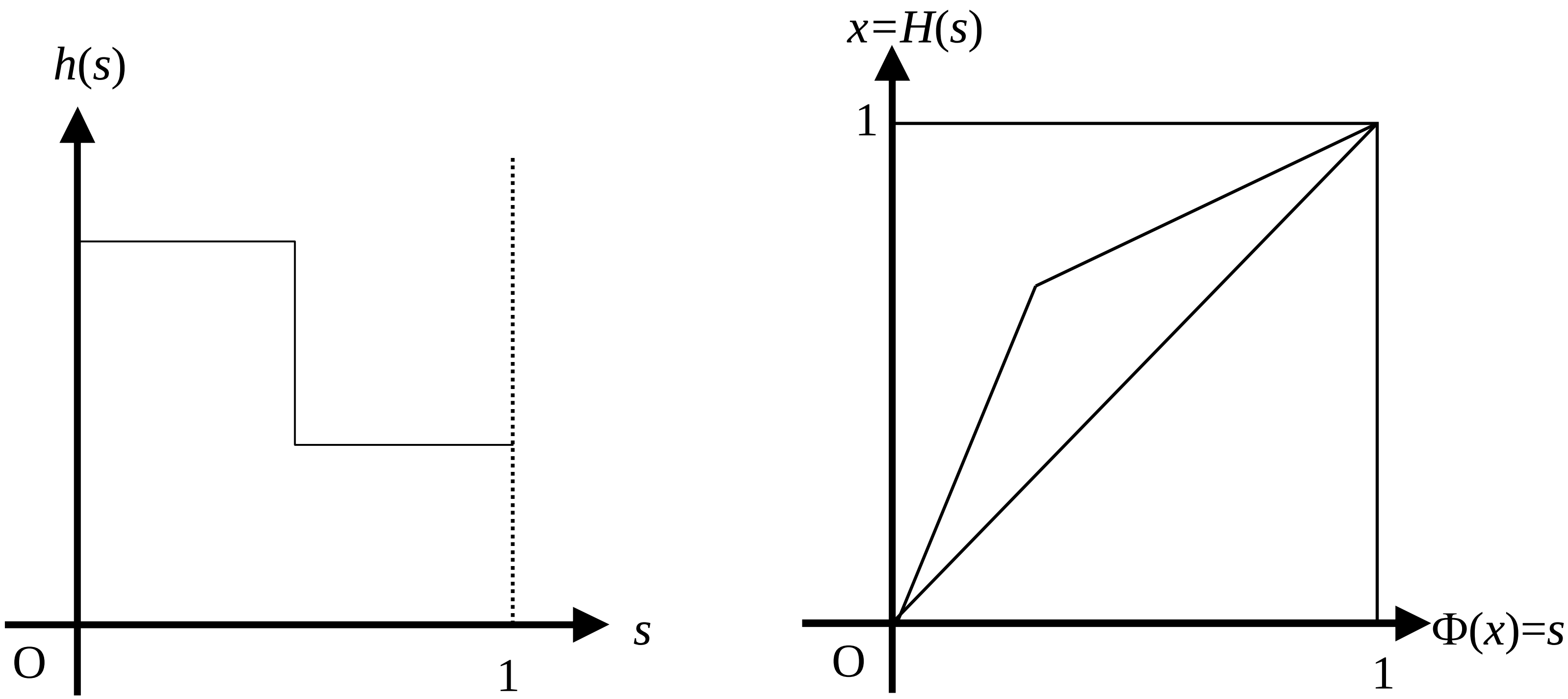
$$x = H(\Phi(x)) \equiv \int_0^{\Phi(x)} h(s)ds, \qquad \text{where } h(s) \equiv \frac{e^{-\theta\gamma(s)}}{\int_0^1 e^{-\theta\gamma(u)}du}.$$



NB: Lorenz Curve also maps a set of countries into a set of goods they produced

Question: When does this mechanism lead to a polarization?

Answer: When γ(●) can be *approximated* by a two-step function. That is, when there are *effectively* only two tradeables**.**



NB: This is different from assuming that there are only two tradeable goods. The uniqueness is lost when you do that; See Matsuyama (1996).

Power-Law (Truncated Pareto) Examples (with World GDP normalized on one):

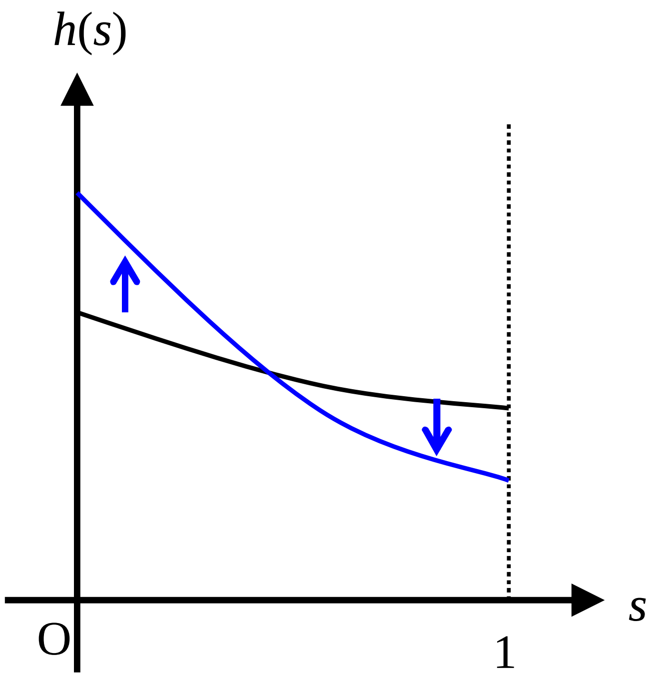
	Example 1: $\gamma(s)=s$	Example 2: $\gamma(s)=\log\left[1+(e^{\theta}-1)s\right]^{\frac{1}{\theta}}$	Example 3: $\gamma(s)=\log\left[1+(e^{\lambda}-1)s\right]^{\frac{1}{\lambda}}$ $(\lambda\neq 0;\neq\theta)$
Inverse Lorenz Curve: $x=H(s)$	$\dfrac{1-e^{-\theta s}}{1-e^{-\theta}}$	$\log\left[1+(e^{\theta}-1)s\right]^{\frac{1}{\theta}}$	$\dfrac{\left[1+(e^{\lambda}-1)s\right]^{1-\frac{\theta}{\lambda}}-1}{e^{\lambda-\theta}-1}$
Lorenz Curve: $s=\Phi(x)$	$\log\left[1-(1-e^{-\theta})x\right]^{-\frac{1}{\theta}}$	$\dfrac{e^{\theta x}-1}{e^{\theta}-1}$	$\dfrac{\left[1+(e^{\lambda-\theta}-1)x\right]^{\frac{\lambda}{\lambda-\theta}}-1}{e^{\lambda}-1}$
Cdf: $x=\Psi(y)$ $=(\Phi')^{-1}(y)$	$\dfrac{1}{1-e^{-\theta}}-\dfrac{1}{\theta y}$	$\dfrac{1}{\theta}\log\left(\dfrac{e^{\theta}-1}{\theta}y\right)$	$\dfrac{\left(\dfrac{y}{y_{Min}}\right)^{\frac{\lambda}{\theta}-1}-1}{e^{\lambda-\theta}-1}=1-\dfrac{1-\left(\dfrac{y}{y_{Max}}\right)^{\frac{\lambda}{\theta}-1}}{1-e^{\theta-\lambda}}$
Pdf: $\psi(y)=\Psi'(y)$	$\dfrac{1}{\theta y^2}$	$\dfrac{1}{\theta y}$	$\left[\dfrac{(\lambda/\theta)-1}{(y_{Max})^{(\lambda/\theta)-1}-(y_{Min})^{(\lambda/\theta)-1}}\right](y)^{\frac{\lambda}{\theta}-2}$
Support: $[y_{Min},y_{Max}]$	$\dfrac{1-e^{-\theta}}{\theta}\leq y\leq\dfrac{e^{\theta}-1}{\theta}$	$\dfrac{\theta}{e^{\theta}-1}\leq y\leq\dfrac{\theta e^{\theta}}{e^{\theta}-1}$	$\left(\dfrac{\lambda}{e^{\lambda}-1}\right)\left(\dfrac{e^{\lambda-\theta}-1}{\lambda-\theta}\right)\leq y$ $\leq\left(\dfrac{\lambda}{e^{\lambda}-1}\right)\left(\dfrac{e^{\lambda-\theta}-1}{\lambda-\theta}\right)e^{\theta}$

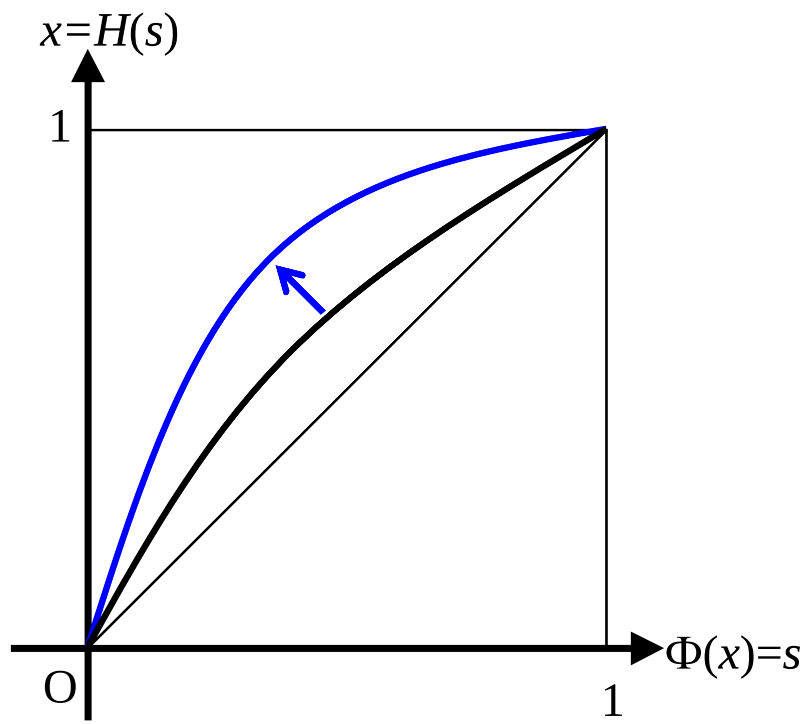
A lower λ (more concentrated use of services in narrower sectors) makes the pdf drop faster.

Log-submodularity and Effect of a higher θ:

Since $h(s) = \hat{h}(s) / \left[\int_0^1 \hat{h}(u)du\right]$, with $\hat{h}(s) \equiv e^{-\theta\gamma(s)}$ being *log-submodular* in θ and *s*, a higher θ rotates $h(s)$ "clockwise."

→ Lorenz curve "bends" more (a Lorenz-dominant shift), hence a greater inequality.





Welfare Effects of Trade

Proposition 3 (the *J*-country case): The welfare of the k-th poorest country is

$$\log\left(\frac{U_k}{U^A}\right)=\sum_{j=1}^{J}\log\left(\frac{\omega_k}{\omega_j}\right)\left(S_j-S_{j-1}\right)+\theta\sum_{j=1}^{J}\Gamma_j\log\left(\frac{\Gamma_j}{\Gamma^A}\right)\left(S_j-S_{j-1}\right).$$

- 1st term: effects on the country's relative productivity, negative for some countries.
- 2nd term; gains from trade (conditional on productivity differences), positive for all.

Proposition 4 (Limit case, *J* → ∞): The welfare of the country at 100x*% is given by

$$\frac{\log\left(U(x^*)/U^A\right)}{\theta}=\gamma(s^*)-\Gamma^A+\int_0^1\gamma(s)\log\left(\frac{\gamma(s)}{\Gamma^A}\right)ds,$$

where $s^*=\Phi(x^*)$ or $x^*=\Phi^{-1}(s^*)$.

- 1st term; Relative productivity effect, negative for some countries.
- 2nd term; gains from trade, conditional on productivity differences, positive for all.

Corollary 1: All countries gain from trade iff $1-\frac{\gamma(0)}{\Gamma^A}\le\int_0^1\left(\frac{\gamma(s)}{\Gamma^A}\right)\log\left(\frac{\gamma(s)}{\Gamma^A}\right)ds$.

$\int_0^1\left(\frac{\gamma(s)}{\Gamma^A}\right)\log\left(\frac{\gamma(s)}{\Gamma^A}\right)ds$: diversity (**Theil index/entropy**) of the tradeables in γ.

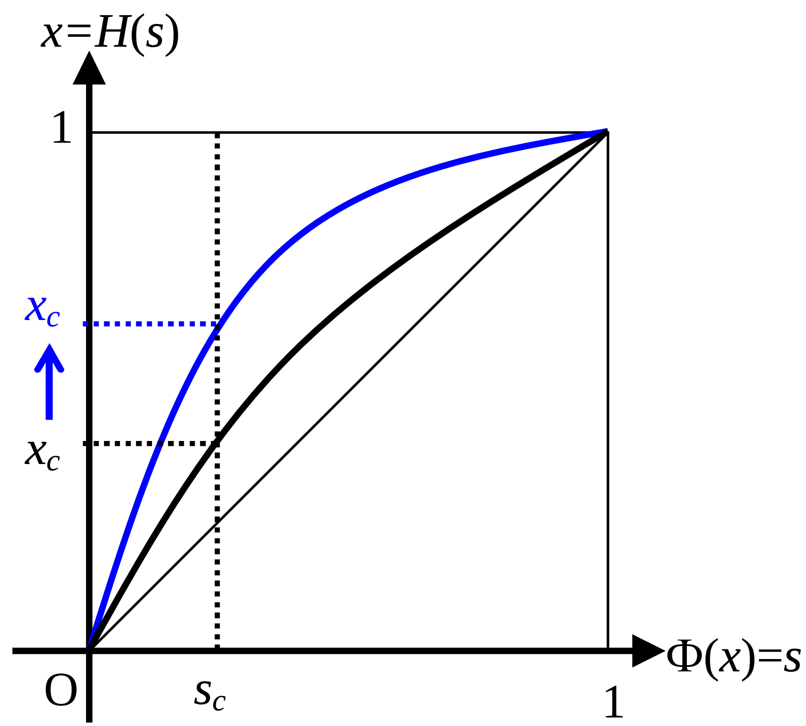
Corollary 2: Suppose $\frac{\gamma(0)}{\Gamma^A} < 1 - \int_0^1 \left(\frac{\gamma(s)}{\Gamma^A}\right) \log\left(\frac{\gamma(s)}{\Gamma^A}\right) ds$. Then, for $s_c > 0$ defined by

$$\gamma(s_c) \equiv \Gamma^A \left[1 - \int_0^1 \left(\frac{\gamma(s)}{\Gamma^A}\right) \log\left(\frac{\gamma(s)}{\Gamma^A}\right) ds\right],$$

a): All countries producing $s \in [0, s_c)$ lose from trade.

b): The fraction of the countries that lose, $x_c = H(s_c;\theta)$, is increasing in θ with $\lim_{\theta\to 0} x_c = s_c$ and $\lim_{\theta\to\infty} x_c = 1$.

Corollary 2: A Graphic Illustration



3. Two Extensions:

3.1 Nontradeable Consumption Goods:

$$\log U = \tau\int_0^1 \log(X_T(s))ds + (1-\tau)\int_0^1 \log(X_N(s))ds$$

τ; the fraction of the consumption goods that are tradeable.

A higher *τ* causes a Lorenz dominant shift.
Globalization through Goods Trade magnifies inequality!

3.2 *Variable Factor Supply (through Factor Mobility or Factor Accumulation):*

$V_j = F(K_j, L)$ with $\omega_j F_K(K_j, L) = \rho$
Correlations between *K/L* and TFPs and per capita income

For $V=F(K, L) = AK^{\alpha}L^{1-\alpha}$ with $0 < \alpha < 1/(1+\theta)$, **a higher *α* → a Lorenz dominant shift.**

Globalization through Factor Mobility or Skill-Biased Technological Change magnifies inequality!

In both extensions, the same techniques ($J \to \infty$ to solve the Lorenz curve analytically & log-submodularity to prove the Lorenz-dominant shifts) work.

In more detail;

3.1. Nontradeable Consumption Goods: Effects of Globalization

$$\log U = \tau \int_0^1 \log(X_T(s))ds + (1-\tau)\int_0^1 \log(X_N(s))ds$$

τ; the fraction of the consumption goods that are tradeable.

Assume the same distribution of γ among the tradeables and the nontradeables. Then,

Proposition 5 (Equilibrium Lorenz curve: the *J*-country case)::
Let S_j be the cumulative share of the J poorest countries. Then, $\{S_j\}_{j=0}^{J}$ solves:

$$\frac{Y_{j+1}}{Y_j} = \frac{\omega_{j+1}}{\omega_j} = \frac{S_{j+1} - S_j}{S_j - S_{j-1}} = \left(\frac{\tau\Gamma(S_j, S_{j+1}) + (1-\tau)\Gamma^A}{\tau\Gamma(S_{j-1}, S_j) + (1-\tau)\Gamma^A} \right)^{\theta\gamma(S_j)} > 1 \text{ with } S_0 = 0 \ \& \ S_J = 1,$$

where $\Gamma(S_{j-1}, S_j) \equiv \dfrac{1}{S_j - S_{j-1}} \displaystyle\int_{S_{j-1}}^{S_j} \gamma(s)ds$.

Again, following the same steps,

Proposition 6 (Equilibrium Lorenz Curve: Limit Case, $J \to \infty$):
The limit equilibrium Lorenz curve, $\lim_{J\to\infty} \Phi^J = \Phi$, solves:

$$\frac{\Phi''(x)}{\Phi'(x)} = \frac{\theta\gamma'(\Phi(x))\Phi'(x)}{1+\Gamma^A / g\gamma\left(\Phi(x)\right)} \text{ with } \Phi(0)=0 \ \& \ \Phi(1)=1$$

whose unique solution is:

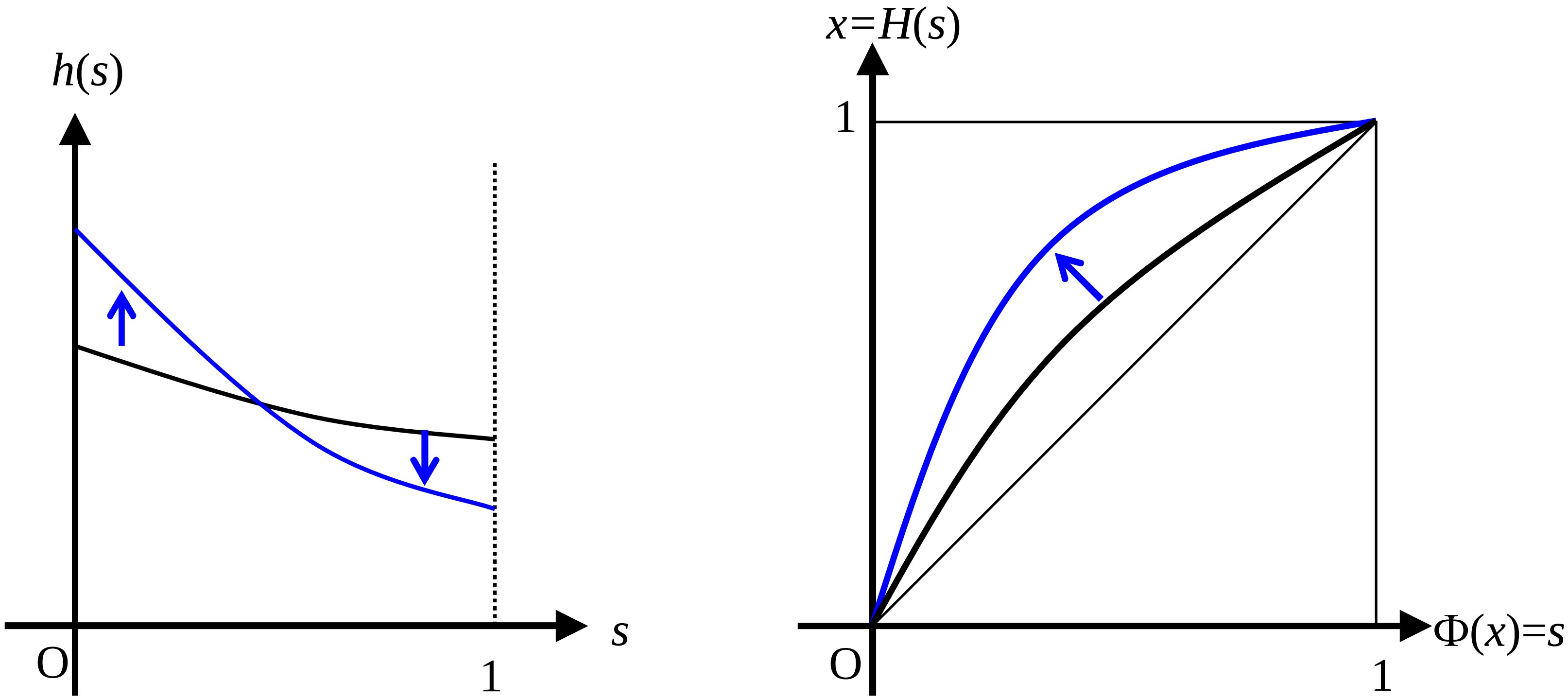
$$x = H\left(\Phi(x); g\right) \equiv \int_0^{\Phi(x)} h(s;g)ds, \text{ where } h(s;g) \equiv \frac{\left(1+g\gamma(s)/\Gamma^A\right)^{\theta\Gamma^A/g} e^{-\theta\gamma(s)}}{\int_0^1 \left(1+g\gamma(u)/\Gamma^A\right)^{\theta\Gamma^A/g} e^{-\theta\gamma(u)}du},$$

where $g \equiv \tau/(1-\tau)$.

Notes:

- $\lim_{\tau\to 1} h(s;g) = \lim_{g\to\infty} h(s;g) = h(s) \equiv \dfrac{e^{-\theta\gamma(s)}}{\int_0^1 e^{-\theta\gamma(u)}du}$; $\quad \lim_{\tau\to 0} h(s;g) = \lim_{g\to 0} h(s;g) = 1$.
- $h(s;g)$ is positive, and strictly decreasing in s for $g > 0$.
  - → $H(\bullet; g)$ is increasing, concave, with $H(0;g)=0$ & $H(1;g)=1$;
  - → $\Phi(x) = H^{-1}\left(x; g\right)$ is increasing, convex, with $\Phi(0)=0$ & $\Phi(1)=1$.

Log-submodularity and Effect of globalization (a higher τ or g) or a higher θ:
The graph of $h(s)$ rotates "clockwise."
→ the Lorenz curve "bends" more, hence a greater inequality.



Proof: $h(s;g) = \hat{h}(s;g) \Big/ \left[\int_0^1 \hat{h}(u;g)du\right]$, where $\hat{h}(s;g) \equiv \left(1 + g\gamma(s)/\Gamma^A\right)^{\theta\Gamma^A/g} e^{-\theta\gamma(s)}$ is *log-submodular* in g & s; (also in θ & s).

3.2 Variable Factor Supply:

$V_j = F(K_j, L)$ with $\omega_j F_K(K_j, L) = \rho$

Two Justifications:

- **Factor Mobility:** In a static setting, the rate of return for mobile factors is equalized as they move across borders to seek the highest return.

(If "countries" are interpreted as "metropolitan areas," K may include not only capital but also labor, with L representing the immobile "land.")

- **Factor Accumulation:** In a dynamic setting, some factors can be accumulated as the representative agent in each country maximizes

$$\int_0^\infty u(C_t)e^{-\rho t}dt \qquad \text{s.t.} \quad Y_t = \left[\int_0^1 \log(X_t(s))ds\right] = C_t + \dot{K}_t$$

Then, the rate of return is equalized in steady state. (In this case, K may include not only physical capital but also human capital.)

Condition for Patterns of Trade:

$$\left(\frac{n_j}{n_{j+1}}\right)^{\theta\gamma(S_j)} = \frac{\omega_j}{\omega_{j+1}} = \frac{F_K(K_{j+1},L)}{F_K(K_j,L)} < 1 \Leftrightarrow \frac{K_{j+1}}{K_j} > 1 \Leftrightarrow \frac{V_{j+1}}{V_j} > 1.$$

For the *j*-th country which produces $s \in (S_{j-1}, S_j)$,

$$n_j = \Gamma_j\left(\frac{\nu V_j}{(1+\nu)f}\right) = \Gamma_j\left(\frac{\nu F(K_j,L)}{(1+\nu)f}\right); \qquad Y_j = \omega_j V_j = \omega_j F(K_j,L) = (S_j - S_{j-1})Y^W.$$

Hence,

$$\frac{F_K(K_j,L)}{F_K(K_{j+1},L)} = \frac{\omega_{j+1}}{\omega_j} = \left(\frac{\Gamma_{j+1}}{\Gamma_j}\frac{F(K_{j+1},L)}{F(K_j,L)}\right)^{\theta\gamma(S_j)} > 1; \qquad \frac{S_{j+1}-S_j}{S_j - S_{j-1}} = \frac{\omega_{j+1}F(K_{j+1},L)}{\omega_j F(K_j,L)}$$

For $V = F(K, L) = AK^{\alpha}L^{1-\alpha}$ with $0 < \alpha < 1-1/\sigma = 1/(1+\theta)$,

$$\frac{Y_{j+1}}{Y_j} = \frac{\omega_{j+1}V_{j+1}}{\omega_j V_j} = \frac{K_{j+1}}{K_j} = \left(\frac{\omega_{j+1}}{\omega_j}\right)^{\frac{1}{1-\alpha}} = \left(\frac{V_{j+1}}{V_j}\right)^{\frac{1}{\alpha}} = \frac{S_{j+1}-S_j}{S_j - S_{j-1}} > 1$$

from which

Proposition 7 (Equilibrium Lorenz curve: the *J*-country case):
Let S_j be the cumulative share of the J poorest countries. Then, $\{S_j\}_{j=0}^{J}$ solves:

$$\frac{Y_{j+1}}{Y_j}=\frac{K_{j+1}}{K_j}=\left(\frac{\omega_{j+1}}{\omega_j}\right)^{\frac{1}{1-\alpha}}=\frac{S_{j+1}-S_j}{S_j-S_{j-1}}=\left(\frac{\Gamma(S_j,S_{j+1})}{\Gamma(S_{j-1},S_j)}\right)^{\frac{\theta\gamma(S_j)}{1-\alpha-\alpha\theta\gamma(S_j)}}>1 \text{ with } S_0=0 \text{ \& } S_J=1,$$

where $\Gamma(S_{j-1},S_j)\equiv\frac{1}{S_j-S_{j-1}}\int_{S_{j-1}}^{S_j}\gamma(s)ds$.

Following the same step as before:

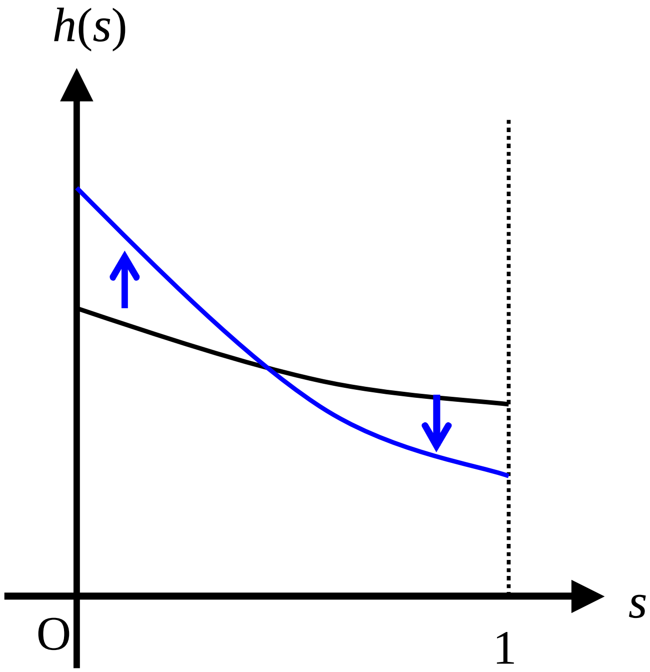
Proposition 8 (Equilibrium Lorenz Curve, Limit Case)
The limit equilibrium Lorenz curve, $\lim_{J\to\infty}\Phi^J=\Phi$, solves:

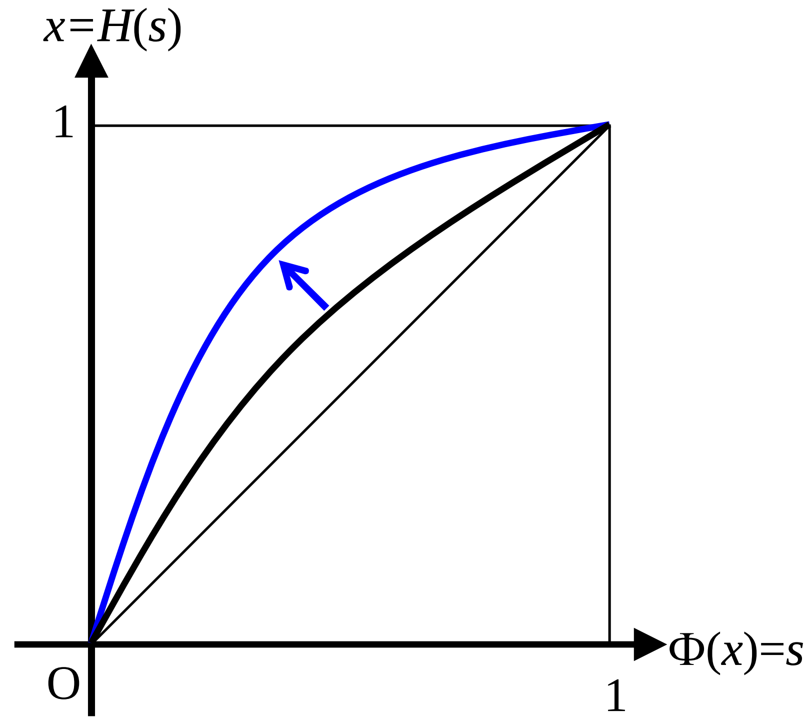
$$\frac{\Phi''(x)}{\Phi'(x)}=\frac{\theta\gamma'(\Phi(x))\Phi'(x)}{1-\alpha-\alpha\theta\gamma(\Phi(x))} \text{ with } \Phi(0)=0 \text{ \& } \Phi(1)=1$$

whose unique solution is:

$$x=H(\Phi(x);\alpha)\equiv\int_0^{\Phi(x)}h(s;\alpha)ds, \qquad \text{where } h(s;\alpha)\equiv\frac{\left(1-\frac{\alpha\theta}{1-\alpha}\gamma(s)\right)^{1/\alpha}}{\int_0^1\left(1-\frac{\alpha\theta}{1-\alpha}\gamma(u)\right)^{1/\alpha}du}.$$

Log-Submodularity and Effect of a higher *α* or a higher *θ*: The graph of $h(s)$ rotates "clockwise." → the Lorenz curve "bends" more, hence a greater inequality.





Proof: $h(s;\alpha)=\dfrac{\hat{h}(s;\alpha)}{\int_0^1 \hat{h}(u;\alpha)du}$, where $\hat{h}(s;\alpha)\equiv\left(1-\dfrac{\alpha\theta}{1-\alpha}\gamma(s)\right)^{1/\alpha}$ is *log-submodular* in α & s (and in θ & s).

Some Concluding Remarks:

- Symmetry-breaking due to two-way causality; Even without ex-ante heterogeneity, cross-country dispersion and correlations in per capita income, TFPs, and *K/L* ratios emerge as stable equilibrium patterns due to interaction through trade.

- Some countries become richer (poorer) than others because they trade with poorer (richer) countries. They are *not* independent observations.

- This type of analysis does not say that ex-ante heterogeneity is unimportant. Instead, it says that even small ex-ante heterogeneity could be magnified to create huge ex-post heterogeneity, a possible explanation of Great Divergence and Growth Miracle

- This paper demonstrates that this type of analysis does not have to be intractable nor lacking in prediction. Equilibrium distribution is *unique, analytically solvable,* varying with parameters in intuitive ways.

- With a finite countries and a continuum of sectors, this model is more compatible with existing quantitative models of trade (Eaton-Kortum, Alvarez-Lucas, etc.)

- A model with many countries can be more tractable than a model with a few countries.