Supplement to "Engel's Law in the Global Economy: Demand-Induced Patterns of Structural Change, Innovation, and Trade"

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Appendix A: (Direct and/or Indirect) Explicit vs. Implicit Additivity and Isoelastically Nonhomothetic CES

This appendix explains in detail why we use the particular class of preferences, isoelastically nonhomothetic CES, eq.(1), and why this must satisfy (direct and indirect) implicit additivity. To this end, we recall different notions of additivity. To simplify the exposition, we only consider the case of a continuum of infinitesimal consumption goods.

A1. (Direct and/or Indirect) Explicit Additivity:

Preference is *directly explicitly additive* if its *direct* utility function can be written *explicitly additively* as:

$$u = M\left[\int_I f_s(c_s)ds\right]$$

where c_s is consumption of $s \in I$, and $M[\cdot]$ is a monotone transformation. Most commonly used nonhomothetic preferences, including Stone-Geary and Constant Ratio of Income Elasticity (CRIE), are directly explicitly additive. Preference is *indirectly explicitly additive* if its *indirect* utility function can be written *explicitly additively* as:

$$u = M\left[\int_I g_s(p_s/E)ds\right]$$

where p_s is the price of $s \in I$, and E is the total expenditure. As shown in Samuelson (1965), the standard homothetic CES, whose direct utility function can be written as:

$$u = \left[\int_I \omega_s (c_s)^{\frac{\eta-1}{\eta}} ds\right]^{\frac{\eta}{\eta-1}}$$

and whose indirect function can be written as:

$$u = \left[\int_I (\omega_s)^{\eta} (p_s/E)^{1-\eta} ds\right]^{\frac{1}{\eta-1}} = E\Big/\left[\int_I (\omega_s)^{\eta} (p_s)^{1-\eta} ds\right]^{\frac{1}{1-\eta}}$$

is the *only* preference that satisfies both direct explicit additivity and indirect explicit additivity.

As Houthakker (1960) and Goldman and Uzawa (1964) and others have pointed out, the direct explicit additivity imposes the strong restriction between the income elasticity and the price elasticity of the goods called Pigou's Law. Formally, let $\varepsilon(s)$ denote the income elasticity of $s \in I$ and $\eta(s, s')$ the Allen-Uzawa elasticity of substitution between $s, s' \in I$. Under the direct explicit additivity, $\varepsilon(s_1)/\eta(s_1, s_3) = \varepsilon(s_2)/\eta(s_2, s_3)$, for any $s_1 \neq s_2 \neq s_3 \in I$; see eq.(2.11) in Hanoch (1975). That is, the ratio of income elasticity of a good and the cross-price elasticity of that good with respect to any other good is constant across all goods. In short, Pigou's Law states that the income elasticity of a good must be proportional to the price elasticity of that good.[1] Pigou's Law is not only rejected empirically, as shown by Deaton (1974) and others. It also makes directly explicitly additive preferences conceptually unsuited for our purpose, because the effects of the income elasticity differences across sectors cannot be disentangled from those of the price elasticity differences across sectors. In particular, nonhomothetic preferences that satisfy direct explicit additivity cannot be CES.

Likewise, indirect explicit additivity imposes the strong restriction between the income elasticity and the price elasticity of the form, $\eta(s_1, s_3) - \eta(s_2, s_3) = \varepsilon(s_1) - \varepsilon(s_2)$, for any $s_1 \neq s_2 \neq s_3 \in I$; see eq.(3.11) in Hanoch (1975). Again, this makes it impossible to isolate the effects of the income elasticity differences across sectors from those of the price elasticity differences across sectors. In particular, nonhomothetic preferences that satisfy indirect explicit additivity cannot be CES.

[1] The Bergson's Law, the homotheticity is equivalent to CES under the direct explicit additivity, is a special case of the Pigou's Law.

A2. (Direct and/or Indirect) Implicit Additivity:

In contrast, Hanoch (1975) showed that the income elasticity difference and the price elasticity difference can be controlled for separately under *implicit additivity*.[2] Preference is *directly implicitly additive* if its *direct* utility function can be written *implicitly additively* as:

$$M\left[\int_I f_s(u, c_s)ds\right] = 1.$$

Preference is *indirectly implicitly additive* if its *indirect* utility function can be written *implicitly additively* as:

$$M\left[\int_I g_s(u, p_s/E)ds\right] = 1.$$

Clearly, direct explicit additivity implies direct implicit additivity, and indirect explicit additivity implies indirect implicit additivity. Implicit additivity imposes less restriction than explicit additivity in both direct and indirect cases because a change in *u* can affect the relative weights attached on different consumption goods under implicit additivity, but not under explicit additivity. In particular, implicit additivity is not subject to Bergson's law, which means that it is possible to have homothetic non-CES, as explored in Matsuyama and Ushchev (2017), as well as nonhomothetic CES, which is our focus here.

A3. Isoelastically Nonhomothetic CES:

For the goal of this paper, it is important to isolate the role of income elasticity differences, which requires the preference to be CES. One can also show that CES, whose direct utility function is given implicitly by:

$$\left[\int_I \omega_s(u)(c_s)^{\frac{\eta-1}{\eta}} ds\right]^{\frac{\eta}{\eta-1}} = 1,$$

and whose indirect utility function is given implicitly by:

[2] This might remind the reader of the problem in macro-finance that intertemporally additive preferences impose the link between the intertemporal elasticity of substitution and the risk tolerance, and that the desire of delinking these parameters motivated Epstein and Zin (1989) to use a class of recursive preferences. I thank J. Markusen and I. Werning for this analogy.

$$\left[\int_I (\omega_s(u))^{\eta}(p_s/E)^{1-\eta}ds\right]^{\frac{1}{\eta-1}} = E\Big/\left[\int_I (\omega_s(u))^{\eta}(p_s)^{1-\eta}ds\right]^{\frac{1}{1-\eta}} = 1$$

is the only preference that satisfies both direct implicit additivity and indirect implicit additivity.[3] In spite of being a CES, this preference is *nonhomothetic* if $\partial \log \omega_s(u)/\partial u$ depends on $s \in I$. Furthermore, if sectors can be indexed such that $\partial \log \omega_s(u)/\partial u$ is monotone increasing $s \in I$, $\omega_s(u)$ becomes *log-supermodular* in *s* and *u*, which facilitates monotone comparative static exercises. In addition, empirically, the slope of the Engel's curve is stable. That is, the income elasticity differences across sectors are independent of the per capita real income, *u*. This requires that the weights of each good be *isoelastic* in *u* (i.e., a power function of *u*), hence $\partial \log \omega_s(u)/\partial \log(u)$ is independent of *u*. This allows us to define, as in Eq.(1), the sector-specific income elasticity, $\varepsilon(s)$, as a fixed parameter for each $s \in I$, which is monotone increasing, $s \in I$.

<u>Appendix B</u>: Two Lemmas

This appendix offers two lemmas, which are used repeatedly in the analysis.

Lemma 1: For a positive value function, $\hat{g}(\cdot, x): I \to R_+$, with a parameter *x*, define a density function on I by $g(s,x) \equiv \frac{\hat{g}(s,x)}{\int_I \hat{g}(t,x)dt}$ and denote its distribution function by $G(s,x)$. If $\hat{g}(s,x)$ is *log-supermodular* in *s* and *x*, *i.e.* $\frac{\partial^2 \log \hat{g}(s,x)}{\partial s \partial x} > 0$,

i) **Monotone Likelihood Ratio (MLR)**: $\frac{g(s,x_1)}{g(s,x_2)}$ is decreasing in s for $x_1 < x_2$;

ii) **First-order Stochastic Dominance (FSD)**: $G(s,x)$ is decreasing in *x*.

For the proof, see Matsuyama (2015, Appendix).[4]

[3]We are not aware of any existing proof of this. However, it can be adopted from the proof of Proposition 4(iii) in Matsuyama and Ushchev (2017). Even though this Proposition states that homothetic direct implicit additivity and homothetic indirect implicit additivity imply homothetic CES, homotheticity does not play any role in the proof.

[4]The results in this lemma are not new. For example, they were used in Matsuyama (2013, 2014) without proof. Furthermore, ii) follows from i). Indeed, they are special cases of more general properties of log-supermodularity known in the literature: see, e.g., Athey (2002) and Vives (1999; Ch.2.7). The proof in Matsuyama (2015, Appendix), however, is written without the language of the lattice theory.

Lemma 2: For $\eta \neq 1$, define $u: R_+ \to R_+$ implicitly by

(26) $\int_I x^{\left(\frac{\eta-1}{\sigma-\eta}\right)}\left[\beta_s(u(x))^{\varepsilon(s)-\eta}\right]^{\left(\frac{\sigma-1}{\sigma-\eta}\right)}ds \equiv 1.$

If $(\varepsilon(s)-\eta)/(1-\eta) > 0$,

i) $u(x)$ is increasing in *x;*

ii) $\zeta(x) \equiv \frac{xu'(x)}{u(x)}$ is decreasing in *x* if $\eta < 1$, and increasing in *x* if $\eta > 1$.

Proof: Rewrite the definition as $x^{\left(\frac{1-\eta}{\sigma-\eta}\right)} = \int_I \left[\beta_s(u(x))^{\varepsilon(s)-\eta}\right]^{\left(\frac{\sigma-1}{\sigma-\eta}\right)}ds$ and differentiating it yields

$$\left(\frac{1-\eta}{\sigma-\eta}\right)x^{\left(\frac{1-\eta}{\sigma-\eta}\right)-1} = \left(\frac{\sigma-1}{\sigma-\eta}\right)\int_I \left[\beta_s\big(u(x)\big)^{\varepsilon(s)-\eta}\right]^{\left(\frac{\sigma-1}{\sigma-\eta}\right)-1}\beta_s(\varepsilon(s)-\eta)\big(u(x)\big)^{\varepsilon(s)-\eta-1}u'(x)ds$$

$$\Leftrightarrow x^{\left(\frac{1-\eta}{\sigma-\eta}\right)} = \left(\frac{\sigma-1}{1-\eta}\right)\left(\frac{xu'(x)}{u(x)}\right)\int_I \left[\beta_s\big(u(x)\big)^{\varepsilon(s)-\eta}\right]^{\left(\frac{\sigma-1}{\sigma-\eta}\right)}(\varepsilon(s)-\eta)ds$$

$$\Leftrightarrow \frac{1}{\zeta(x)} = \frac{(\sigma-1)}{x^{\left(\frac{1-\eta}{\sigma-\eta}\right)}}\int_I \left(\frac{\varepsilon(s)-\eta}{1-\eta}\right)\left[\beta_s\big(u(x)\big)^{\varepsilon(s)-\eta}\right]^{\left(\frac{\sigma-1}{\sigma-\eta}\right)}ds$$

which can be further rewritten as:

(*) $$\frac{1}{\zeta(x)} = (\sigma-1)\int_I \left(\frac{\varepsilon(s)-\eta}{1-\eta}\right)g(s,x)ds = (\sigma-1)\int_I \left(\frac{\varepsilon(s)-\eta}{1-\eta}\right)dG(s,x) > 0$$

where $g(s,x) \equiv \frac{\left[\beta_s\big(u(x)\big)^{\varepsilon(s)-\eta}\right]^{\left(\frac{\sigma-1}{\sigma-\eta}\right)}}{\int_I \left[\beta_t\big(u(x)\big)^{\varepsilon(t)-\eta}\right]^{\left(\frac{\sigma-1}{\sigma-\eta}\right)}dt}$ is a density function, and $G(s,x)$ is its cumulative distribution function.

First, (*) shows $\zeta(x) \equiv \frac{xu'(x)}{u(x)} > 0$, hence $u(x)$ is increasing. Second, because $u(x)$ is increasing, $\left[\beta_s\big(u(x)\big)^{\varepsilon(s)-\eta}\right]^{\left(\frac{\sigma-1}{\sigma-\eta}\right)}$ is log-supermodular in *s* and *x*. Hence, from ii) of Lemma 1, $G(s,x)$ satisfies FSD. For $\eta < 1$, $\frac{\varepsilon(s)-\eta}{1-\eta}$ is increasing in *s,* so that RHS of (A1) is increasing in *x*, hence $\zeta(x)$ is decreasing in *x*. For $\eta > 1$, $\frac{\varepsilon(s)-\eta}{1-\eta}$ is decreasing in *s*, so that RHS of (A1) is decreasing in *x*, hence $\zeta(x)$ is increasing in *x*. **Q.E.D.**

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