

Endogenous Ranking and Equilibrium Lorenz Curve Across (ex-ante) Identical Countries

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1. *Introduction:*

- In cross-section of countries, the rich tend to have higher TFPs and K/L ratios than the poor.
- These findings are typically interpreted as the *causality* from TFPs and/or K/L ratios to per capita income, under the maintained hypotheses
 - These countries offer independent observations
 - Variations across countries would disappear without any exogenous variations.
- Here, I offer a **trade economist's perspective** by showing how interaction through trade among countries might lead to:
 - Endogenous country ranking
 - Joint distributions and correlations in per capita income, TFPs, & K/L ratios emerging as stable equilibrium patterns.
- **Key Features** of the model:
 - A finite number (J) of (ex-ante) identical countries
 - A unit interval $[0,1]$ of tradeable goods *à la* Dornbusch-Fischer-Samuelson
 - Endogenous productivity due to the variety of nontradeable differentiated intermediates, “producer services,” *à la* Dixit-Stiglitz
 - Tradeable goods sectors differ in the dependence on local producer services.

- **Key Mechanism: Two-way causality** between patterns of trade and productivity;
 - More variety of local services gives a country CA in tradeable sectors that depend more on such services.
 - Having CA in tradeable sectors that depend more on the local services means a larger market for such services and hence more variety.
- In any *stable* equilibrium,
 - **Endogenous comparative advantage:** Different countries sort themselves into different sets of tradeable goods; A unit interval $[0,1]$ is partitioned into J subintervals.
 - **Strict ranking** of countries in income, TFP, and K/L ratios.
 - **Equilibrium distribution:** As $J \rightarrow \infty$, unique and analytically solvable, and all key parameters entering in log-submodular way, easy to show their changes cause a Lorenz-dominant shift of the cross-country distribution.
- Welfare effects of trade
 - When is trade Pareto-improving?
 - When trade is not Pareto-improving, what fractions of countries would lose from trade?

A Preview of Some Main Results:**Proposition 1 (Equilibrium Lorenz curve: the J -country case)**

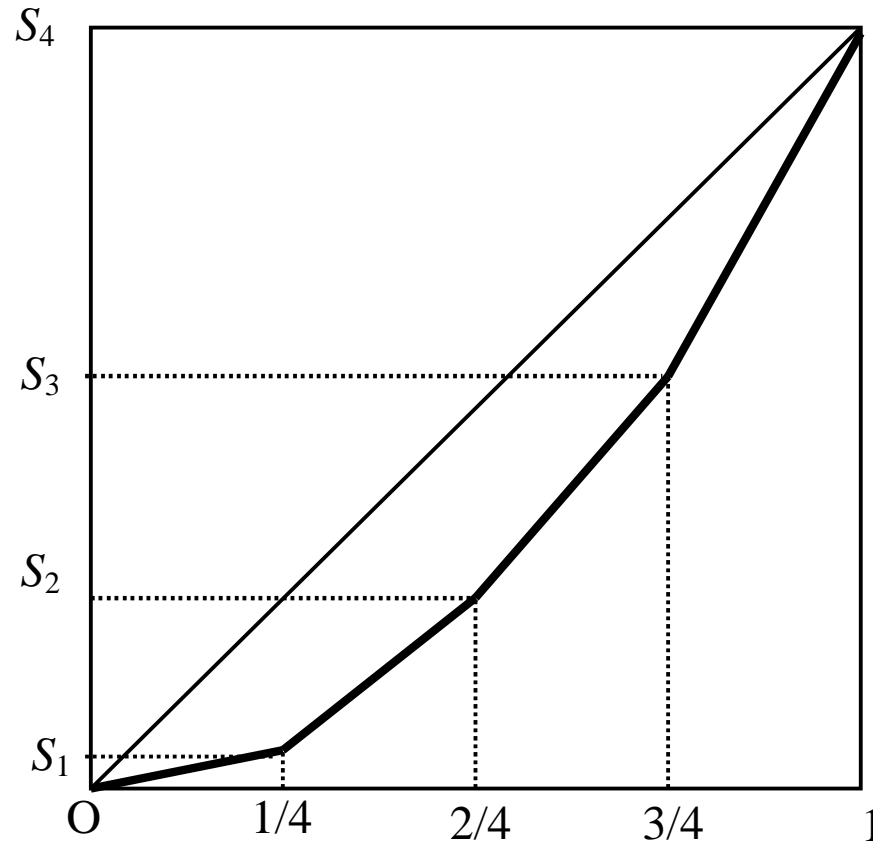
Let S_j be the share of the j poorest countries in the world income. Then, $\{S_j\}_{j=0}^J$ solves

$$\frac{S_{j+1} - S_j}{S_j - S_{j-1}} = \left(\frac{\Gamma(S_j, S_{j+1})}{\Gamma(S_{j-1}, S_j)} \right)^{\theta \gamma(S_j)} > 1 \text{ with } S_0 = 0 \text{ \& } S_J = 1,$$

where $\gamma(s)$ is the share of local differentiated services in the tradeable good- s (strictly increasing in s); $\theta > 0$ is the degree of differentiation, and

$$\Gamma(S_{j-1}, S_j) \equiv \frac{1}{S_j - S_{j-1}} \int_{S_{j-1}}^{S_j} \gamma(s) ds.$$

Equilibrium Lorenz curve, Φ^J : A Graphic Illustration for $J = 4$



Proposition 2 (Equilibrium Lorenz curve: Limit Case, $J \rightarrow \infty$)

The limit equilibrium Lorenz curve, $\lim_{J \rightarrow \infty} \Phi^J = \Phi$, solves:

$$\frac{\Phi''(x)}{\Phi'(x)} = \theta\gamma'(\Phi(x))\Phi'(x) \text{ with } \Phi(0) = 0 \text{ \& } \Phi(1) = 1$$

whose unique solution is given by:

$$x = H(\Phi(x)) \equiv \int_0^{\Phi(x)} h(s) ds, \quad \text{where } h(s) \equiv \frac{e^{-\theta\gamma(s)}}{\int_0^1 e^{-\theta\gamma(u)} du}.$$

We will obtain some analytical results for the limit case.

- Power-law examples
- Lorenz-dominant shifts caused by parameter changes
- Welfare effects of trade

We will also consider two extensions to allow for

- *Nontradeable consumption goods* to study the effects of globalization in trade
- *Variable factor supply* to study the effects of factor mobility and accumulation

2. *Basic Model: All Factors in Fixed Supply, All Consumer Goods Tradeable*

J (inherently) identical countries in the World Economy

Representative Consumers:

- Endowed with V units of the (nontradeable) primary factor of production, which may be a composite of capital, labor, etc., as $V = F(K, L, \dots)$.
- Cobb-Douglas preferences over **Tradeable Consumer Goods**, $s \in [0,1]$

$$\log U = \int_0^1 \log(X(s)) dB(s) = \int_0^1 \log(X(s)) ds$$

Note: The goods indexed by the cumulative expenditure share, WLOG.

Tradeable Consumer Goods Sectors $s \in [0,1]$: *Competitive*

Cobb-Douglas unit cost function: $C(s) = \zeta(s)(\omega)^{1-\gamma(s)} (P_N)^{\gamma(s)}$

ω : the price of the primary factor of production (TFP in equilibrium).

P_N : the Dixit-Stiglitz price index of nontradeable producer services, defined by

$$P_N = \left\{ \int_0^n [p(z)]^{-\frac{1}{\theta}} dz \right\}^{-\theta} \quad \left(\theta = \frac{1}{\sigma - 1} > 0 \right)$$

n : Equilibrium variety of producer services

θ : the degree of differentiation

$\gamma(s)$: the share of services in sector- s , increasing in $s \in [0,1]$

Nontradeable Producer Services Sector: *Monopolistically Competitive*

- Primary factor required to supply q units of each variety: $T(q) = f + mq$
- **Constant Mark-Up Pricing:** $p(z) = (1+\nu)\omega m$ $(0 < \nu \leq \theta)$
 - Unconstrained (Dixit-Stiglitz) monopoly pricing: $\nu = \theta$
 - **Limit pricing:** $\nu < \theta$
- **Free Entry-Zero Profit:** $\nu m q = f$

Unit Cost in Sector- s :

$$C(s) = \zeta(s)(\omega)^{1-\gamma(s)} \left\{ \int_0^n [p(z)]^{-\frac{1}{\theta}} dz \right\}^{-\theta\gamma(s)} = \zeta(s) \{(1+\nu)m\}^{\gamma(s)} (n)^{-\theta\gamma(s)} \omega$$

- Decreasing in n ; productivity gains from variety *à la* Ethier-Romer
- High-indexed sectors gain more from greater variety
- This effect is stronger for a larger θ .

In stable equilibrium, ω and n will end up being different across countries.

Single-country ($J = 1$) or Autarky Case: The economy produces all $s \in [0,1]$.

$$\text{Let } \Gamma^A \equiv \int_0^1 \gamma(s) ds .$$

- **Producer Services Market:** $npq = n(1+v)m\omega q = \Gamma^A Y,$
- **Primary Factor Market:** $\omega V = (1 - \Gamma^A)Y + n\omega(f + mq),$
- **With the Zero Profit condition:** $vmq = f,$

→

$$n^A \propto \Gamma^A;$$

Equilibrium variety depends on the market size for services, proportional to the average share of services across all consumer goods sector.

$$Y^A = \omega^A V = \omega^A F(K, L, \dots).$$

ω = the price of the primary factor composite = TFP

Two-Country ($J = 2$) Case: Home & Foreign (*). Suppose $n < n^*$. Then,

- $\frac{C(s)}{C^*(s)} = \left(\frac{n}{n^*}\right)^{-\theta\gamma(s)} \left(\frac{\omega}{\omega^*}\right)$, increasing in s .

- H exports $s \in [0, S)$ & F exports $s \in (S, 1]$, where $\frac{C(S)}{C^*(S)} = \left(\frac{n}{n^*}\right)^{-\theta\gamma(S)} \left(\frac{\omega}{\omega^*}\right) = 1$

A country with a higher n has comparative advantage in higher-indexed sectors.

- $\frac{\omega}{\omega^*} = \left(\frac{n}{n^*}\right)^{\theta\gamma(S)} < 1$. productivity gains from variety

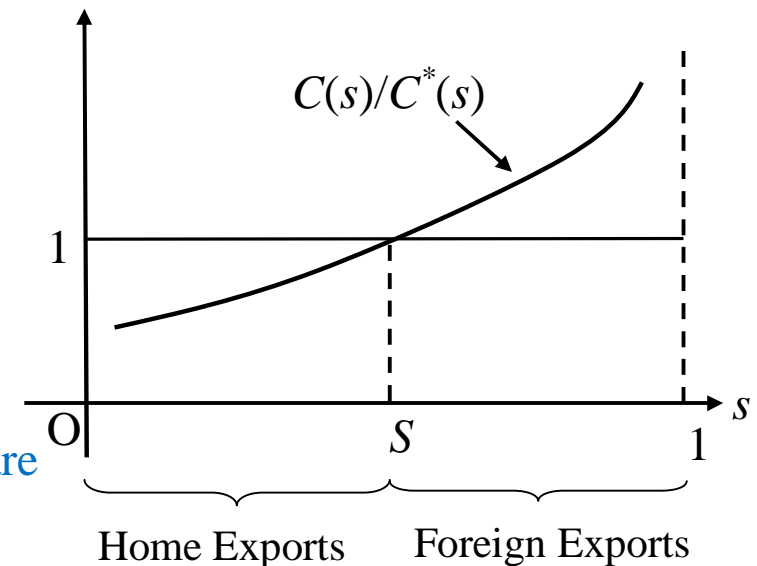
- $S(Y + Y^*) = Y = \omega V$ & $(1 - S)(Y + Y^*) = Y^* = \omega^* V$

A country's share = the world's expenditure share of the consumer goods it produces.

- $n \propto \Gamma^-(S) \equiv \frac{1}{S} \int_0^S \gamma(s) ds < n^* \propto \Gamma^+(S) \equiv \frac{1}{1 - S} \int_S^1 \gamma(s) ds$

In each country, variety is proportional to the average share of services among its (active) tradeable sectors.

$$\Rightarrow \frac{S}{1 - S} = \frac{Y}{Y^*} = \frac{\omega}{\omega^*} = \left(\frac{\Gamma^-(S)}{\Gamma^+(S)}\right)^{\theta\gamma(S)} < 1.$$



A Symmetric Pair of Stable Asymmetric Equilibria

- Home produces $s \in [0, S]$ and Foreign produces $s \in [S, 1]$,

$$\frac{S}{1-S} = \frac{Y}{Y^*} = \frac{\omega}{\omega^*} = \left(\frac{\Gamma^-(S)}{\Gamma^+(S)} \right)^{\theta\gamma(S)} < 1;$$

- Foreign produces $s \in [0, S]$ and Home produces $s \in [S, 1]$,

$$\frac{1-S}{S} = \frac{Y}{Y^*} = \frac{\omega}{\omega^*} = \left(\frac{\Gamma^+(S)}{\Gamma^-(S)} \right)^{\theta\gamma(S)} > 1.$$

Instability of Symmetric Equilibrium: $n = n^*$ ($= n^A$)

Stable Equilibrium Patterns in the J -Country World:

Index the countries so $\{n_j\}_{j=1}^J$ is monotone increasing. Then,

- $\frac{C_j(s)}{C_{j+1}(s)} = \left(\frac{n_j}{n_{j+1}}\right)^{-\theta\gamma(s)} \left(\frac{\omega_j}{\omega_{j+1}}\right)$, is strictly increasing in s :

- The unit interval is partitioned into J -subintervals: the j -th exports $s \in (S_j, S_{j+1})$, where

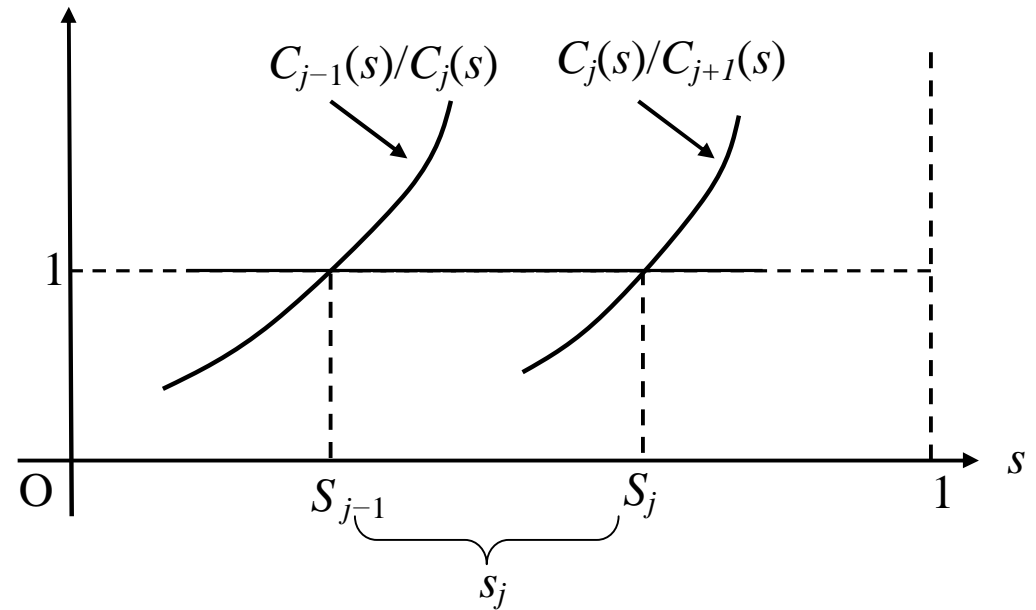
$$\{S_j\}_{j=1}^J \text{ is given by } S_0 = 0, S_J = 1 \text{ and } \frac{C_j(S_j)}{C_{j+1}(S_j)} = \left(\frac{n_j}{n_{j+1}}\right)^{-\theta\gamma(S_j)} \left(\frac{\omega_j}{\omega_{j+1}}\right) = 1.$$

- $\{\omega_j\}_{j=1}^J$ is monotone increasing.

- $Y_j = \omega_j F(K, L, \dots) = (S_j - S_{j-1}) Y^W$

- $n_j \propto \Gamma_j \equiv \frac{1}{S_j - S_{j-1}} \int_{S_{j-1}}^{S_j} \gamma(s) ds,$

hence monotone increasing, as assumed.



This can be summarized as:

Proposition 1 (the J -country case):

$\{S_j\}_{j=0}^J$ solves the nonlinear 2nd-order difference equation with the 2 terminal conditions:

$$\frac{S_{j+1} - S_j}{S_j - S_{j-1}} = \left(\frac{\Gamma(S_j, S_{j+1})}{\Gamma(S_{j-1}, S_j)} \right)^{\theta\gamma(S_j)} > 1 \text{ with } S_0 = 0 \text{ \& } S_J = 1,$$

where $\Gamma(S_{j-1}, S_j) \equiv \frac{1}{S_j - S_{j-1}} \int_{S_{j-1}}^{S_j} \gamma(s) ds.$

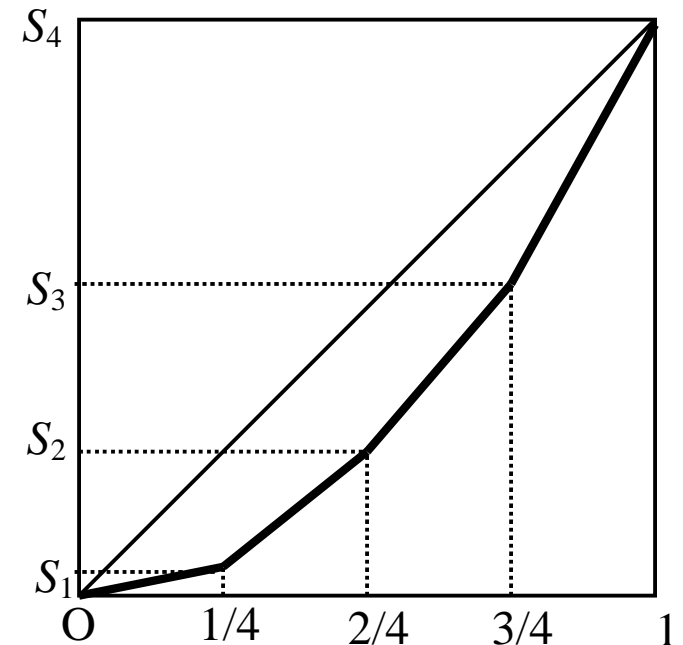
The Lorenz curve, $\Phi^J : [0,1] \rightarrow [0,1]$, is the piece-wise linear function, $\Phi^J(j/J) = S_j$. Clearly,

- Φ^J is strictly increasing & convex;
- $\Phi^J(0) = 0$ & $\Phi^J(1) = 1$.

But, it is not analytically solvable.

- Uniqueness?
- Comparative statics?
- Welfare evaluations?

These problems disappear as $J \rightarrow \infty$.



Calculating the limit Lorenz Curve: $\Phi^J \rightarrow \Phi$, as $J \rightarrow \infty$

$$\frac{S_{j+1} - S_j}{S_j - S_{j-1}} = \left(\frac{\Gamma(S_j, S_{j+1})}{\Gamma(S_{j-1}, S_j)} \right)^{\theta \gamma(S_j)} \quad \text{with } \Gamma(S_{j-1}, S_j) \equiv \frac{1}{S_j - S_{j-1}} \int_{S_{j-1}}^{S_j} \gamma(s) ds$$

By setting $x = j/J$ and $\Delta x = 1/J$,

$$S_{j+1} - S_j = \Phi(x + \Delta x) - \Phi(x) = \Phi'(x)\Delta x + \Phi''(x) \frac{|\Delta x|^2}{2} + o(|\Delta x|^2),$$

$$S_j - S_{j-1} = \Phi(x) - \Phi(x - \Delta x) = \Phi'(x)\Delta x - \Phi''(x) \frac{|\Delta x|^2}{2} + o(|\Delta x|^2),$$

from which

$$\text{LHS} = \frac{S_{j+1} - S_j}{S_j - S_{j-1}} = 1 + \frac{\Phi''(x)}{\Phi'(x)} \Delta x + o(|\Delta x|).$$

Likewise,

$$\Gamma(S_j, S_{j+1}) = \frac{\int_{\Phi(x)}^{\Phi(x+\Delta x)} \gamma(s) ds}{\Phi(x+\Delta x) - \Phi(x)} = \gamma(\Phi(x)) + \frac{1}{2} \gamma'(\Phi(x)) \Phi'(x) \Delta x + o(|\Delta x|)$$

$$\Gamma(S_j, S_{j-1}) = \frac{\int_{\Phi(x-\Delta x)}^{\Phi(x)} \gamma(s) ds}{\Phi(x) - \Phi(x-\Delta x)} = \gamma(\Phi(x)) - \frac{1}{2} \gamma'(\Phi(x)) \Phi'(x) \Delta x + o(|\Delta x|)$$

from which

$$\begin{aligned} \text{RHS} &= \left(\frac{\Gamma(S_j, S_{j+1})}{\Gamma(S_{j-1}, S_j)} \right)^{\theta\gamma(S_j)} = \left(1 + \frac{\gamma'(\Phi(x))}{\gamma(\Phi(x))} \Phi'(x)\Delta x + o(|\Delta x|) \right)^{\theta\gamma(\Phi(x))} \\ &= 1 + \theta\gamma'(\Phi(x))\Phi'(x)\Delta x + o(|\Delta x|) \end{aligned}$$

Combining these yields

$$1 + \frac{\Phi''(x)}{\Phi'(x)} \Delta x + o(|\Delta x|) = 1 + \theta\gamma'(\Phi(x))\Phi'(x)\Delta x + o(|\Delta x|).$$

Hence, as $J \rightarrow \infty$, $\Delta x = 1/J \rightarrow 0$,

$$\frac{\Phi''(x)}{\Phi'(x)} = \theta\gamma'(\Phi(x))\Phi'(x).$$

By integrating once,

$$\log(\Phi'(x)) - \theta\gamma(\Phi(x)) = c_0 \quad \Leftrightarrow \quad \exp(-\theta\gamma(\Phi(x)))\Phi'(x) = e^{c_0}$$

By integrating once again,

$$\int_0^{\Phi(x)} e^{-\theta\gamma(s)} ds = c_1 + e^{c_0} x.$$

From $\Phi(0) = 0$ & $\Phi(1) = 1$, $\Phi : [0,1] \rightarrow [0,1]$, is determined *uniquely* by

$$\int_0^{\Phi(x)} e^{-\theta\gamma(s)} ds = \left[\int_0^1 e^{-\theta\gamma(u)} du \right] x.$$

To summarize:

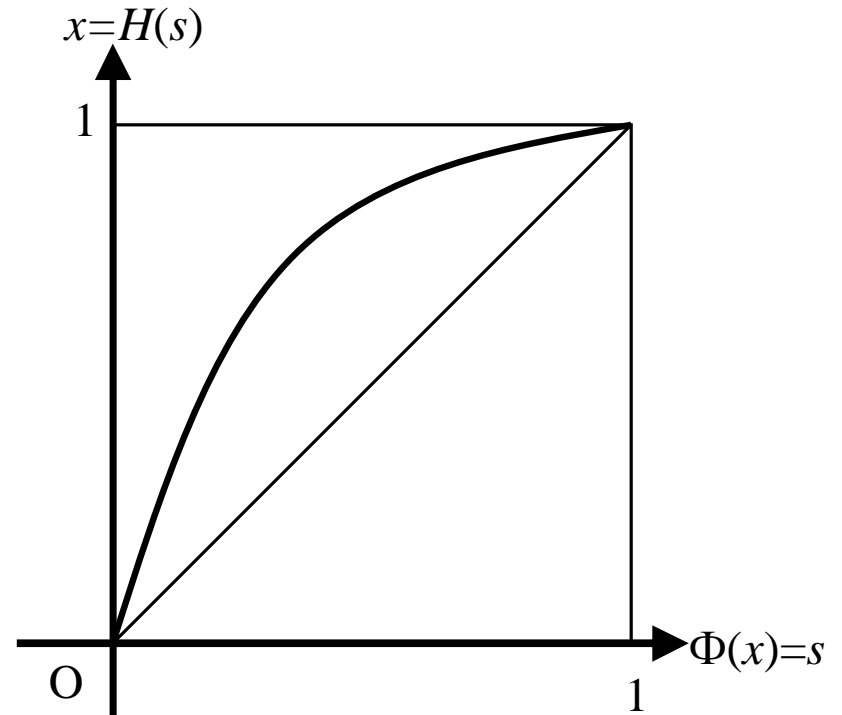
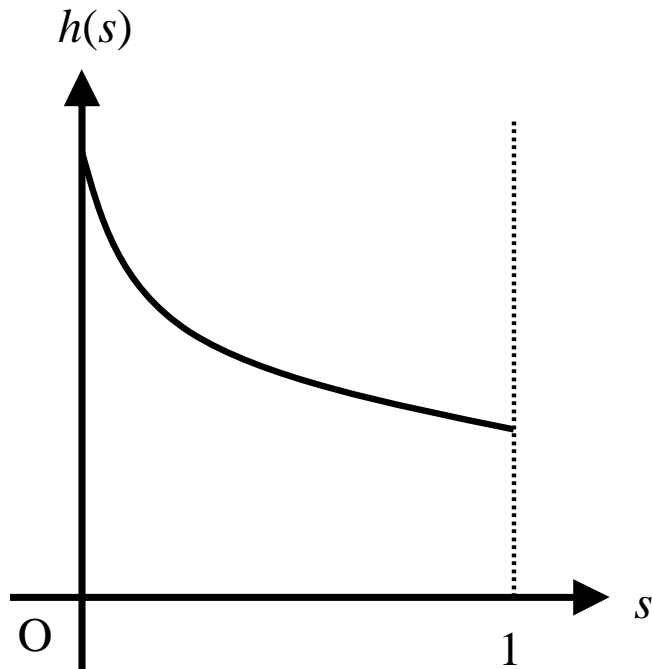
Proposition 2 (Limit Case; $J \rightarrow \infty$)

The limit equilibrium Lorenz curve, $\lim_{J \rightarrow \infty} \Phi^J = \Phi$, solves:

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Its unique solution is

$$x = H(\Phi(x)) \equiv \int_0^{\Phi(x)} h(s) ds, \quad \text{where } h(s) \equiv \frac{e^{-\theta \gamma(s)}}{\int_0^1 e^{-\theta \gamma(u)} du}.$$



Power-Law (Truncated Pareto) Examples:

	Example 1: $\gamma(s) = s$	Example 2: $\gamma(s) = \log[1 + (e^\theta - 1)s]^{1/\theta}$	Example 3: $\gamma(s) = \log[1 + (e^\lambda - 1)s]^{1/\lambda}$ ($\lambda \neq 0; \neq \theta$)
Inverse Lorenz Curve: $x = H(s)$	$\frac{1 - e^{-\theta s}}{1 - e^{-\theta}}$	$\log[1 + (e^\theta - 1)s]^{1/\theta}$	$\frac{[1 + (e^\lambda - 1)s]^{1 - \frac{\theta}{\lambda}} - 1}{e^{\lambda - \theta} - 1}$
Lorenz Curve: $s = \Phi(x)$	$\log[1 - (1 - e^{-\theta})x]$	$\frac{e^{\theta x} - 1}{e^\theta - 1}$	$\frac{[1 + (e^{\lambda - \theta} - 1)x]^{\frac{\lambda}{\lambda - \theta}} - 1}{e^\lambda - 1}$
Cdf: $x = \Psi(y)$ $= (\Phi')^{-1}(y)$	$\frac{1}{1 - e^{-\theta}} - \frac{1}{\theta y}$	$\frac{1}{\theta} \log\left(\frac{e^\theta - 1}{\theta} y\right)$	$\frac{\left(\frac{y}{y_{Min}}\right)^{\frac{\lambda}{\theta} - 1} - 1}{e^{\lambda - \theta} - 1} = 1 - \frac{1 - \left(\frac{y}{y_{Max}}\right)^{\frac{\lambda}{\theta} - 1}}{1 - e^{\theta - \lambda}}$
Pdf: $\psi(y) = \Psi'(y)$	$\frac{1}{\theta y^2}$	$\frac{1}{\theta y}$	$\left[\frac{(\lambda / \theta) - 1}{(y_{Max})^{(\lambda / \theta) - 1} - (y_{Min})^{(\lambda / \theta) - 1}} \right] (y)^{\frac{\lambda}{\theta} - 2}$
Support: $[y_{Min}, y_{Max}]$	$\frac{1 - e^{-\theta}}{\theta} \leq y$ $\leq \frac{e^\theta - 1}{\theta}$	$\frac{\theta}{e^\theta - 1} \leq y \leq \frac{\theta e^\theta}{e^\theta - 1}$	$\left(\frac{\lambda}{e^\lambda - 1}\right) \left(\frac{e^{\lambda - \theta} - 1}{\lambda - \theta}\right) \leq y$ $\leq \left(\frac{\lambda}{e^\lambda - 1}\right) \left(\frac{e^{\lambda - \theta} - 1}{\lambda - \theta}\right) e^\theta$

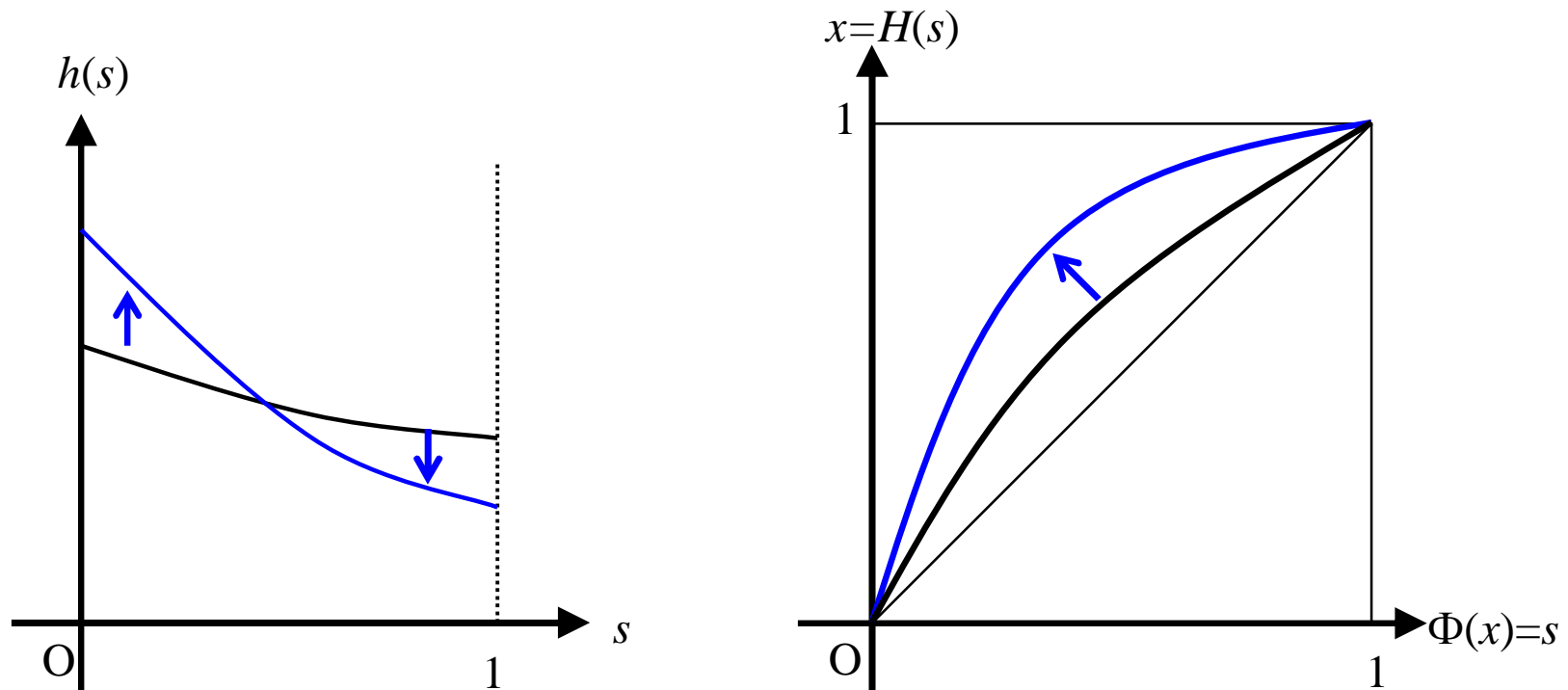
A lower λ makes the density function drop faster.

Log-submodularity and Effect of a higher θ :

Since $h(s) = \hat{h}(s) / \left[\int_0^1 \hat{h}(u) du \right]$, with $\hat{h}(s) \equiv e^{-\theta\gamma(s)}$ being *log-submodular* in θ and s ,

a higher θ rotates $h(s)$ “clockwise.”

→ Lorenz curve “bends” more (a Lorenz-dominant shift), hence a greater inequality.



Welfare Effects of Trade

Proposition 3 (the J -country case): The welfare of the k -th poorest country is

$$\log\left(\frac{U_k}{U^A}\right) = \sum_{j=1}^J \log\left(\frac{\omega_k}{\omega_j}\right)(S_j - S_{j-1}) + \theta \sum_{j=1}^J \Gamma_j \log\left(\frac{\Gamma_j}{\Gamma^A}\right)(S_j - S_{j-1}).$$

- 1st term: effects on the country's relative productivity, negative for some countries.
- 2nd term; gains from trade (conditional on productivity differences), positive for all.

Proposition 4 (Limit case, $J \rightarrow \infty$): The welfare of the country at 100 x^* % is given by

$$\frac{\log(U(x^*)/U^A)}{\theta} = \gamma(s^*) - \Gamma^A + \int_0^1 \gamma(s) \log\left(\frac{\gamma(s)}{\Gamma^A}\right) ds,$$

where $s^* = \Phi(x^*)$ or $x^* = \Phi^{-1}(s^*)$.

- 1st term; Relative productivity effect, negative for some countries.
- 2nd term; gains from trade, conditional on productivity differences, positive for all.

Corollary 1: All countries gain from trade iff $\frac{\gamma(0)}{\Gamma^A} \geq 1 - \int_0^1 \left(\frac{\gamma(s)}{\Gamma^A}\right) \log\left(\frac{\gamma(s)}{\Gamma^A}\right) ds$.

Corollary 2: Suppose $\frac{\gamma(0)}{\Gamma^A} < 1 - \int_0^1 \left(\frac{\gamma(s)}{\Gamma^A}\right) \log\left(\frac{\gamma(s)}{\Gamma^A}\right) ds$. Then, for $s_c > 0$ defined by

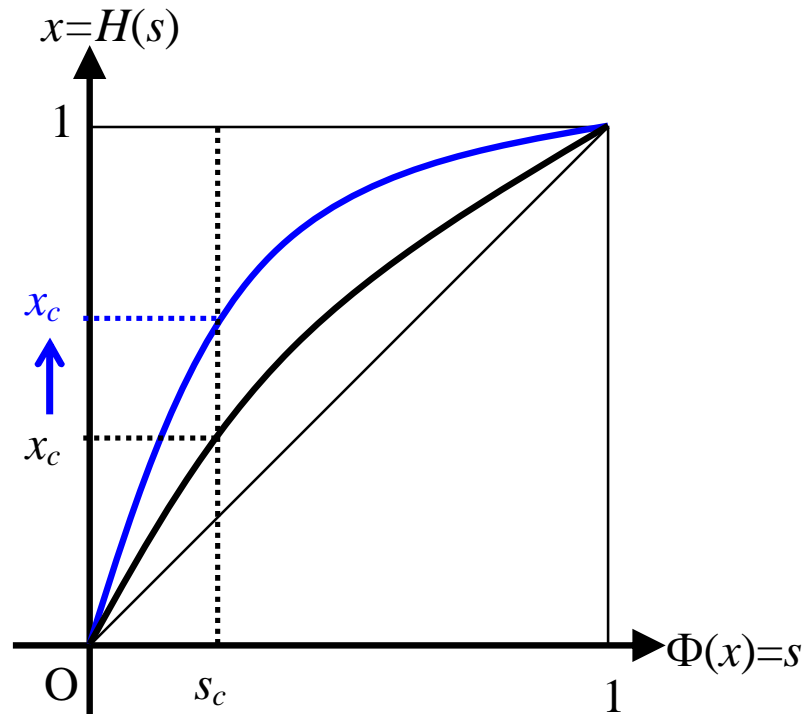
$$\gamma(s_c) \equiv \Gamma^A \left[1 - \int_0^1 \left(\frac{\gamma(s)}{\Gamma^A}\right) \log\left(\frac{\gamma(s)}{\Gamma^A}\right) ds \right],$$

a): All countries producing $s \in [0, s_c)$ lose from trade.

b): The fraction of the countries that lose, $x_c = H(s_c; \theta)$, is increasing in θ with

$$\lim_{\theta \rightarrow 0} x_c = s_c \text{ and } \lim_{\theta \rightarrow \infty} x_c = 1.$$

Corollary 2: A Graphic Illustration



3. *Two Extensions:*

3.1 *Nontradeable Consumption Goods:*

$$\log U = \tau \int_0^1 \log(X_T(s)) ds + (1 - \tau) \int_0^1 \log(X_N(s)) ds$$

τ ; the fraction of the consumption goods that are tradeable.

A higher τ causes a Lorenz dominant shift.

Globalization through Goods Trade magnifies inequality!

3.2 *Variable Factor Supply (through Factor Mobility or Factor Accumulation):*

$$V_j = F(K_j, L) \text{ with } \omega_j F_K(K_j, L) = \rho$$

Correlations between K/L and TFPs and per capita income

For $V = F(K, L) = AK^\alpha L^{1-\alpha}$ with $0 < \alpha < 1/(1 + \theta)$,

A higher α causes a Lorenz dominant shift.

Globalization through Factor Mobility or Skill-Biased Technological Change magnifies inequality!

In both extensions, the same techniques ($J \rightarrow \infty$ to solve the Lorenz curve analytically & log-submodularity to prove the Lorenz-dominant shifts) work.

Concluding Messages:

- Countries are not close economies; they trade with one another. The world economy is the only closed economy.
- Symmetry-breaking due to two-way causality; Even without ex-ante heterogeneity, interaction through trade could generate cross-country dispersion and correlations in per capita income, TFPs, and K/L ratios as stable equilibrium patterns.
- Some countries become richer (poorer) than others because they trade with poorer (richer) countries. They are *not* independent observations.
- This type of analysis does not suggest that ex-ante heterogeneity is unimportant. Instead, it suggests that even small ex-ante heterogeneity could be magnified to create huge ex-post heterogeneity.
- This paper demonstrates that this type of analysis does not have to be intractable nor lacking in prediction. Equilibrium distribution is *unique, analytically solvable*, varying with parameters in intuitive ways.
- A model with many countries can be more tractable than a model with a few countries.