Economic Growth & Development: Part 4
Vertical Innovation Models

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Introduction

- In the previous models, R&D develops products that are new, i.e., imperfect substitutes of the existing products. The economy grows through “expanding variety.”

- We now study models where a newly developed product is a perfect substitute of the existing product. The new product is a “better or improved” version of the old one, which it is going to replace. Products are “vertically differentiated.” The economy grows through “quality improvement.”

- Each product has its own “life-cycle.” It first replaces older vintages, but it will eventually be replaced by a new product in the future. Temporary monopoly power.

- Subtle welfare implication. On one hand, innovators do not value the monopoly profit earned by the producer of the old vintage that its successful innovation will destroy; this works in the direction of over-investment. On the other hand, they also know that the return to innovation is only temporary; this works in the direction of under-investment.

- These models may also be interpreted as models of process innovations. Each innovation comes up with a new way of producing the goods at a reduced cost.
Lab-Equipment Version: (Acemoglu Ch.14.1)

**Final Good Production:**

\[ Y(t) = \frac{1}{1 - \beta} \bar{X}(t)^{1-\beta} L^\beta, \text{ where } \bar{X}(t) \equiv \left[ \int_0^1 \left[ \sum_{q \in I(v,t)} q^{\zeta} x(v,t|q) \right]^{1 - \frac{1}{\sigma}} d\nu \right]^{\frac{\sigma}{\sigma - 1}} \]

- A continuum of industries, \( v \in [0,1] \), each producing a particular line of intermediates.
- \( I(v,t) \): The range of quality available for product line \( v \) at time \( t \).
- \( q \): the quality index of each product
- \( x(v,t|q) \): the units of the product used of quality \( q \) in product line \( v \) at time \( t \).

Within each product line, products of different quality are perfect substitutes. It turns out that, in equilibrium, only the product of highest quality available, denoted by \( q(v,t) \), is used at each moment. We also assume \( \beta = 1/\sigma \) and index quality such that \( \zeta \frac{1 - \beta}{1 - \beta} = 1 \).

\[ \Rightarrow Y(t) = \frac{1}{1 - \beta} \left[ \int_0^1 q(v,t) [x(v,t|q(v,t))]^{1-\beta} d\nu \right] L^\beta. \]
“Quality Ladder”: \( q(v,t) = \lambda^{n(v,t)} q(v,0). \)

- \( n(v,t); \) # of successful innovations between 0 and \( t \) in product line \( v \), a random variable.
- Within each product line, a new innovation improves the quality by factor of \( \lambda > 1 \).

R&D and Production Technologies for Intermediates:

- R&D is cumulative in the sense that it builds on the experiences of previous R&D.
- Only with \( q(v,t) \) currently available, it is feasible to invent quality, \( \lambda q(v,t) \).
- Investing \( Z(v,t) \) units of the final good generate a flow rate of success (Poisson arrival rate) equal to \( \eta Z(v,t) \left[ q(v,t) \right]^{\zeta_2}. \)

- Only new entrants conduct such R&D, not by the incumbent, which currently produces \( q(v,t) \). The incumbent has weaker incentives, because it would replace its own product, thus destroying the profits that they are currently making. Arrow’s replacement effect.
- Once invented, one unit of product of quality \( q \) can be produced with \( \psi(q)^{\zeta_3} \) units of the final good.
- Assume \( \zeta_2 = \zeta_3 = 1. \) (We need some restriction on \( \zeta_1, \zeta_2, \) and \( \zeta_3 \) to ensure the BGP; \( \zeta_1(1 - \beta) = \zeta_2 = \zeta_3 = 1 \) is one such restriction, but not the only one.)
Demand for an intermediate: \[ x(\nu, t) = L \left( \frac{q(\nu, t)}{p^x(\nu, t)} \right)^{1/\beta}. \]

Monopoly pricing: Each quality leader has quality advantage of \( (\lambda)^{\xi_1} = (\lambda)^{1/(1-\beta)} \) and cost disadvantage of \( (\lambda)^{\xi_3} = \lambda \) over the previous leader. In order to replace the previous leader, its price must satisfy \( p^x(\nu, t) / \lambda^{1/(1-\beta)} < \psi / \lambda \Rightarrow p^x(\nu, t) < \lambda^{\beta/(1-\beta)} \psi \). This constraint is not binding if \( \lambda \) is sufficiently large (the drastic innovation case) such that,

\[
\lambda > \left( \frac{1}{1-\beta} \right)^{1-\beta}. 
\]

In this case, the leader sets \( p^x(\nu, t)(1-\beta) = \psi q(\nu, t) \). Normalize \( \psi = 1-\beta \). Then,

\[
p^x(\nu, t) = q(\nu, t), \ x(\nu, t) = L, \text{ and } \pi(\nu, t) = \beta q(\nu, t)L \text{ for all } \nu \text{ & } t
\]

as long as it remains the quality leader.

\[
\Rightarrow Y(t) = \frac{Q(t)}{1-\beta} L, \ V(t) = (1-\beta)Q(t)L, \text{ & } w(t) = \frac{\beta Q(t)}{1-\beta}
\]

where \( Q(t) \equiv \int_0^1 q(\nu, t) d\nu \) is the average quality across sectors, which is the engine of growth.
Value of a Quality Leader: Even if each product is forever protected by patent, its value is destroyed when it is replaced by innovation of a better product.

\[
\dot{r(t)V(v,t|q)} = V(v,t|q) + \pi(v,t|q) - z(v,t|q)V(v,t|q)
\]

where \( z(v,t|q) = \eta Z(v,t|q) / q \) is the flow rate at which a successful innovation takes place in \( v \) at \( t \).

R&D (Innovation): Free entry

\[
\eta V(v,t|q) \leq q / \lambda \quad \text{and} \quad \eta V(v,t|q) = q / \lambda \quad \text{if} \quad z(v,t|q) > 0.
\]

Note: Innovators make zero profit. This means that, when they innovate across sectors, they are indifferent about how much they invest in each sector.
**Evolution of $Q(t)$:** Suppose $z(v,t|q) = z(t)$ for all $v$ at $t$. In an interval of time, $\delta t$,

- $z(t)\delta t$ sectors experience one innovation, which will increase their quality by $\lambda$.  
- The measure of sectors experiencing more than one innovation is $o(\delta t)$.

\[ \Rightarrow Q(t + \delta t) = \lambda Q(t) z(t) \delta t + Q(t)(1 - z(t)\delta t)) + o(\delta t) \Rightarrow \dot{Q}(t) = (\lambda - 1) z(t) Q(t). \]

Since $z(v,t|q) = z(t) \Rightarrow z(t) q(v,t) = \eta Z(v,t|q) \Rightarrow z(t) Q(t) = \eta Z(t)$,

\[ \dot{Q}(t) = (\lambda - 1) \eta Z(t). \]

No aggregate fluctuation because there are many (a continuum of) sectors.

**Characterizing BGP:** Look for BGP along which

- $r(t) = r^*$ such that $\bar{Q} / Q = \bar{Y} / Y = \bar{C} / C = (r^* - \rho) / \theta > 0$;

- From the aggregate resource constraint, $Y(t) = C(t) + X(t) + Z(t)$,

\[ \frac{L}{1 - \beta} = \frac{C(t)}{Q(t)} + (1 - \beta)L + \frac{Z(t)}{Q(t)} = \frac{C(t)}{Q(t)} + (1 - \beta)L + \frac{z(t)}{\eta} \Rightarrow z(t) = z^* > 0. \]
Free Entry: \[ V(v, t|q) = \frac{q}{\eta \lambda} \equiv V(q) \]

Valuation of a Firm: \[ V(q) = \frac{\pi(q)}{r^* + z^*} = \frac{\beta q L}{r^* + z^*} \]

Combining these, \[ r^* + z^* = \lambda \eta \beta L \]

\[ \Rightarrow \frac{r^* - \rho}{\theta} = g^* = \frac{Q}{Q} = (\lambda - 1)z^* = (\lambda - 1)(\lambda \eta \beta L - r^*) \]

\[ \Rightarrow g^* = \frac{\lambda \eta \beta L - \rho}{\theta + (\lambda - 1)^{-1}} > 0. \]

From \[ \frac{L}{1 - \beta} = \frac{C(t)}{Q(t)} + (1 - \beta)L + \frac{z^*}{\eta}, \]

\[ C(t) = \left[ \frac{L}{1 - \beta} - (1 - \beta)L - \frac{z^*}{\eta} \right] Q(t). \]

For the existence, we need \[ \lambda \eta \beta L > \rho > (1 - \theta) \frac{\lambda \eta \beta L - \rho}{\theta + (\lambda - 1)^{-1}}. \]
Notes:
- Again, the scale, efficiency, and taste parameters have the growth effects.
- Starting from any \( Q(0) > 0 \), there is an equilibrium path along which the economy grows at the constant rate, \( g^* \). But, I am not convinced that this is the only equilibrium path (although Acemoglu asserts that it is).
- The optimal growth is also a balanced growth path. Unlike the horizontal innovation models, the optimal growth rate can be higher or lower than the growth rate in the equilibrium balanced growth path.

Tax Policy on R&D spending:
By discouraging R&D, this increases the value of an incumbent as \( V(q) = \frac{(1 + \tau)q}{\eta \lambda} \).

Since \( V(q) = \frac{\beta q L}{r^* + z^*} \),

\[
\Rightarrow r^* + z^* = \frac{\lambda \eta \beta L}{1 + \tau} \Rightarrow \frac{r^* - \rho}{\theta} = g^* = (\lambda - 1)z^* = (\lambda - 1)\left(\frac{\lambda \eta \beta L}{1 + \tau} - r^* \right)
\]

\[
\Rightarrow g^* = \frac{1}{\theta + (\lambda - 1)^{-1} \left(\frac{\lambda \eta \beta L}{1 + \tau} - \rho \right)}.
\]

Hence, it reduces the growth rate.
Aghion-Howitt Model: (Acemoglu Ch.14.2)
Innovation by Both Incumbents & Entrants: (Acemoglu Ch.14.3)

So far, R&D is done only by entrants. This model allows R&D both by incumbents and entrants. The lab-equipment version (Ch.14.1) is modified as follows:

- A small incremental improvement (“tinkering”) can be done only by incumbents.
- The parameters are also changed so that, incumbents grow in size as they improve the quality of their products, to obtain the firm size dynamics.
- A drastic innovation is done only by entrants in equilibrium. (Again, incumbents have no incentives to do so in equilibrium due to the Arrow’s replacement effect).

**Final Goods Production:** Index quality such that \( \zeta_1(1 - \beta) = \beta \).

\[
\Rightarrow Y(t) = \frac{1}{1 - \beta} \left[ \int_0^1 q(v, t)^\beta x(v, t|q(v, t)) ]^{1 - \beta} dv \right] L^\beta
\]

**Intermediate Inputs Production:** Once invented, one unit of product of quality \( q \) can be produced with \( \psi(q) \zeta_3 = (1 - \beta)(q) \zeta_3 \) units of the final good, where \( \zeta_3 = 0; \) i.e., it is equal to \( \psi = 1 - \beta \), independent of \( q \).
Small quality improvement (tinkering): R&D available only to incumbents:

“Quality Ladder”: \( q(v, t) = \lambda^{n(v, t)} q(v, s) \), where \( \lambda > 1 \)
- \( n(v, t) \); # of successful improvement between \( s \) and \( t > s \) in product line \( v \), a random variable.
- \( s \) is the date at which this incumbent took over this product line, by making a drastic innovation.
- Tinkering upgrades the current quality level, \( q(v, t) \), to the next level, \( \lambda q(v, t) \).
- Investing \( z(v, t)q(v, t) \) units of the final good by the incumbent generates a flow success rate (Poisson arrival rate) equal to \( \frac{\phi z(v, t)q(v, t)}{[q(v, t)]^{\zeta_2}} = \phi z(v, t) \), with \( \zeta_2 = 1 \).

Drastic innovation (Creative destruction): R&D pursued only by entrants:
- For the current quality \( q(v, t) \), a successful drastic innovation leads to \( \kappa q(v, t) \), \( \kappa > \lambda \).
- Each unit of the final good invested by an entrant in R&D generates a flow success rate of \( \eta(\hat{z}(v, t))/q(v, t) \), where \( \hat{z}(v, t) \) is the total R&D spending by all entrants, divided by \( q(v, t) \), so that the flow success rate is \( \eta(\hat{z}(v, t))\hat{z}(v, t) \).
- \( \eta(z) \) is strictly decreasing. This captures external diminishing returns, which each entrant takes as given. The negative externalities are mild so that \( z\eta(z) \) is strictly increasing. Assume \( \lim_{z \to \infty} \eta(z) = 0 \); \( \lim_{z \to 0} \eta(z) = \infty \), to ensure the interior solution.
**Demand for an intermediate:** \( x(v, t) = q(v, t)L (p^x(v, t))^{-1/\beta} \).

**Monopoly pricing:** Each incumbent has at least quality advantage of \((\kappa)^{\xi_1} = (\kappa)^{\beta/(1-\beta)}\) over the previous leader, but no cost disadvantage, because of \((\kappa)^{\xi_3} = 1\). Hence, the leader must set its price such that \( p^x(v, t) < (\kappa)^{\beta/(1-\beta)}\psi \).

Assume that the innovation by entrants is drastic enough that

\[ \kappa > \left( \frac{1}{1-\beta} \right)^{\frac{1-\beta}{\beta}}. \]

Then, the leader sets its monopoly price unconstrained, \( p^x(v, t) = \psi / (1-\beta) = 1 \).

\[ \Rightarrow x(v, t) = q(v, t)L, \text{ and } \pi(v, t) = \beta q(v, t)L \text{ for all } v \& t \]

\[ \Rightarrow Y(t) = \frac{L}{1-\beta} Q(t), \; X(t) = (1-\beta)Q(t)L, \text{ and } w(t) = \frac{\beta}{1-\beta} Q(t) \]

where \( Q(t) \equiv \int_0^1 q(v, t)dv \) is again the average quality across sectors.
Value of an Incumbent producing $q$: Keep the notation simple by $V(v,t|q) = V(q)$;

$$r(t)V(q) = V(q) + \pi(q) + \max\{ (\phi\zeta)(V(\lambda q) - V(q)) - zq \} - \eta(\hat{z}(q))\hat{z}(q)V(q)$$

R&D by Entrants (Creative Destruction): $\eta(\hat{z}(v,t|q))V(v,t|\kappa q) = q$ with $\hat{z}(v,t|q) > 0$.

There is always some R&D by entrants, since $\lim_{z \to 0} \eta(z) = \infty$.

R&D by Incumbents (Tinkering):

$$\phi(V(v,t|\lambda q) - V(v,t|q)) \leq q; \quad \phi(V(v,t|\lambda q) - V(v,t|q)) = q \quad \text{if } z(v,t|q) > 0.$$ 

Evolution of $Q(t)$: suppose $z(v,t|q) = z(t)$ and $\hat{z}(v,t|q) = \hat{z}(t)$ for all $v$, $q$, and $t$. Then,

$$Q(t + \delta t) = (\lambda\phi\zeta(t)\delta t)Q(t) + \kappa\eta(\hat{z}(t))\hat{z}(t)Q(t) + (1 - \phi\zeta(t)\delta t - \kappa\eta(\hat{z}(t))\hat{z}(t)\delta t)Q(t) + o(\delta t)$$

$$\Rightarrow \frac{\dot{Q}(t)}{Q(t)} = (\lambda - 1)\phi\zeta(t) + (\kappa - 1)\eta(\hat{z}(t))\hat{z}(t).$$
**Balanced Growth Path:** Let us look for the BGP where

- $r(t) = r^*$ such that $C/C = (r^* - \rho) / \theta \equiv g^* > 0$
- $\dot{z}(v, t|q) = \dot{z} > 0$ and $z(v, t|q) = z > 0$.
- $V(v, t|q) = vq$

Positive R&D by Incumbents $\Rightarrow V(q) = \frac{q}{\phi(\lambda - 1)}$.

Valuation of a Firm $\Rightarrow r^*V(q) = \beta L q - \eta(\dot{z})\dot{z}V(q) \Rightarrow V(q) = \frac{\beta L q}{r^* + \eta(\dot{z})\dot{z}}$

$\Rightarrow r^* = \phi(\lambda - 1)\beta L - \eta(\dot{z})\dot{z};$

Free Entry by Entrants $\Rightarrow \eta(\dot{z}) = \frac{q}{V(\kappa q)} = \frac{\phi(\lambda - 1)}{\kappa}$

$\Rightarrow g^* = \frac{r^* - \rho}{\theta} = \frac{\phi(\lambda - 1)\beta L - \eta(\dot{z})\dot{z} - \rho}{\theta} = \frac{\eta(\dot{z})\kappa \beta L - \eta(\dot{z})\dot{z} - \rho}{\theta}$.

We also have: $g^* = \dot{Q}/Q = (\lambda - 1)\phi z + (\kappa - 1)\eta(\dot{z})\dot{z}$,

which determines $z.$
Again, for the existence of this BGP, we must verify:
- Incumbents have an incentive to do R&D.
- \((1 - \theta)r^* < \rho < r^*\).

**Effects of \(\lambda\) & \(\kappa\):** From \(g^* = \frac{\phi(\lambda - 1) \beta L - \eta(\hat{z}) \hat{z} - \rho}{\theta}\) and \(\eta(\hat{z}) = \frac{\phi(\lambda - 1)}{\kappa}\),

\[
\lambda \uparrow \rightarrow \hat{z} \downarrow \rightarrow \eta(\hat{z}) \hat{z} \downarrow \rightarrow g^* \uparrow; \quad \kappa \uparrow \rightarrow \hat{z} \uparrow \rightarrow \eta(\hat{z}) \hat{z} \uparrow \rightarrow g^* \downarrow.
\]

- More creative destruction reduces the growth rate.
- This is because it reduces the incremental innovation by the incumbents:
  \(\kappa \uparrow, \eta(\hat{z}) \hat{z} \uparrow, \text{ and } g^* \downarrow \rightarrow z \downarrow \text{ from } g^* = (\lambda - 1)\phi z + (\kappa - 1)\eta(\hat{z}) \hat{z}.\)

Indeed, Acemoglu (Proposition 14.6) show that, while taxation on R&D spending by incumbents are growth-reducing, taxation on R&D spending by entrants are growth-enhancing, in strong contrast with the baseline model of Ch.14.1.

**My partial intuition:** Encouraging R&D by the entrants discourages R&D by the incumbents, while encouraging R&D by the incumbents will not discourage R&D by the entrants, because they do R&D to become incumbents.

Acemoglu also discusses the model’s implication on firm size dynamics.
Step-By-Step Innovation: (Acemoglu Ch.14.4)